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Orderability of all noncompact images

Eric K. van Douwen

Abstract

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We characterize the noncompact spaces whose every noncompact image is orderable as the noncompact continuous images of ω_1. We find other useful characterizations as well. We also characterize the continuous images of ω_1+1.

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1. Introduction

The following theorem shows that very few noncompact spaces have the property that all their noncompact continuous images are orderable.

Theorem 1.1. *If X is not compact then every noncompact continuous image of X is orderable iff X is a continuous image of ω_1.*

A corollary shows that "noncompact" is essential.

Theorem 1.2. *Every continuous image of X is orderable iff X is compact and countable.*

Theorem 1.1 was discovered by accident: In order to solve a problem in Boolean algebra, van Douwen [5], I needed a good description of all continuous images of ω_1+1.

Theorem 1.3. *A space Y is a continuous image of ω_1+1 iff one of the following conditions holds*:

(1) *Y is compact and $|Y| = \omega_1$ and there is a pairwise disjoint family $\mathcal{U}$ of open compact countable subsets of Y with $|Y - \bigcup \mathcal{U}| = 1$, or*

(2) *Y is compact and there are a compatible linear order $\leqslant$ on Y and a point ∞ in Y such that each interval in $\{(\leftarrow, y]: y \in (\leftarrow, \infty)\} \cup \{[\infty, \rightarrow)\}$ is countable.*

A more detailed statement, see Section 5, leads to the conclusion that the if-part of Theorem 1.1 holds. For the proof of the only-if-part we need to generalize the notion of a cub (=closed unbounded) subset of an ordinal: We call a subset of any space a *cub* if it is closed and noncompact. We call a space a *bear* if it is noncompact and has no disjoint cubs. (It is well known that an ordinal ξ $(=[0, \xi))$, in the order topology, is a bear iff $\text{cof}(\xi) \geqslant \omega_1$.) With this terminology we have the following extension of Theorem 1.1. The interesting part of the proof is the implication (2)$\Rightarrow$(3).

Theorem 1.4. *For a noncompact space X the following are equivalent.*

(1) *X is a continuous image of ω_1,*

(2) *every noncompact continuous image of X is orderable,*

(3) *X is a scattered first countable orderable bear,*

(4) *X is a locally countable orderable bear, and*

(5) *X is countably compact and X has a compatible linear order $\leqslant$ such that initial segments are countable.*

Corollary 1.5. *Each noncompact continuous image of ω_1 has a closed subspace homeomorphic to ω_1.*

2. Preliminaries

All ordinals carry the order topology. We let $D(\omega_1)$ denote any discrete space of cardinality ω_1, and let $\alpha D(\omega_1)$ denote its one-point compactification. We frequently use the fact that $\alpha D(\omega_1)$ embeds in no orderable space.

For a space X we let X' denote the set of isolated points of X.

For spaces X and Y (and points x of X and y of Y) we let $X \approx Y$ (or $\langle X, x\rangle \approx \langle Y, y\rangle$) signify that there is a homeomorphism from X onto Y (which maps x to y).

For a closed subset F of a space X we use X/F to denote the quotient space obtained from X by collapsing F to a single point. Note that if X is normal then so is X/F. Also note that $(\omega_1+1)/(\omega_1+1)' \approx \alpha D(\omega_1) \approx \omega_1/\omega_1'$.

A space is called *scattered* if every nonempty subspace (or, equivalently, every nonempty closed subspace) has an isolated point. We need the fact that each continuous image of a compact scattered space is scattered. (Proof: a closed subspace of the image is the image of a space with an isolated point under a continuous closed irreducible map.) We also need the fact that compact scattered spaces are zero-dimensional.

3. Bears

In this section we collect some useful facts about bears. Some of this material will be used later.

Proposition 3.1. *Let X be a bear. Then X has the following properties:*

(1) *X is normal and countably compact.*

(2) *X is locally compact.*

(3) *If f is a continuous map from X onto a noncompact space Y then f is perfect and Y is a bear.*

(4) *If X is a dense subspace of some space Y then $X = Y$ or Y is compact and $Y - X$ consists of exactly one point.*

Proof. That (1) holds should be clear. That the noncompact continuous image of a bear is again a bear should also be clear.

Furthermore note that if a space is not compact then every point (even every compact subset) of it has a neighbourhood with a noncompact complement. This immediately proves (2) and with some extra effort also (3).

Finally, to prove (4) note that $X \cup \{y\}$ is compact for every $y \in Y - X$ (if U is a neighbourhood of y then $X - U$ must be compact). $\square$

Corollary 3.2. *Bear topologies are minimally noncompact.*

Proof. If $\mathcal{B}$ is a bear topology on a set X and $\mathcal{A} \subseteq \mathcal{B}$ is a noncompact topology then apply (3) of Proposition 3.1 to the identity map from $\langle X, \mathcal{B}\rangle$ to $\langle X, \mathcal{A}\rangle$. $\square$

Bears share with ordinals of uncountable cofinality the property that the filter of cubs is countably complete.

