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On the Impossibility of Packing Space with Different Cubes

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It is impossible to pack 3-space with cubes in such a way that no two neighbouring cubes are the same size and that no ball contains infinitely many of the cubes.

INTRODUCTION

In 1964, Daykin [3] asked, "Can space be filled by disjoint integer cubes, no two cubes being the same size, and the lengths of the cubes being integers?" The two-dimensional version of this problem is easily solved by packing in a spiral around a "perfect squared rectangle." This might suggest that the problem could hang on the number-theoretic properties of the integers, as a certain amount of what could be described as number theory is involved in the theory of squared rectangles. (See, for instance, Dehn [4], or Brooks, Smith, Stone, and Tutte [1].)

However, a negative answer to Daykin's question may be obtained [2] without using any property of the integers except that any strictly decreasing sequence of positive integers must be finite. This suggests that we may be able to get rid of the edge-length condition entirely, and still obtain a negative answer. In this paper, we will see that this is correct, although we must introduce the condition of "local finiteness" to prevent the trivial (and inelegant!) construction of a packing by a "greedy algorithm" in which we just put cubes into any gap that shows.

As in [2], the method used will be based upon that used in [1] to prove the impossibility of packing a cube with different integer cubes. As a first step, we will prove, rather surprisingly, that there is essentially only one way to cover a unit cube with cubes of different sizes none of whose edges

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are less than or equal to $\frac{1}{2}$ and that, in this covering, all the neighbours are larger than the central cube. We will then continue and find, in a hypothetical packing of space with different cubes, a sequence of cubes whose centers converge—an obvious contradiction of local finiteness.

While in outline this paper resembles [2], the construction must necessarily be more complicated, as we want to achieve a significantly faster decrease in the size of the cubes in the sequence constructed. This means a great increase in the number of cubes and parts of cubes involved. Therefore, while in [2] I remarked that an arithmetization of this sort of problem was "impractical and unclear," I have here been forced to arithmetize!

1. Definitions

By a *cube*, we will mean a set A of points in $\mathbb{R}^3$ of the form

$$A = \{(x, y, z)\colon \mathbf{A}_\mathrm{d} \leqslant x \leqslant \mathbf{A}_\mathrm{u}, \mathbf{A}_\mathrm{l} \leqslant y \leqslant \mathbf{A}_\mathrm{r}, \mathbf{A}_\mathrm{f} \leqslant z \leqslant \mathbf{A}_\mathrm{b}\},$$

where $(\mathbf{A}_\mathrm{u} - \mathbf{A}_\mathrm{d}) = (\mathbf{A}_\mathrm{r} - \mathbf{A}_\mathrm{l}) = (\mathbf{A}_\mathrm{b} - \mathbf{A}_\mathrm{f}) = \mathrm{l}(A) > 0$. $\mathbf{A}_\mathrm{u}$, $\mathbf{A}_\mathrm{d}$, $\mathbf{A}_\mathrm{r}$, $\mathbf{A}_\mathrm{l}$, $\mathbf{A}_\mathrm{b}$, and $\mathbf{A}_\mathrm{f}$ are thus the coordinates defining the planes containing the upper, downmost, right, left, back, and front faces, respectively, of the cube A; $\mathrm{l}(A)$ is the edge length of A. We shall also use these subscripted letters to refer to the faces themselves; in this usage, they will not be boldface. *Squares* in $\mathbb{R}^2$ will be defined analogously, using only the first four subscripts.

Two cubes (resp. squares) will be said to be *neighbours* if their faces (edges) have an intersection of nonzero area (length); and *weak neighbours* if they intersect. A collection of cubes is a *packing of space* if their union is all of $\mathbb{R}^3$ and any two cubes in the collection intersect at most in their boundaries.

Given a cube A in a collection $\mathscr{C}$, the set of elements of $\mathscr{C}$ that are neighbours of A is called its *star*, written $\mathrm{St}(A)$. A collection will be called *locally finite* if every ball in $\mathbb{R}^3$ intersects at most finitely many members of the collection. If no two neighbouring cubes in a collection have the same edge length, we will call the collection *locally heterogeneous*. A cube whose boundary is contained in the union of the boundaries of its neighbours will be said to be *covered*.

