Convergence and Numerical Analysis of a Family of Two-Step Steffensen’s Methods

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Abstract—We provide sufficient conditions for the semilocal convergence of a family of two-step Steffensen’s iterative methods on a Banach space. The main advantage of this family is that it does not need to evaluate neither any Fréchet derivative nor any bilinear operator, but having a high speed of convergence. Some numerical experiments are also presented. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Nonlinear equations, Iterative methods, Semilocal convergence.

1. INTRODUCTION

We consider the problem of approximating locally the unique solution \( x^* \) of a nonlinear equation,

\[
    f(x) = 0, \tag{1}
\]

where \( f \) is a continuous operator defined on closed convex domain \( D \) of a Banach space \( E_1 \) with values in a Banach space \( E_2 \).

We use the two-step Steffensen’s method given by

\[
    y_n = x_n - [g_1(x_n), g_2(x_n); f]^{-1} f(x_n),
    \quad x_{n+1} = y_n - [g_1(x_n), g_2(x_n); f]^{-1} f(y_n), \quad n \geq 0. \tag{2}
\]

to generate a sequence converging to \( x^* \).

Here, \( [x, y; f] \in L(E_1, E_2) \) is a divided difference of order one for the operator \( f \) on the points \( x, y \in D \). If \( f \) is Fréchet differentiable then,

\[
    [x, y; f] = \int_0^1 f'(x + t(y - x)) \, dt, \quad \forall x, y \in D. \tag{3}
\]

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In [1], it is proved that these type of divided differences are characterized by the following property
\[ [x, y; f] = 2[x, 2y - x; f] - [y, 2y - x; f], \]
for all \( x, y \in D, \) \( x \neq y, \) and \( 2y - x \in D. \)

In the numerical section, we give explicit expressions of some divided differences of this type in \( \mathbb{R}^n. \)

The operators \( g_1 : D \subset E_1 \rightarrow E_2 \) and \( g_2 : D \subset E_1 \rightarrow E_2 \) are given and continuous on \( D. \)

Several choices for \( g_1 \) and \( g_2 \) are possible.

1. Secant type \([2,3]\): \( g_1(x_n) = x_{n-1} \) and \( g_2(x_n) = x_n. \)

2. Improved secant type \([4,5]\): \( g_1(x_n) = x_n + \alpha_n(x_{n-1} - x_n) \) and \( g_2(x_n) = x_n, \) where \( \alpha_n \in (0, 1] \) is a control of the good approximation to the derivative. In order to control the stability in practice, the \( \alpha_n \) can be computed, such that
\[ tol_c \ll \|\alpha_n(x_{n-1} - x_n)\| \leq tol_u, \]
where \( tol_c \) is related with the computer precision and \( tol_u \) is a free parameter.

3. Steffensen’s type \((f : D \subset E_1 \rightarrow E_1)\): \( g_1(x_n) = x_n \) and \( g_2(x_n) = x_n + f(x_n) \) \([2,6,7]\).

4. Improved Steffensen’s type \([8]\) \((f : D \subset E_1 \rightarrow E_1)\): \( g_1(x_n) = x_n + f(x_n) \) and \( g_2(x_n) = x_n + \alpha_n f(x_n). \) In this case, \( tol_c \ll \|\alpha_n f(x_n)\| \leq tol_u. \)

On one hand, the Steffensen and secant forms of our scheme are related with the schemes studied in \([9,10]\). On the other hand, the full two-step version is related with the scheme analyzed in \([11]\), that writes
\[ y_n = x_n - [x_n, y_{n-1}; f]^{-1} f(x_n), \]
\[ x_{n+1} = y_n - [x_n, y_{n-1}; f]^{-1} f(y_n), \quad n \geq 0, \]
starting from two initial guesses \( x_0 \) and \( y_{-1}. \)

In this collection of papers, as well as in the monograph \([12]\), the authors present an unified theory of convergence based on the so-called “method of nondiscrete mathematical induction”. The theory has been starting from paper \([13]\) where the classical closed graph theorem and nonlinear mapping are studied.

The basic idea is to define the rate of convergence as a function, not as a number, giving more information than the classical methods, which makes possible to give error estimates that are sharp even throughout the whole process. By the same token, the new rate of convergence is less intuitive and its computation could be more difficult than in the classical notion. In the present paper, we do not consider the “method of nondiscrete mathematical induction” in order to analyze the convergence of the scheme \((2)\).

In a different way, using the fixed-point theorem, the convergence of Steffensen’s type methods has been analyzed by Argyros in \([14]\). We propose a semilocal theorem for the introduced iterative method \((2)\), with a similar Argyros’ strategy.