Proposition 3.3. *If X is a bear then the intersection of countably many cubs is again a cub.*

Proof. We let $\mathcal{C}$ denote the collection of all cubs in X and we put

$$\nu = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{C} \text{ and } \bigcap \mathcal{A} \text{ is not a cub}\}$$

and

$$\mu = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{C} \text{ and } \bigcap \mathcal{A} = \emptyset\}.$$

Clearly $\nu \leqslant \mu$. To show that $\mu \leqslant \nu$ take $\mathcal{A} \subseteq \mathcal{C}$ such that $\bigcap \mathcal{A}$ is not a cub, i.e., $\bigcap \mathcal{A}$ is compact or empty. By local compactness we can find an open set U around $\bigcap \mathcal{A}$ with compact closure. Then $\{A - U : A \in \mathcal{A}\}$ is a collection of cubs with empty intersection.

Using the equality $\mu = \nu$ we may show by induction that $\mu \geqslant n$ for every $n \in \omega$; it follows that $\mu = \nu \geqslant \omega$. $\square$

4. Scattered spaces

A common method of proving results about scattered spaces is to prove them by induction on the scattering height. We find it pleasant that the proofs of this section are straightforward from the definition. This is our justification for including our proof of the following result of Mazurkiewicz and Sierpiński from [2].

Lemma 4.1. *Every compact and countable space is homeomorphic to* $\alpha+1$ *for some* $\alpha \in \omega_1$.

Proof. Let X be compact and countable, so X is scattered and zero-dimensional. Let Ω denote the family of those clopen subsets of X that are homeomorphic to $\alpha+1$ for some $\alpha \in \omega_1$.

Note that a clopen subset of a successor ordinal is homeomorphic to a successor ordinal (use the same ordering), so that every clopen subset of every element of Ω is again an element of Ω.

Consider the set A of those $x \in X$ that have a neighbourhood in Ω. It suffices to show that $A = X$ for then we can find a pairwise disjoint cover of X consisting of elements of Ω. Using the fact that $[0, \alpha] \oplus [0, \beta] \approx [0, \alpha+1+\beta]$ we may then conclude that $X \in \Omega$.

So assume $A \neq X$ and pick a relatively isolated point a of $X - A$. Also pick a clopen neighbourhood U of a such that $U \cap A = \{a\}$. Now a is not isolated in X and $U - \{a\} \subseteq A$, so we can find a pairwise disjoint nonempty $U_i \in \Omega$ $(i \in \omega)$ such that $U - \{a\} = \bigcup_{i \in \omega} U_i$. Next we take a strictly increasing sequence $\langle \alpha_i : i \in \omega \rangle$ in $\{-1\} \cup \omega_1$ such that $U_i \approx [\alpha_i + 1, \alpha_{i+1}]$ for all i. Let $\sigma = \sup_i \alpha_i$, then

$$[0, \sigma) \approx \bigoplus_{i \in \omega} [\alpha_i + 1, \alpha_{i+1}] \approx \bigoplus_{i \in \omega} U_i = U - \{\alpha\}.$$

It follows that $U \approx [0, \sigma]$ because one-point compactifications are unique. Thus, the assumption $A \neq X$ leads to a contradiction. □

The following two corollaries follow because they are easily seen to hold for compact ordinal spaces.

Corollary 4.2. *If* X *is compact and countable and* $p, q \in X$ *then there is a compatible ordering on* X *with* p *as its first and* q *as its last element.*

Corollary 4.3. *If* X *is compact and countable and* $p \in X$ *then there is* $\gamma \in \omega_1$ *such that the pointed spaces* $\langle X, p \rangle$ *and* $\langle [0, \gamma], \gamma \rangle$ *are homeomorphic.*

We conclude this section with a lemma that may, but should not, be new. We shall need it at one point later on.

Lemma 4.4. *Let X be an infinite compact scattered space. Then*
(1) $\chi(X)=|X|$ *and*
(2) *if* $|X|$ *is regular then* $\chi(x, X)=|X|$ *for some* $x\in X$.

Proof. Since X is compact we have $\chi(x, X)=\psi(x, X)\leq|X|$ for all x.

We prove the reverse inequalities by induction on the cardinality of the space. They should be clear for countable compact spaces.

Let A be the set of those x in X for which every neighbourhood has cardinality $|X|$. As X is compact the set A is not empty. Let a be a relatively isolated point of A and pick a closed neighbourhood Y of a such that $Y\cap A=\{a\}$. Now $|Y|=|X|$ and $\chi(x, Y)\leq\chi(x, X)$ for all $x\in Y$ so we may as well assume that $Y=X$.