Any cube in a locally finite collection has a neighbour of minimal size; for each cube A in the collection, select a cube $A' \in \mathrm{St}(A)$ such that

$$\mathrm{l}(A') = \min_{X \in \mathrm{St}(A)} \mathrm{l}(X).$$

(Note that this may not be uniquely defined.)

If a cube is covered and smaller than all of its neighbours, we say that it is the *core* of a *singular star*. If a cube A is covered and satisfies

$$\mathrm{l}(A) < 2\,\mathrm{l}(A'),$$

we say that A is the core of a *semisingular* star.

Edges and *vertices* of cubes (and vertices of squares) are intersections of faces (resp. edges). We may write them as follows:

$$A_i \cap A_j = A_{ij},$$

$$A_i \cap A_j \cap A_k = A_{ijk}.$$

Thus, e.g., A_{lf} will be the left front edge of cube A. Note that not all combinations are meaningful; a cube has no vertex A_{rlb}!

If a cube A is covered, there is, for every pair (A_i, A_{ijk}) exactly one cube that has an intersection of nonzero area with A_i and contains A_{ijk}. We shall call this cube $N(A_{i(jk)})$.

A neighbour B of a cube (square) A will be said to *overhang* the edge (resp. vertex) A_{ij} if the interior of some face (edge) of B intersects A_{ij}.

Given any cube A, $\mathrm{c}(A)$ is the *center* $\left(\dfrac{\mathbf{A}_{\mathrm{u}} + \mathbf{A}_{\mathrm{d}}}{2}, \dfrac{\mathbf{A}_{\mathrm{l}} + \mathbf{A}_{\mathrm{r}}}{2}, \dfrac{\mathbf{A}_{\mathrm{b}} + \mathbf{A}_{\mathrm{f}}}{2}\right)$.

2. Configurations of Semisingular Stars

In this section, we will demonstrate that there is essentially only one semisingular star configuration. This configuration is in fact singular; its uniqueness as a singular star configuration is very easy to see and has the status of folklore. It is rather less obvious that the relaxation of conditions to semisingularity does not in fact allow any new (locally heterogeneous) configurations.

2.1. Lemma. *If a square A in a locally heterogeneous collection is the core of a semisingular star, the neighbour*(*s*) *on any edge of A extend beyond exactly one end of that edge.*

Proof. The neighbour(s) on one edge of A cannot be flush with both adjacent edges, for if there were one neighbour, it would be the same size as A (Fig. 1a), whereas if there were two or more, one of them would have an edge length less than $\frac{1}{2}\,\mathrm{l}(A)$ (Fig. 1b).

Furthermore, at most one neighbour can extend beyond A at any vertex, or an overlap (Fig. 1c) would result. So, by the pigeonhole principle, the

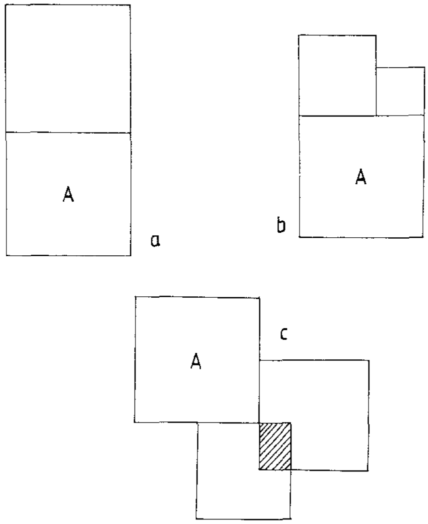

FIGURE 1

neighbours on any one side of A extend beyond exactly one adjacent edge. ∎

2.1.1. COROLLARY. *Let X_i, X_j be adjacent faces of the core X of a semisingular star in a locally heterogeneous collection of cubes; for precisely one of the two X_k adjacent to both X_i and X_j,*

$$\mathbf{N}(X_{i(jk)})_k = \mathbf{X}_k.$$

Proof. Consider a plane cutting X parallel and very close to X_j, containing no face of a neighbour of X. Its intersections with the various cubes of the collection form a locally heterogeneous collection of squares; applying the lemma, we get the desired result (Fig. 2). ∎

2.2. THEOREM. *In a locally heterogeneous collection of cubes, let X be the core of a semisingular star. Then X is the core of a singular star; and the configuration of* St(X) *is, up to rotation, reflection, and variation in the sizes of the cubes, shown in Fig.* 3.