As we will see in the numerical section, where we study the numerical behavior of scheme \((2)\), considering our \( \alpha_n \) correction have two advantages. First, it is an strategy to obtain numerical third-order without stability problems and without evaluating neither derivatives nor bilinear operators, and second, there are examples where the hypothesis of convergence are verified for the proposed iterative method \((2)\), but not for the classical Steffensen’s type schemes.

2. CONVERGENCE ANALYSIS

DEFINITION 1. The first-order divided difference of \( f \) on the points \( x, y \in E_1 \) is an application, \( [x, y; f] \in L(E_1, E_2), \) which satisfies the following.

1. \( [x, y; f](x - y) = f(x) - f(y), \)
2. if \( f \) is Fréchet differentiable, then, \( [x, x; f] = f'(x). \)
We consider the operator, \( P : D \subseteq E_1 \to E_2 \), defined by
\[
P(z) = z - \left[ g_1(z), g_2(z) ; f^{-1}(f(z) + f(z - [g_1(z), g_2(z) ; f^{-1}(f(z))]) \right],
\] (5)
in which case, method (2) can be written as
\[
x_{n+1} = P(x_n), \quad n \geq 0 \quad (x_0 \in D).
\] (6)

Let \( a, b_1, b_2, c, \) and \( \alpha \) be nonnegative numbers. Let \( x_0 \in D \) be, such that the inverse of
\[
[g_1(x_0), g_2(x_0) ; f],
\]
exists on \( D \).

Define parameters \( d, \delta, \beta_1, \beta_2, \epsilon_1, \epsilon_2 \) by
\[
d = a (b_1 + b_2),
\]
\[
\delta = \left\| g_1(x_0), g_2(x_0) ; f^{-1}(f(x_0)) \right\|,
\]
\[
\beta_i = a \| x_0 - g_i(x_0) \|,
\]
\[
e = 2a \epsilon + \beta_1 + \beta_2 + d \epsilon,
\]
\[
\epsilon_i = \frac{\| x_0 - g_i(x_0) \|}{1 - b_i} \quad (0 \leq b_i < 1).
\]

Moreover, define the following real functions on \([0, \infty)\),
\[
h_1(r) = r - \frac{1}{d}, \quad (7)
\]
\[
h_2(r) = (1 - d \epsilon)^2 \epsilon + (1 - d \epsilon)(2\alpha(\alpha + d) + c\epsilon) + c(\alpha(\alpha + d)) - (1 - d \epsilon)^3, \quad (8)
\]
\[
h_3(r) = r \frac{h_2(r)}{(1 - d \epsilon)^3} + \delta, \quad (9)
\]

and
\[
h_4(r) = r - \max(\epsilon_1, \epsilon_2). \quad (10)
\]

For our convergence theorem, we assume there is an interval \( B \), such that
\[
h_1(r) < 0, \quad (11)
\]
\[
h_2(r) < 0, \quad (12)
\]
\[
h_3(r) \leq 0, \quad (13)
\]
\[
h_4(r) \geq 0, \quad (14)
\]

and
\[
U(x_0, r) \subseteq D, \quad (15)
\]

for all \( r \in B \).

Next, we show that this can happen.
THEOREM 1. Let \( f : D \subseteq E_1 \rightarrow E_2, g_1 : D \subseteq E_1 \rightarrow E_2, \) and \( g_2 : D \subseteq E_1 \rightarrow E_2 \) be continuous operators, \( D \) closed convex subset of \( E_1 \), and \( E_1, E_2 \) Banach spaces. Let \( x_0 \in D \), such that exists \( \Gamma_0 := [g_1(x_0), g_2(x_0); f]^{-1} \). Assume the following.

(a) \( f \) twice and \( g_i \), \( i = 1, 2 \), once Fréchet differentiable on \( D \).
(b) there exist \( a, b_1, b_2, c > 0 \), \( x_0 \in D \), such that

\[
\begin{align*}
\| [g_1(x_0), g_2(x_0); f]^{-1} (x, y; f) - (v, w; f) \| & \leq a (\| x - v \| + \| y - w \|), \\
\| g_1(x) - g_1(y) \| & \leq b_1 \| x - y \|, \\
\| g_2(x) - g_2(y) \| & \leq b_2 \| x - y \|, \\
\| \Gamma_0 (f(y) - f(y')) \| & \leq c \| x - y \|, \\
\| \Gamma_0 (g_1(x), g_2(x); f) - (g_1(x_0), g_2(x_0); f) \| & \leq \alpha,
\end{align*}
\]

for all \( x, y, v, w \in D \).
(c) Conditions (11)--(15) hold on some interval \( B \subseteq \mathbb{R}^+ \cup \{0\} \).

Then, the following hold.