By this assumption the family $\mathcal{U}$ of clopen sets of cardinality less than $|X|$ covers $X-\{a\}$; also, $X-\{a\}$ can be covered by $\chi(a, X)$ compact sets so $\mathcal{U}$ has a subcover $\mathcal{V}$ of size at most $\chi(a, X)$.

If $|X|$ is regular then necessarily $|\mathcal{V}|=|X|$, so $\chi(a, X)=|X|$.

If $|X|$ is singular then $\sup_{V\in\mathcal{V}}|V|=|X|$ and so, by the inductive assumption

$$\chi(X)\geq\sup\{\chi(V): V\in\mathcal{V}\}=\sup\{|V|: V\in\mathcal{V}\}=|X|. \qquad\square$$

If κ is a singular cardinal then every point of the space $\kappa+1$ has character less than κ, so regularity of $|X|$ is necessary in (2).

Note also that in a similar way one may prove that compact scattered spaces are zero-dimensional.

5. Continuous images of ω_1+1

The following two theorems characterize all continuous images of ω_1+1.

Theorem 5.1. *The following are equivalent for a space Y.*
(1) *There is a continuous surjection $f: \omega_1+1\to Y$ such that $|f^{\leftarrow}\{f(\omega_1)\}|=\omega_1$ and*
(2) *Y is compact, $|Y|\leq\omega_1$ and there is a pairwise disjoint family $\mathcal{U}$ of compact, open and countable subsets of Y with $|Y-\bigcup\mathcal{U}|=1$.*

Theorem 5.2. *The following are equivalent for a space Y.*
(1) *There is a continuous surjection $f: \omega_1+1\to Y$ such that $|f^{\leftarrow}\{f(\omega_1)\}|\leq\omega$,*
(2) *Y is compact and uncountable and there are a compatible ordering $<$ and a point ∞ such that $[\infty,\to)$ and every interval $(\leftarrow, y]$ with $y<\infty$ are countable, and*
(3) *there is an exactly two-to-one continuous surjection from ω_1+1 onto Y.*

As a corollary we discover when an uncountable continuous image of ω_1+1 is orderable.

Corollary 5.3. *The following are equivalent for an uncountable continuous image Y of ω_1+1.*

(1) *Y is orderable,*

(2) *for every continuous surjection $f:\omega_1+1\to Y$ we have $|f^{\leftarrow}\{f(\omega_1)\}|\leq\omega$,*

(3) *the one-point compactification of $D(\omega_1)$ does not embed into Y and*

(4) *ω_1+1 embeds into Y.*

The proof will also yield the following result:

Theorem 5.4. *The following are equivalent for a space Y.*

(1) *There is a continuous surjection $f:\omega_1+1\to Y$ such that $|f^{\leftarrow}\{f(\omega_1)\}|=1$ and*

(2) *Y is compact, uncountable and orderable in such a way that each interval not containing the last element is countable.*

Remark 5.5. If $<$, Y and ∞ are as in (2) of Theorem 5.2 then there is a strictly increasing embedding $e:\omega_1+1\to Y$ with $e(0)=\min Y$ and $e(\omega_1)=\infty$. This is mentioned for use in van Douwen [5].

We give a combined proof of implications (1)⇒(2) in Theorems 5.1 and 5.2.

Let $f:\omega_1+1\to Y$ be continuous. We put $\infty=f(\omega_1)$ and for $\xi\leq\omega_1$ we put $F(\xi)=f^{\leftarrow}\{f(\xi)\}$. Note that $F(\xi)$ is closed and that $F(\xi)$ is countable if $\omega_1\notin F(\xi)$. Consider the following subset Γ of ω_1 defined by

$$\alpha\in\Gamma \quad\text{iff}\quad \text{for all } \xi<\alpha\text{: if } \omega_1\notin F(\xi) \text{ then } F(\xi)\subseteq\alpha.$$

Clearly Γ is a closed and unbounded subset of ω_1. Also note that every $\alpha\in\Gamma$ is saturated with respect to F in that if $\omega_1\notin F(\xi)$ then $F(\xi)\subseteq\alpha$ if $\xi<\alpha$ and $F(\xi)\subseteq[\alpha,\omega_1)$ if $\xi\geq\alpha$.

Case 1: $|F(\omega_1)|=\omega_1$. In this case $F(\omega_1)\cap\omega_1$ is cub in ω_1 and so $\Delta=\Gamma\cap F(\omega_1)$ is also cub. Let $\mathscr{C}$ be the collection of convex components of $\omega_1-\Delta$. Because each $\alpha\in\Delta$ is saturated we get the following equality for each $C\in\mathscr{C}$:

$$C-F(\omega_1)=f^{\leftarrow}[f^{\rightarrow}C-\{\infty\}].$$

It follows that $\mathscr{V}=\{f^{\rightarrow}C-\{\infty\}: C\in\mathscr{C}\}$ is a pairwise disjoint collection of open subsets of Y. Note also that $\mathscr{V}$ covers $Y-\{\infty\}$. As each $V\in\mathscr{V}$ is countable and locally compact we may decompose it into countably many compact open sets. These collections put together will form our collection $\mathscr{U}$.