Proof. We first show that the set of neighbours on any face of X must contain a cube with two adjacent faces flush with the corresponding faces of X; then we show that this cube must be larger than X. The configuration for the entire star follows.

Consider any face of X, without loss of generality X_f. We show that some $N(X_{f(ij)})$ must overhang neither X_{fi} nor X_{fj} (and hence have

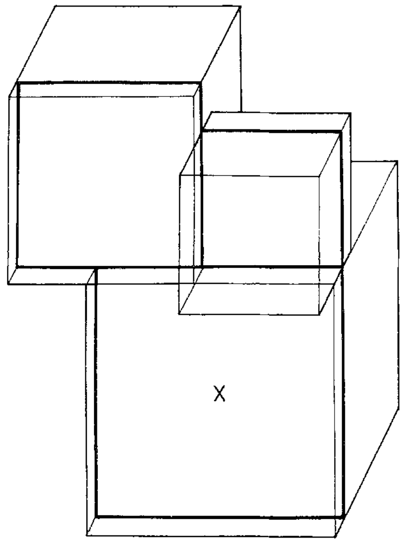

FIGURE 2

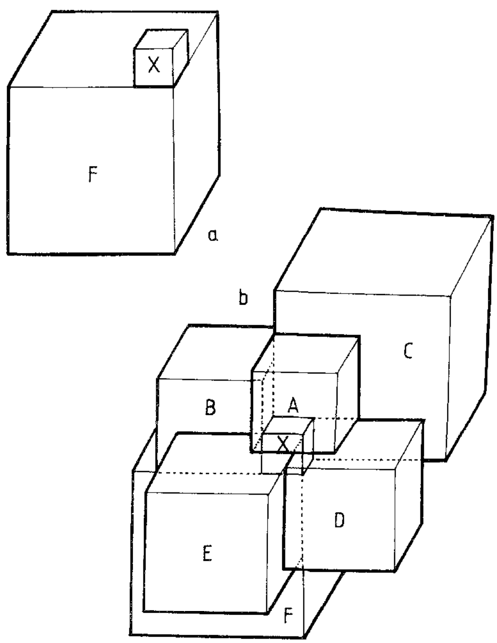

FIGURE 3

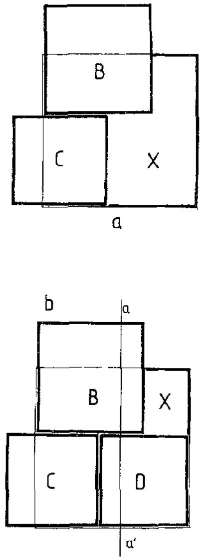

FIGURE 4

corresponding faces flush with both X_i and X_j). For suppose this not to be the case; by Corollary 2.1.1 and the pigeonhole principle, each $N(X_{\mathrm{f}(ij)})$ must overhang exactly one of $X_{\mathrm{f}i}$, $X_{\mathrm{f}j}$. Let B be the largest $N(X_{\mathrm{f}(ij)})$, and $X_{\mathrm{f}j}$ the edge that it overhangs; without loss of generality, we will assume these to be $N(X_{\mathrm{f(ul)}})$ and X_{uf}, so $\mathbf{B}_{\mathrm{l}} = \mathbf{X}_{\mathrm{l}}$, $\mathbf{B}_{\mathrm{u}} > \mathbf{X}_{\mathrm{u}}$. Let C be $N(X_{\mathrm{f(ld)}})$; by Corollary 2.1.1, $\mathbf{C}_{\mathrm{d}} = \mathbf{X}_{\mathrm{d}}$. Then, by our hypothesis, $\mathbf{C}_{\mathrm{l}} < \mathbf{X}_{\mathrm{l}} = \mathbf{B}_{\mathrm{l}}$. But

$$\begin{aligned} \mathbf{C}_{\mathrm{r}} &= \mathbf{C}_{\mathrm{l}} + \mathrm{l}(C) && \text{(from the definition of a cube)} \\ &< \mathbf{B}_{\mathrm{l}} + \mathrm{l}(B) \\ &= \mathbf{B}_{\mathrm{r}} && \text{(Fig. 4a).} \end{aligned}$$