1. \( \| P'(x) \| \leq c(r) < 1 \),

\[
\begin{align*}
\text{and} \\
r & \geq \frac{\delta}{1 - c(r)},
\end{align*}
\]

for all \( r \in B \), where

\[
c(r) = \frac{1}{(1 - dr) c} + \frac{1}{(1 - dr)^2} (2\alpha (cr + \delta) + ce) + \frac{1}{(1 - dr)^3} c(\alpha (cr + \delta)).
\]

2. \( P \) has a unique fixed-point \( x^* \) in \( U(x_0, r) \).

3. Iteration \( x_{n+1} = P(x_n) \) \( n \geq 0 \) converges to \( x^* \). Moreover, the following error bound holds, for all \( n \geq 0 \),

\[
\| x_n - x^* \| \leq c^n (r) r, \quad r \in B.
\]

PROOF. Let \( x, g_1(x), g_2(x) \in U(x_0, r) \) \( r \in B \). Using (3) and (5), we obtain

\[
P' (x) = I - \left( [g_1(x), g_2(x); f]^{-1} \right)' (f(x) + f(y)) \\
- [g_1(x), g_2(x); f]^{-1} (f'(x) + (f(y)')) \\
= [g_1(x), g_2(x); f]^{-1} [g_1(x_0), g_2(x_0); f] [g_1(x_0), g_2(x_0); f]^{-1} \\
\times \left( [g_1(x), g_2(x); f] - f'(x) - (f(y)') \\
- \int_0^1 f''(g_1(x) + t(g_2(x) - g_1(x))) (g_1(x) + t(g_2(x) - g_1(x))) \, dt \\
\cdot [g_1(x), g_2(x); f]^{-1} [g_1(x_0), g_2(x_0); f] [g_1(x_0), g_2(x_0); f]^{-1} \\
\cdot (f(x) + f(y))).
\]

By (11), (16), (17), (18), and the following estimate,

\[
\| [g_1(x_0), g_2(x_0); f]^{-1} (g_1(x), g_2(x); f) - [g_1(x_0), g_2(x_0); f] \| \\
\leq a (\| g_1(x) - g_1(x_0) \| + \| g_2(x) - g_2(x_0) \|) \leq a (b_1 r + b_2 r) = dr < 1,
\]

the proof is completed.
and from the Banach lemma on invertible operators, [2], it follows that \([g_1(x), g_2(x); f]^{-1}\) exists \((x \in U(x_0, r)) (r \in B)\) and that
\[
\|g_1(x), g_2(x); f]^{-1} g_1(x_0), g_2(x_0); f\| \leq (1 - dr)^{-1}. \tag{26}
\]
Conditions (17) and (18) imply that \(f'(x) = [x, x; f] (x \in D)\) [2] and that
\[
\|g_1(x_0), g_2(x_0); f]^{-1} g_1(x_0), g_2(x_0); f\| \leq (1 - dr)^{-1} (1 - dr) - 1.
\]
Then, by (16)-(20), (27) and (26), the second member inside the brackets in (25) is bounded above by
\[
\alpha (1 - dr)^{-1} 2 (c r + d). \tag{28}
\]
Moreover,
\[
(f(y))' = f'(y) \left( I - \left( g_1(x), g_2(x); f]^{-1} f(x) - g_1(x), g_2(x); f]^{-1} f'(x) \right) \right.
\]
\[
= f'(y) \left( g_1(x), g_2(x); f]^{-1} g_1(x), g_2(x); f]^{-1} \left( g_1(x), g_2(x); f] - [x, x; f] \right) \right.
\]
\[
- \left( g_1(x), g_2(x); f]^{-1} f(x) \right)
\]
\[
= f'(y) \left( g_1(x), g_2(x); f]^{-1} g_1(x_0), g_2(x_0); f] \right.
\]
\[
\times g_1(x_0), g_2(x_0); f]^{-1} \left( g_1(x), g_2(x); f] - [x, x; f] \right) \right.
\]
\[
- \left( g_1(x), g_2(x); f]^{-1} f(x) \right)
\]
\[
= f'(y) \left( g_1(x), g_2(x); f]^{-1} g_1(x_0), g_2(x_0); f] \right.
\]
\[
\times g_1(x_0), g_2(x_0); f]^{-1} \left( g_1(x), g_2(x); f] - [x, x; f] \right) \right.
\]
\[
- \int_0^1 f''(g_1(x) + t (g_2(x) - g_1(x))) (g_2'(x) + t (g_2'(x) - g_2'(x))) dt
\]
\[
\cdot \left( g_1(x), g_2(x); f]^{-1} g_1(x_0), g_2(x_0); f] - [x, x; f] \right) \right) \).
\]
Then, from (16)-(20), (27) and (26) we deduce,
\[
\|g_1(x_0), g_2(x_0); f]^{-1} f(y)\| \leq c \frac{1}{(1 - dr)^2} ((1 - dr) e + \alpha (c r + d)). \tag{29}
\]
Finally, using relations (26)-(29), (25) gives
\[
\|P'(x)\| \leq c(r) \quad (r \in B),
\]
where function \(c(r)\) is given by (23). From (11) and (12), it is easy to check that \(c(r) \in [0, 1)\) and from (13) that (22) holds.
On the other hand, it follows from (14), (22), and (2) (for \(n = 0\)), that
\[
\|x_1 - x_0\| \leq (1 - c(r)) r \leq r,
\]
and