Case 2: $|F(\omega_1)|\leq\omega$. Now fix $\alpha\in\Gamma$ such that $F(\omega_1)\cap\omega_1\subseteq\alpha$ and fix a clopen-in-$f^{\rightarrow}[0,\alpha]$ set Ω such that $\infty\in\Omega$ and $f(\alpha)\notin\Omega$. Also let $C=f^{\leftarrow}\Omega$ and $A=\{\infty\}\cup(Y-\Omega)$. Note that A and Ω are closed and that $A\cap\Omega=\{\infty\}$. Now Ω is compact and countable so we can order it by a compatible linear order $\leq_\Omega$ which has ∞ as its first element.

We shall order A in such a way that ∞ is its last element; in this way Y will be ordered in the required fashion.

To begin we order $f^{\leftarrow}([0,\alpha]-C)$ in some way so as to make $f(\alpha)$ its last element. Next choose, for every $\beta\in[\alpha,\omega_1)\cap\Gamma$, a compatible order $\leqslant_\beta$ on $f^{\leftarrow}[\beta,\beta^+]$—where $\beta^+=\min\Gamma-[0,\beta]$—for which $f(\beta)$ is the first and $f(\beta^+)$ the last element.

Now combine these orderings into an ordering $\leqslant_A$ of A in such a way that $f^{\leftarrow}([0,\alpha]-C)$ and each set $f^{\leftarrow}[\beta,\beta^+]$ retain their original ordering, and if $\beta<\gamma$ then $f^{\leftarrow}[\beta,\beta^+]$ comes before $f^{\leftarrow}[\gamma,\gamma^+]$; also $f^{\leftarrow}([0,\alpha]-C)$ comes before all sets $f^{\leftarrow}[\beta,\beta^+]$.

We leave it to the reader to check that all requirements are met.

We turn to the proofs of the other implications.

Proof of (2)⇒(1) in Theorem 5.1. If Y is countable then $Y\approx[0,\alpha]$ for some $\alpha\in\omega_1$ and $[0,\alpha]$ is obtained from ω_1+1 by the collapsing $[\alpha,\omega_1]$ to a point.

So assume $|Y|=\omega_1$ and enumerate the family $\mathcal{U}$ in some one-to-one manner: $\langle U_\xi:\xi\in\omega_1\rangle$. We define $f:\omega_1+1\to\omega_1+1$ by induction as follows: $f(0)=0$ and if $\eta\leqslant\omega_1$ is a limit then $f(\eta)=\sup_{\xi<\eta}f(\xi)$. If $\xi<\omega_1$ then we choose the minimal ordinal α_ξ such that $U_\xi\approx[0,\alpha_\xi]$ and we put $f(\xi+1)=f(\xi)+1+\alpha_\xi+1$; note that $(f(\xi),f(\xi+1))\approx U_\xi$.

Now as f is continuous the set $F=f^{\leftarrow}[0,\omega_1]$ is closed and $(\omega_1+1)-F\approx\bigoplus_{\xi<\omega_1}U_\xi$. It follows that both Y and $(\omega_1+1)/F$ are the one-point compactification of $\bigoplus_{\xi<\omega_1}U_\xi$ and so $Y\approx(\omega_1+1)/F$. □

Proof of (2)⇒(3) in Theorem 5.2. The set $[\infty,\to)$ is homeomorphic to $[0,\alpha]$ for some $\alpha\in\omega_1$; map $[0,\alpha+\alpha]$ onto $[\infty,\to)$ by an obvious two-to-one map.

It remains to map $(\alpha+\alpha,\omega_1)$ onto $Y-[\infty,\to)$ by a two-to-one map (we shall map ω_1 onto $Y-[\infty,\to)$ of course).

First we take a strictly increasing map $e:\omega_1\to Y$ such that $e(0)=\min Y$ and $e(\eta)=\sup_{\xi<\eta}e(\xi)$ if η is a limit.

Each interval $[e(\eta),e(\eta+1)]$ is compact and countable; we shall find for each η an ordinal γ_η and a map $F_\eta:[0,\gamma_\eta]\to[e(\eta),e(\eta+1)]$ such that $|F^{\leftarrow}\{x\}|=2$ if $e(\eta)<x<e(\eta+1)$ and $|F^{\leftarrow}\{e(\eta)\}|=|F^{\leftarrow}\{e(\eta+1)\}|=1$.