Now consider the cube D that touches B_{d}, C_{r}, and X_{f}. Applying Lemma 2.1 to the cross section $\overline{aa'}$ in Fig. 4b, we see that $\mathbf{D}_{\mathrm{d}} = \mathbf{X}_{\mathrm{d}}$, and thus that

$$\begin{aligned} \mathrm{l}(D) &= \mathbf{B}_{\mathrm{d}} - \mathbf{X}_{\mathrm{d}} \\ &= \mathrm{l}(C), \end{aligned}$$

contradicting our assumption of local heterogeneity.

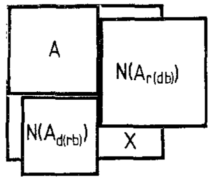

FIGURE 5

Therefore there must exist a cube A touching X_f, with two adjacent faces (without loss of generality A_u and A_l) flush with the corresponding faces of X. We will show that $l(A) > l(X)$; for suppose this not to be the case. Either

$$(\mathbf{N}(A_{d(rb)}))_r = \mathbf{A}_r$$

or

$$(\mathbf{N}(A_{r(db)}))_d = \mathbf{A}_d,$$

because if they both overhung A_{rd} they would intersect (Fig. 5). Assume the former (the two cases are equivalent by symmetry). Either $N(A_{d(rb)})$ is larger than A, or it is smaller. In the first case,

$$\begin{aligned}(\mathbf{N}(A_{d(rb)}))_l &= (\mathbf{N}(A_{d(rb)}))_r - l(N(A_{d(rb)}))\\ &< \mathbf{A}_r - l(A)\\ &= \mathbf{A}_l\\ &= \mathbf{X}_l,\end{aligned}$$

and hence $N(A_{d(rb)}) = N(X_{f(dl)})$ (Fig. 6a). In the second case,

$$(\mathbf{N}(A_{d(rb)}))_l > \mathbf{X}_l,$$

and so $N(A_{d(rb)})$ and $N(X_{f(dl)})$ are different cubes (Fig. 6b). Then

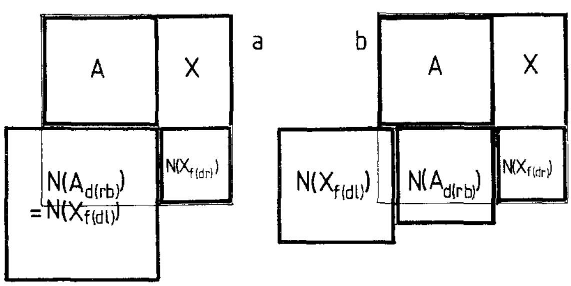

FIGURE 6

$$
\begin{aligned}
(\mathbf{N}(X_{\mathrm{f(dl)}}))_{\mathrm{l}} &= \mathbf{A}_{\mathrm{r}} - \mathrm{l}(N(A_{\mathrm{d(rb)}})) - \mathrm{l}(N(X_{\mathrm{f(dl)}})) \\
&< \mathbf{A}_{\mathrm{r}} - \tfrac{1}{2}\mathrm{l}(X) - \tfrac{1}{2}\mathrm{l}(X) && \text{(semisingularity)} \\
&< \mathbf{A}_{\mathrm{r}} - \mathrm{l}(A) && \text{(hypothesis, } \mathrm{l}(A) < \mathrm{l}(X)\text{)} \\
&= \mathbf{A}_{\mathrm{l}}.
\end{aligned}
$$

In either case, $(\mathbf{N}(X_{\mathrm{f(dl)}}))_{\mathrm{l}} < \mathbf{X}_{\mathrm{l}}$; and so, by Corollary 2.1.1, $(\mathbf{N}(X_{\mathrm{f(dr)}}))_{\mathrm{r}} = \mathbf{X}_{\mathrm{r}}$. But this requires that

$$\mathrm{l}(N(X_{\mathrm{f(dr)}})) + \mathrm{l}(A) \leqslant \mathrm{l}(X),$$

impossible under our assumption of semisingularity and local heterogeneity. Thus we conclude that $\mathrm{l}(A) > \mathrm{l}(X)$.