Once this is done we map ω_1 onto $Y-[\infty,\to)$ as follows: First define $\langle\delta_\eta:\eta<\omega_1\rangle$ by $\delta_0=1$, $\delta_{\eta+1}=\delta_\eta+1+\gamma_\eta$ and $\delta_\eta=\sup_{\xi<\eta}\delta_\xi$ if η is a limit. Now use the F_η to map $[\delta_\eta+1,\delta_{\eta+1}]$ onto $[e(\eta),e(\eta)+1]$. Furthermore, map 0 to $e(0)$ and if η is a limit map δ_η to $e(\eta)$. The resulting map F is continuous and exactly two-to-one as the reader may easily verify.

To find F_η and γ_η we take the quotient of $[e(\eta),e(\eta)+1]\times 2$ obtained by identifying $\langle e(\eta),0\rangle$ and $\langle e(\eta),1\rangle$ at one end and $\langle e(\eta+1),0\rangle$ and $\langle e(\eta+1),1\rangle$ at the other end. This quotient space is compact and countable, hence homeomorphic with some successor $\gamma_\eta+1$. The map F_η now suggests itself. □

6. Continuous images of ω_1

We here prove Theorem 1.4, i.e., prove the following conditions on a noncompact space X to be equivalent:

(1) X is a continuous image of ω_1,

(2) every noncompact continuous image of X is orderable,

(3) X is a scattered first countable orderable bear,

(4) X is a locally countable orderable bear,

(5) X has a compatible order all initial closed segments of which are compact and countable.

We also prove Corollary 1.5 which says that each noncompact continuous image of ω_1 has a closed subspace homeomorphic to ω_1.

Proof of (1)⇒(2). Let Y be a noncompact continuous image of X. Then Y is a noncompact continuous image of ω_1, hence is a bear by Proposition 3.1(3). It follows from Proposition 3.1(2) that Y is locally compact; let $\alpha Y = Y \cup \{\infty\}$ be its one-point compactification. Extend f to a map $F: \omega_1 + 1 \to \alpha Y$ by stipulating that $F(\omega_1) = \infty$.

If F is continuous then αY is orderable by Theorem 5.2 since $|F^{\leftarrow}\{F(\omega_1)\}| = 1$, hence its open subspace Y is also orderable.

To see that F is continuous we note that, by Proposition 3.1(3), f is perfect so that $f^{\leftarrow}K$ is compact whenever $K \subseteq Y$ is compact. Because the complements of compact subsets of Y form a neighbourhood base at ∞ we conclude that F is continuous at ∞. Also, $F \restriction \omega_1 = f$ is continuous so that F is continuous. □

Proof of (2)⇒(3). We do this in six steps. Clearly X itself is orderable; let $\leqslant$ be a compatible ordering.

Fact 1. *X is countably compact.*

Proof. We assume X has a countably infinite closed discrete set D, and prove $\neg(2)$. Since every countable orderable space is first countable, and since there is a countable regular (necessarily) noncompact space which is not first countable, e.g., $\omega \times (\omega+1)/(\omega \times \{\omega\})$, it suffices to prove X can be mapped continuously onto each countable space or, equivalently, that X admits an infinite pairwise disjoint open cover.

First note X is zero-dimensional: if not there are $a, b \in X$ with $a < b$ and $[a, b]$ connected. Then $X/\{a, b\}$, which obviously is normal and not compact, is not orderable since its nondegenerate connected subspace $[a, b]/\{a, b\}$ is not disconnected by any point.

Since X is normal and zero-dimensional we can find a (necessarily infinite) discrete clopen family $\mathcal{U}$ which separates D. Then $\mathcal{U} \cup \{X \setminus \bigcup \mathcal{U}\}$ is our cover.

Fact 2. *Every copy of ω_1 in X is closed.*

Proof. We suppose not and prove $\neg(2)$. Simply assume ω_1 is a nonclosed subspace of X, and let $p \in \bar{\omega}_1 - \omega_1$. Then $\omega_1 \cup \{p\}$ is compact, by Proposition 3.1(4), hence $\omega_1 \cup \{p\} \approx \omega_1 + 1$. So we may assume $\omega_1 + 1$ is a subspace of X.

The quotient space $X/(\omega_1+1)'$ is regular and noncompact since we collapsed the compact set $(\omega_1+1)'$ to a point. It is not orderable since it has a subspace homeomorphic to $\alpha D(\omega_1)$: the quotient $(\omega_1+1)/(\omega_1+1)'$.

Fact 3. *Every noncompact closed subspace of* X *has a copy of* ω_1.

Proof. Let F be a noncompact closed subspace of X. Let F^+ be any compact linearly orderable space, with compatible order $\leq^+$, in which F embeds densely. In what follows intervals are with respect to $\leq^+$.

Pick any $p \in F^+ - F$. We may assume $p \in \overline{F \cap (\leftarrow, p)}$. By Fact 1, F is countably compact so $\sup(A) \in F \cap (\leftarrow, p)$ for each countable $A \subseteq F \cap (\leftarrow, p)$. Hence we can find a strictly increasing function $e: \omega_1 \to F \cap (\leftarrow, p)$ such that $e(\lambda) = \sup_{\xi < \lambda} e(\xi)$ for every limit λ. Clearly e is an embedding.

Fact 4. X *is a bear.*

Proof. We assume X has two disjoint cubs A and B, and prove $\neg(2)$. By Facts 2 and 3 we may assume $A = \omega_1$. Then X/ω_1' is normal since A' is closed in the normal space X. It is not orderable since its subspace ω_1/ω_1' is homeomorphic to $\alpha D(\omega_1)$. And it is not compact since B is a cub of it.

Fact 5. X *is scattered.*

Proof. We assume not, and prove $\neg(2)$. Since X is not scattered, it has a closed crowded (=has no isolated points) subspace, hence it has two disjoint closed crowded subspaces A and B. Because of Fact 4 we may assume A is compact. Then A admits a continuous map onto the closed unit interval, cf. Juhász [1, 3.16, Case 1]. It follows that A admits a continuous map f onto a space Y which does not embed in any orderable space, e.g., the letter O or the letter T. The adjunction space $X \cup_f Y$ is normal and not compact since the quotient map from $X \oplus Y$ is closed (since A is compact) and has compact fibres, i.e., is perfect, and it is not orderable since Y embeds in it.

Fact 6. X *is first countable.*

Proof. Suppose there is $q \in X$ which has no countable neighbourhood base. We will contradict Fact 4. We may assume q has no countable neighbourhood base in $(\leftarrow, q]$. Then $(\leftarrow, q)$ is a noncompact countably compact subspace of X, hence by Fact 3 there is a copy A of ω_1 in $(\leftarrow, q)$. Since A is closed, by Fact 2, and $q \notin A$, there is $p < q$ with $A \subseteq (\leftarrow, p)$. Now $[p, q)$ is a noncompact countably compact subspace of X. So we can also find a copy of ω_1, closed in X, in $[p, \to)$. Hence X has two disjoint cubs.

This finishes the proof of (2)$\Rightarrow$(3). $\square$

Proof of (3)⇒(4). Since X is locally compact by Proposition 3.1(2), X is locally countable by Lemma 4.4. □

Proof of (4)⇒(5). There are $p, q \in X$ with q an immediate successor of p since X, being scattered, has an isolated point. Without loss of generality $[q, \rightarrow)$ is uncountable since X is uncountable, e.g., because X, being a bear, is countably compact by Proposition 3.1(1). Define

$$A = \{x \in [p, \rightarrow) : [p, x] \text{ is countable}\}$$

and

$$B = \{x \in [p, \rightarrow) : [p, x] \text{ is uncountable}\}.$$

Using the fact that X is locally countable one easily proves that A and B are open, and also that $(\forall x \in A)(\exists y \in A)[x < y]$. Hence A and B are disjoint closed sets and as it has no maximal element, A is not compact. Since X is a bear it follows that the closed sets $(\leftarrow, p]$ and B are compact. But $(\leftarrow, p]$ and B are also open, hence we can find a new compatible order on X which agrees with $<$ on $(\leftarrow, p) \cup A$ and on B and in which B precedes $(\leftarrow, p] \cup A$. Since $(\leftarrow, p]$ and B are countable, being compact subspaces of a locally countable space, initial closed segments of X in this new order are countable, hence are also compact since X is countably compact. □

Proof of (5)⇒(1). Clearly X is locally compact, hence αX exists. Let $<$ be the compatible order of X given by (5). Then we may extend $<$ to a compatible order $<'$ on αX by stipulating $x <' \infty$ for all $x \in X$. This order satisfies the condition of Theorem 5.2(3) that each interval not containing the last point ∞ is countable. Hence there is a continuous surjection $f : \omega_1 + 1 \twoheadrightarrow Y$ with $|f^{\leftarrow}\{f(\omega_1)\}| = 1$. Since $\{\omega_1\}$ is not a G_δ of $\omega_1 + 1$ we must have $f(\omega_1) = \infty$. (This also would follow from the proof of Theorem 5.2.) Hence $f \restriction \omega_1$ maps ω_1 onto X. □

7. When all continuous images are orderable

We here prove Theorem 1.2, i.e., we prove that all continuous images of X are orderable iff X is compact and countable. Sufficiency is clear since each compact countable space is orderable by Lemma 4.1. Now assume all continuous images of X are orderable. Then in particular X itself is orderable.

Fact 1. *X is compact.*

Proof. If not then X is a continuous image of ω_1 by Theorem 1.1, hence ω_1 embeds as a closed subspace by Corollary 1.5. Then X/ω_1' is a regular continuous image of X which is not orderable since $\omega_1/\omega_1' \approx \alpha D(\omega_1)$. □

Fact 2. *X is first countable.*

Proof. If not then ω_1+1 embeds in X since X is compact and orderable. But then the continuous image $X/(\omega_1+1)'$ is not orderable since $(\omega_1+1)/(\omega_1+1)' \approx \alpha D(\omega_1)$ embeds in it. □

Fact. 3. *X is scattered.*

Proof. The proof of Fact 5 in Section 6 works. □

It now follows from Lemma 4.4 that X is countable.