This argument applies to each face of X; so X must meet each of its neighbours as shown in Fig. 3a. Let one neighbour be positioned as shown; then either the neighbour on X_{r} or the neighbour on X_{b} must overhang X_{rb}, and each succeeding stage of the construction is "forced" at points where seven octants have already been filled. ▮

2.2.1. COROLLARY. *If X is a covered element of a locally heterogeneous collection of cubes, and* $\mathrm{St}(X)$ *does not have the singular star configuration shown in Fig.* 3, *then* $\mathrm{l}(X') \leqslant \frac{1}{2}\mathrm{l}(X)$.

Note. While it is not necessary to do so for the purpose of this paper, we can in fact make the inequality in the corollary above strict; however, we cannot reduce the constant. In fact, for any $\varepsilon > 0$, we can cover a unit cube with cubes of edge length greater than $(\frac{1}{2} - \varepsilon)$ in a locally heterogeneous manner.

3. CONFIGURATIONS NEAR A SINGULAR STAR

In this section, we will examine the possible configurations of some of the cubes near a singular star and prove the existence of a cube with certain useful properties.

3.1. THEOREM. *Let A be the smallest neighbour of the core X of a singular star in a locally heterogeneous collection of cubes; then there exists a cube $Y(A)$ in the collection with the following properties:*

(a) *$Y(A)$ is a weak neighbour of A*

(b) $\mathrm{l}(Y(A)) < \frac{1}{2}\mathrm{l}(A)$

(c) *$Y(A)$ is not the core of a singular star.*

Proof. Assume, without loss of generality, that the star is oriented as in Fig. 3, with A the smallest neighbour of X; and let $G = N(A_{\mathrm{r(db)}})$,

$H = N(A_{r(df)})$. Then $\mathbf{G}_b = \mathbf{C}_f = \mathbf{A}_b$; $\mathbf{H}_f \leqslant \mathbf{A}_f$. One of the following must hold:

(i) $G \neq H$, $\mathrm{l}(G) + \mathrm{l}(H) \leqslant \mathrm{l}(A)$

$$\Rightarrow \quad \mathrm{l}(H) < \tfrac{1}{2}\mathrm{l}(A) \text{ or } \mathrm{l}(G) < \tfrac{1}{2}\mathrm{l}(A) \qquad \text{(Fig. 7a).}$$

(ii) $G \neq H$, $\mathrm{l}(G) + \mathrm{l}(H) > \mathrm{l}(A)$

$$\begin{aligned} \Rightarrow \quad \mathbf{H}_f &\leqslant \mathbf{G}_b - \mathrm{l}(G) - \mathrm{l}(H) \\ &< \mathbf{G}_b - \mathrm{l}(A) \\ &= \mathbf{A}_b - \mathrm{l}(A) = \mathbf{A}_f \qquad \text{(Fig. 7b).} \end{aligned}$$

(iii) $G = H$

$$\Rightarrow \quad \mathbf{H}_b = \mathbf{A}_b, \mathbf{H}_f \leqslant \mathbf{A}_f; \quad \text{by local heterogeneity, } \mathbf{H}_f < \mathbf{A}_f \text{ (Fig. 7c).}$$

In case (i), one of the cubes $\{G, H\}$ must have properties (a) and (b); in fact, it must also have property (c). For H, this is easily seen; in order to meet A in the fashion shown in Fig. 3, it must have $\mathbf{H}_f = \mathbf{A}_f$; but then, as

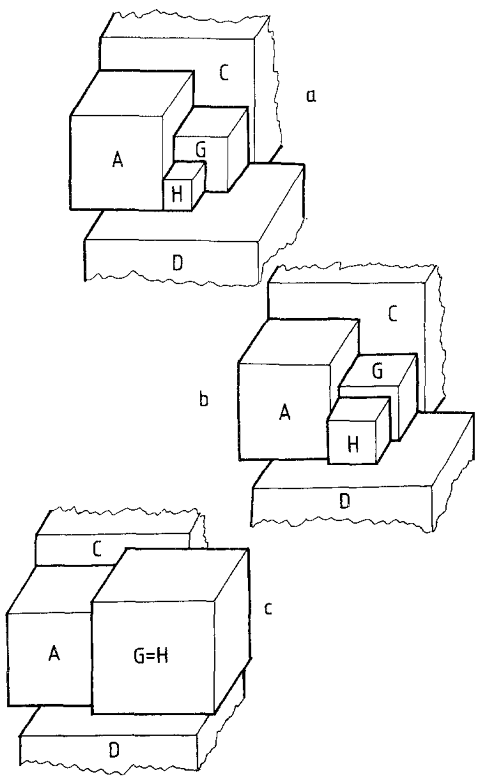

FIGURE 7

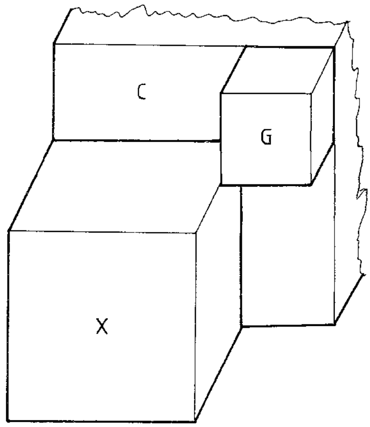

FIGURE 8

$l(D) > l(A)$, $\mathbf{D}_f < \mathbf{H}_f$, and D_u must extend before and behind H, in a way not compatible with the singular star configuration (Fig. 7a).

In the case of G, the argument is a little less direct. The singular star configuration requires $\mathbf{G}_u = \mathbf{C}_u$, $\mathbf{G}_r = \mathbf{C}_r$; but then (Fig. 3):

$$\mathbf{A}_l = \mathbf{C}_l, \qquad \mathbf{A}_r = \mathbf{G}_l, \qquad \mathbf{G}_r = \mathbf{C}_r$$
$$l(A) + l(G) = l(C).$$

But also (Fig. 8)

$$\mathbf{X}_d = \mathbf{C}_d, \qquad \mathbf{X}_u = \mathbf{A}_d = \mathbf{D}_u = \mathbf{G}_d, \qquad \mathbf{G}_u = \mathbf{C}_u$$
$$\Rightarrow \quad l(G) + l(X) = l(C)$$
$$\Rightarrow \quad l(X) = l(A),$$

which contradicts our assumption of local heterogeneity. Thus, G must also have property (c).

In cases (ii) and (iii), H_l extends forward beyond A_f. Letting $I = N(A_{f(dr)})$, $J = N(A_{f(dl)})$, we repeat the previous argument to show that either one of $\{I, J\}$ satisfies (a) and (b), or J_b extends to the left beyond A_l. If one of $\{I, J\}$ does satisfy (a) and (b), it also satisfies (c) by the same argument as that used for H.

If J_b extends to the left beyond A_l (Fig. 9), then let $K = N(A_{l(df)})$, L be the cube that touches E_u, B_f, and A_l, and $M = N(K_{f(ur)})$. One of the following must hold:

(i) $K \neq L$ (Fig. 9),

$$\begin{aligned} \Rightarrow \quad l(K) + l(L) &\leq \mathbf{B}_f - \mathbf{J}_b \\ &= (\mathbf{A}_b - l(X)) - \mathbf{A}_f \\ &= l(A) - l(X) \\ &< l(A). \end{aligned}$$

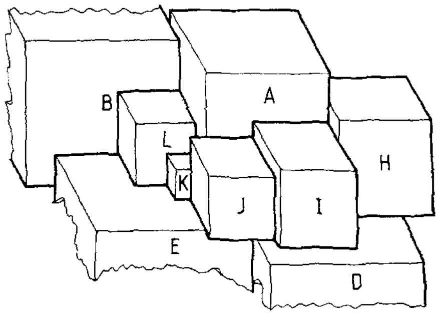

FIGURE 9

Hence K or L satisfies (a) and (b). E_u extends in front of and behind K, A extends in front of and behind L, so both K and L must satisfy (c).