I have not seriously investigated the question of when all zero-dimensional continuous images of a zero-dimensional are orderable.

8. ω_1 itself

We call a subset of a noncompact space X *stationary* if it intersects every cub of X. We say that a space X satisfies the *wPDL* (*weak Pressing Down Lemma*) if X' is not compact and if for every stationary $S \subseteq X'$ the following holds:

> For every map $\phi : S \to \mathcal{P}(X)$ such that $x \in \overline{\phi(x)} - \phi(x)$ for all $x \in S$ there are distinct s and t in S such that $\phi(s) \cap \phi(t) \neq \emptyset$.

Since we have given an ω_1-free characterization of the noncompact continuous images of ω_1 the following amounts to an ω_1-free characterization of ω_1.

Theorem 8.1. *A noncompact continuous image of ω_1 is homeomorphic to ω_1 iff it satisfies the wPDL.*

Proof. *Necessity*: it is easy to see that the Pressing Down Lemma implies the weak Pressing Down Lemma, since ω_1' is not compact.

Sufficiency: By Theorem 1.4, X admits a compatible order the initial segments of which are compact and countable. By the proof of Corollary 1.5 (in Section 6) there is an increasing embedding e of ω_1 into X as a closed subspace. Let $S \subseteq X' \cap e^{\to}\omega_1$ be defined by

$$S = \{x \in e^{\to}\omega_1 : x \in \overline{(x, \to)}\}.$$

For $s \in S$ we can find a countable $\phi(s) \subseteq (x, \to)$ with $s \in \overline{\phi(s)}$. There is a cub of $e^{\to}\omega_1$ such that for all $s \in S$ and all $x \in C$ if $s < x$ then $\phi(s) \subseteq (\leftarrow, x)$. Then $\phi(s) \cap \phi(t) = \emptyset$ for all distinct s and t in $S \cap C$. Hence $S \cap C$ is not stationary. So let D be a cub of X disjoint from $S \cap C$. Then $C \cap D$ is a cub, being the intersection of two cubs, hence $e^{\leftarrow}(C \cap D)$ is a cub of ω_1, hence is homeomorphic to ω_1. It follows that we may assume without loss of generality that $S = \emptyset$. An analysis of the proof of (5)$\Rightarrow$(1) in Section 6 will now show that X and ω_1 are homeomorphic. □

The fact that every cub of ω_1 is homeomorphic to ω_1 suggests another characterization of ω_1: up to homeomorphism ω_1 is the only noncompact nondiscrete first countable orderable space which is homeomorphic to each of its cubs. More generally, a noncompact orderable space is homeomorphic to a regular cardinal iff it is homeomorphic to each of its cubs. This is trivial of course. It leads to the following very nontrivial question.

Question 8.2. Is there a noncompact nonorderable space which is homeomorphic to each of its cubs? Is there such a space which is a bear?

Such a space must be nondiscrete (since ω is orderable).

9. Nonimages of ω_1

Theorem 1.4 characterizes the noncompact continuous images of ω_1 as the locally countable orderable spaces without disjoint cubs. Under $\diamond$ orderability is essential, since in [3] Ostaszewski has constructed from $\diamond$ a locally countable space without disjoint cubs which is hereditarily separable. There are two natural reasons that noncompact continuous images of ω_1 are not hereditarily separable: one, ω_1-free, is that continuous images of ω_1 are ω-bounded (by (5) of Theorem 1.4) or, more generally, since countably compact orderable spaces are ω-bounded and the other, not ω_1-free, that the nonseparable space ω_1 embeds in noncompact continuous images of ω_1. This suggests the question of whether a locally countable ω-bounded space without disjoint cubs is a continuous image of ω_1. Our following example shows that this is not the case, at least under $\clubsuit^2$.

Definition 9.1. Let Λ be the set of limit ordinals in ω_1 and $n \geq 1$. $\clubsuit^n$ is the statement that there is $\langle S_\lambda^k : k \in n, \lambda \in \Lambda \rangle$, with S_λ^k cofinal in λ for $k \in n$ and $\lambda \in \Lambda$, such that if X_k is an uncountable subset of ω_1 for $k \in n$ then

$$\{\lambda \in \Lambda : S_\lambda^k \subseteq X_k \text{ for } k \in n\}$$

is stationary.

So $\clubsuit^1 \equiv \clubsuit$. I do not know if $\clubsuit^1 \Leftrightarrow \clubsuit^2$. However, $\diamond \Rightarrow \clubsuit^2$. Note also that $\clubsuit^{n+1} \Rightarrow \clubsuit^n$ for every n.