(ii) $K = L$, $M \neq J$, $l(M) + l(J) \leqslant l(A)$ (Fig. 10a),

$$\Rightarrow \quad l(M) \leqslant l(A) - l(J) < \tfrac{1}{2} l(A).$$

M satisfies (a) (although, as shown, it may be a *weak* neighbour of A) and

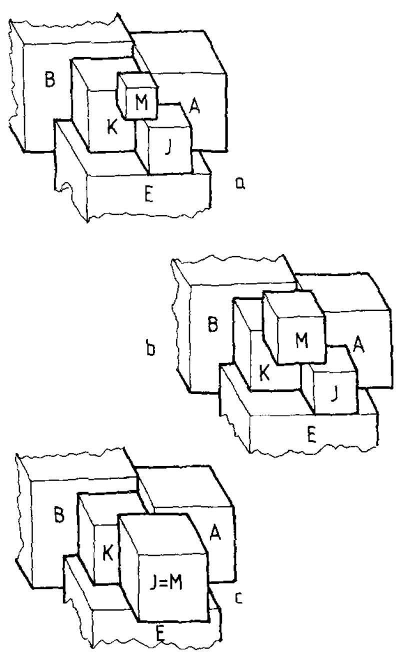

FIGURE 10

(b); it also satisfies (c), as A_f and K_f together extend to the left and right of M.

(iii) $K=L$, $M \neq J$, $l(M)+l(J)>l(A)$ (Fig. 10b),

$$\begin{aligned} \Rightarrow \qquad \mathbf{M}_u &= \mathbf{M}_d + l(M) \\ &= \mathbf{J}_d + l(J) + l(M) \\ &> \mathbf{A}_d + l(A) \\ &= \mathbf{A}_u. \end{aligned}$$

Hence M_b extends above K_u.

(iv) $K=L$, $M=J$ (Fig. 10c).

This case requires that $l(M) \geqslant l(K)$; local heterogeneity strengthens this to $l(M)>l(K)$. Therefore, M_b extends above K_u. In the first two cases, we have found a cube that satisfies the three conditions; in the second two, M_b extends above K_u. Furthermore,

$$\begin{aligned} \mathbf{B}_u &= \mathbf{B}_d + l(B) \\ &> \mathbf{B}_d + l(A) \\ &= \mathbf{B}_d + l(X) + l(K) \\ &= \mathbf{X}_d + l(X) + l(K) \\ &= \mathbf{X}_u + l(K) \\ &= \mathbf{K}_d + l(K) = \mathbf{K}_u \qquad \text{(Fig. 11).} \end{aligned}$$

Therefore, let $P=N(K_{u(fr)})$ and $Q=N(K_{u(br)})$; local heterogeneity forces these to be different cubes (Fig. 12). But

$$l(P)+l(Q)=l(K)<l(A);$$

so either P or Q must satisfy conditions (a) and (b).

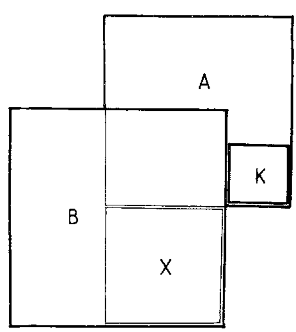

FIGURE 11

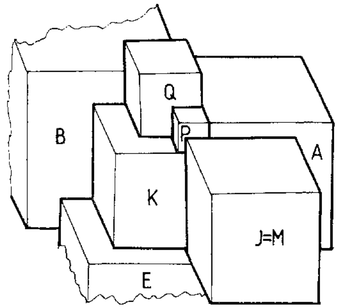

FIGURE 12

$\mathbf{A}_1$ extends behind and in front of Q; so Q satisfies condition (c). Suppose that P did not satisfy (c); then the singular star configuration would require that $\mathbf{M}_u = \mathbf{P}_u = \mathbf{A}_u$. But in case (iii), $\mathbf{M}_u > \mathbf{A}_u$, while in case (iv)

$$\begin{aligned}
\mathbf{M}_u &= \mathbf{J}_u \\
&= \mathbf{J}_d + l(J) \\
&= \mathbf{A}_d + l(J) \\
&\neq \mathbf{A}_d + l(A) \qquad \text{(by local heterogeneity)} \\
&= \mathbf{A}_u.
\end{aligned}$$

Therefore P satisfies (c), concluding the proof, as we can always find a cube $Y(A)$ that satisfies all three conditions. ∎

4. Nonexistence of Locally Finite, Locally Heterogeneous Packings of Space with Cubes

In this section, we show that any locally finite, locally heterogeneous collection of cubes fails to fill space, and obtain a limit on the distance within which this failure will occur. (This limit is generally an improvement on the limit obtained in [1].)