Example 9.2 ($\clubsuit^2$). A noncompact locally countable ω-bounded space without disjoint cubs in which ω_1 does not embed.

Proof. Let $\langle S_\lambda^k : k \in 2, \lambda \in \Lambda \rangle$ be a $\clubsuit^2$-sequence. We may assume S_λ^k converges to λ for $k \in 2$ and $\lambda \in \Lambda$. We assume the reader is familiar with Ostaszewski's construction, [3], or similar constructions. Then we do not have to explain how to construct a space Ω with underlying set the countable ordinals such that in Ω

(1) $(\forall \lambda \in \Lambda)[\lambda \in \overline{S_\lambda^0} \cap \overline{S_\lambda^1}]$;

(2) $(\forall\lambda \in \Lambda)[S_\lambda^0$ does not converge to $\lambda]$;

(3) $(\forall\lambda \in \Lambda)[[0, \lambda+1]$ is compact open in $\lambda]$.

We only mention that after ensuring (1) and (2) one lets $[0, \lambda+1]$ be the one-point compactification of $[0, \lambda]$ if $[0, \lambda]$ is not yet compact, as subspace of what is to become Ω.

Clearly Ω is locally countable and ω-bounded but not compact, because of (3), and has no disjoint cubs because of (1).

Now suppose there is an embedding $e: \omega_1 \to \Omega$. The set

$$C = \{\eta \in \omega_1 : (\forall \xi \in \omega_1)[e(\xi) < \eta \Leftrightarrow \xi < \eta]\}$$

is easily seen to be a cub because e is an injection. Since $e \restriction C$ is strictly increasing it follows that for each strictly increasing function $s: \omega \to e^{\to}C$, the sequence s converges in Ω (to a point of $e^{\to}C$), hence for each $S \subseteq e^{\to}C$ and each λ, if S converges to λ in ω_1 then S converges to some point in Ω. This contradicts (1) and (2). □

We finish this section with the following example due to Rajagopalan, Soundarajan and Jakel [4]. Our description is simpler.

Example 9.3. A noncompact continuous image of ω_1 not homeomorphic to ω_1.

Proof. Let $\omega_1'' = (\omega_1')'$. Let $<$ denote the usual order on ω_1. Define an auxiliary order

$$\xi \prec \eta \quad \text{iff} \quad \begin{array}{l} \text{either } \xi < \eta \text{ and } (\forall \lambda \in \omega_1'')[\{\xi, \eta\} \not\subseteq (\lambda, \lambda+\omega]] \\ \text{or } \eta < \xi \text{ and } (\exists \lambda \in \omega_1'')[\{\xi, \eta\} \subseteq (\lambda, \lambda+\omega]]. \end{array}$$

Thus almost all points stay in the same order except for every $\lambda \in \omega_1''$ for which the interval $(\lambda, \lambda+\omega]$ is turned upside down.

We show $\prec$ also is compatible. So let $\mathcal{T}_<$ and $\mathcal{T}_\prec$ denote the order topologies induced by $<$ and $\prec$, respectively. Let $\xi \in \omega_1$.

Case 1: $\xi \in \omega_1''$. In this case $[0, \xi]_< = [0, \xi]_\prec$ and because $\xi + \omega$ is ξ's successor with respect to $\prec$ the interval is $\mathcal{T}_\prec$-clopen. Furthermore, if $\eta < \xi$ then $\eta + \omega < \xi$ and $(\eta+\omega, \xi]_< = (\eta+\omega, \xi]_\prec$ so $\{(\eta+\omega, \xi]_< : \eta < \xi\}$ is a local base at ξ for both $\mathcal{T}_<$ and $\mathcal{T}_\prec$.

Case 2: $\xi \notin \omega_1''$. If ξ is a successor ordinal then it will stay isolated in $\mathcal{T}_\prec$; its immediate neighbours stay its immediate neighbours.

Now if ξ is a limit then it is of the form $\eta + \omega$ for some $\eta \in \omega_1'$. If $\eta \in \omega_1''$ then $(\eta, \xi]$ is turned upside down but otherwise not disturbed and if $\eta \notin \omega_1''$ then $(\eta, \xi]$ is left as it is. In either case ξ retains its old neighbourhoods.

Next let X be the quotient obtained from ω_1 by collapsing $\{\lambda, \lambda+\omega\}$ to a single point for all $\lambda \in \omega_1''$. Then X is regular, and in fact orderable since $\prec$ induces a compatible linear order on X (since $\lambda + \omega$ is the immediate $\prec$-successor of λ if $\lambda \in \omega_1''$). Let $q: \omega_1 \to X$ be the quotient map. Then $q^{\to}\omega_1''$ is a cub in X, and the map $\phi: q^{\to}\omega_1'' \to \mathcal{P}(X)$ defined by $\phi(q(\lambda)) = q^{\to}(\lambda, \lambda+\omega)$ shows that X does not satisfy the wPDL. □

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