4.1. THEOREM. *Let $\mathscr{C}$ be a locally heterogeneous collection of cubes such that each element of $\mathscr{C}$ is covered; then there exists a sequence of cubes $\{Z_i\} \subset \mathscr{C}$ such that the sequence $\{c(Z_i)\}$ of their centers converges.*

Proof. Define Z_i inductively as follows:

(0) Z_0 is not the core of a singular star.

(i) If Z_i' is not the core of a singular star, $Z_{i+1} = Z_i'$.

(ii) If Z_i' is the core of a singular star, $Z_{i+1} = Y((Z_i')')$.

Under our hypotheses, we can always find such a sequence; Theorem 3.1 and the inductive definition guarantee that no Z_i is the core of a singular star.

If Z_i' is not the core of a singular star, Corollary 2.2.1 proves that $\mathrm{l}(Z_{i+1}) \leqslant \frac{1}{2}\mathrm{l}(Z)$. If Z_i' is the core of a singular star, Theorem 3.1 proves the same inequality.

If Z_i' is not the core of a singular star,

$$\begin{aligned} d(c(Z_i), c(Z_{i+1})) &< \operatorname{radius}(Z_i) + \operatorname{radius}(Z_{i+1}) \\ &\leqslant \frac{\sqrt{3}}{2}\mathrm{l}(Z_i) + \frac{\sqrt{3}}{4}\mathrm{l}(Z_i) \\ &= \frac{3\sqrt{3}}{4}\mathrm{l}(Z_i). \end{aligned}$$

If Z_i' is the core of a singular star,

$$\begin{aligned} d(c(Z_i), c(Z_{i+1})) &< \operatorname{radius}(Z_i) + \operatorname{diam}(Z_i') + \operatorname{diam}((Z_i')') + \operatorname{radius}(Z_{i+1}) \\ &\leqslant \frac{\sqrt{3}}{2}\mathrm{l}(Z_i) + \frac{\sqrt{3}}{2}\mathrm{l}(Z_i) + \sqrt{3}\,\mathrm{l}(Z_i) + \frac{\sqrt{3}}{4}\mathrm{l}(Z_i) \\ &= \frac{9\sqrt{3}}{4}\mathrm{l}(Z_i). \end{aligned}$$

In either case, a bound on the distance between the centers of successive cubes is given by

$$\begin{aligned} d(c(Z_i), c(Z_{i+1})) &< \frac{9\sqrt{3}}{4}\mathrm{l}(Z_i) \\ &\leqslant \frac{9\sqrt{3}}{4}\left(\frac{1}{2}\right)^i \mathrm{l}(Z_0); \end{aligned}$$

the sequence is therefore convergent, with

$$d(c(Z_0), \lim_{i \to \infty} c(Z_i)) < \frac{9\sqrt{3}}{2}\mathrm{l}(Z_0). \quad \blacksquare$$

By closer inspection of the geometry, it is possible to improve this limit to the extent of reducing the constant; for the purposes of this paper, that obtained here is sufficient.

4.2. THEOREM. *There is no locally finite, locally heterogeneous packing of space with cubes.*

Proof. If a collection of cubes fills space, every element must be covered. But if such a collection were to be locally heterogeneous, it would fail to be locally finite at the limit point of the sequence of centers described above. ∎

4.2.1. COROLLARY. *Let A be a cube not the core of a singular star in a locally finite, locally heterogeneous collection of cubes. Then the distance from the center of A to the complement of the union of the collection is less than* $(9\sqrt{3}/2)\,\mathrm{l}(A)$.

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