\[ \| g_i(x_0) - x_0 \| \leq (1 - b_i) r < r, \quad i = 1, 2, \]

which imply \( x_1, g_1(x_0), g_2(x_0) \in U(x_0, r) \).

Assume for all positive integers smaller or equal to \( n \) that

\[ x_n \in U(x_0, r), \quad \| x_n - x_0 \| \leq (1 - c^n(r)) r < r. \]

First, for \( i = 1, 2 \), we get

\[ \| x_n - x_0 \| \leq (1 - c^n(r)) r < r. \]

That is, \( g_i(x_n) \in U(x_0, r), \ i = 1, 2. \)

Using (6) and (21), we obtain

\[ \| x_{n+1} - x_n \| = \| P(x_n) - P(x_{n-1}) \|
\]

\[ \leq \max_{y \in [x_{n-1}, x_n]} \| P'(y) \| \| x_n - x_{n-1} \|
\]

\[ \leq c(r) \| x_n - x_{n-1} \| \quad (r \in B). \] (30)

By an induction strategy, we have

\[ \| x_{n+1} - x_n \| \leq c^n(r) \| x_1 - x_0 \| = c^n(r) (1 - c^n(r)) r \]

and

\[ \| x_{n+1} - x_0 \| \leq \| x_{n+1} - x_n \| + \| x_n - x_0 \|
\]

\[ \leq c^n(r) (1 - c(r)) r + (1 - c^n(r)) r
\]

\[ = (1 - c^{n+1}(r)) r \leq r. \]

That is, \( x_n \in U(x_0, r) \) and \( \| x_n - x_0 \| \leq (1 - c^n(r)) r \leq r \), for all \( n \).

From (30), now, we obtain

\[ \| x_{n+m} - x_n \| \leq (1 - c^m(r)) c^n(r) r. \] (31)

In particular, \( \{ x_n \} \) is a Cauchy sequence in the Banach space \( E_1 \) and therefore, exists \( x^* = \lim_{n \to \infty} x_n \).

Using the continuity of \( f, g_1, \) and \( g_2, (6) \) gives \( P(x^*) = x^* \).

To show the uniqueness, assume \( y^* \in U(x_0, r) \) and \( P(y^*) = y^* \). Then,

\[ \| x^* - y^* \| = \| P(x^*) - P(y^*) \| \leq \sup_{x \in [x^*, y^*]} \| P'(z) \| \| x^* - y^* \| \leq c(r) \| x^* - y^* \| , \]

since \( c(r) < 1 \), we deduce \( x^* = y^* \).

Finally, taking \( m \to \infty \) in (31), we obtain (24).

That completes the proof of the theorem.

\[ \Box \]

3. NUMERICAL ANALYSIS

We start with a simple example that shows the convergence hypothesis. Let the real equation

\[ f(x) = x - x^2, \]

defined in \([-0.1, 0.1]\] and \( x_0 = 0.01 \).
Taking for instance, \( g_1(x) = x \) and \( g_2(x) = x - f(x) \), it is easy to check that
\[
\begin{align*}
  a &= 1.010203, \\
  b_1 &= 0.2, \\
  b_2 &= 1, \\
  c &= 1.4546923, \\
  d &= 1.2122436, \\
  \alpha &= 1.7061206, \\
  \beta_1 &= 0, \\
  \beta_2 &= 0.010001, \\
  \epsilon_1 &= 0, \\
  \epsilon_2 &= 0.012375, \\
  \delta &= \beta_2.
\end{align*}
\]

Then, conditions (11)-(14) are verified, for all \( r \in [0.0154232, 0.0342567] \).

In general, since
\[
P : U(x_0, r) \rightarrow U(x_0, r)
\]
is a contraction with contraction factor \( c(r) < 1 \), and using the contraction mapping principle [2], the region of accessibility to \( x^* \) can be extended also to a ball around \( x_0 \) in a different way.

**Theorem 2.** Under the hypothesis of Theorem 1 iteration \( y_{n+1} = P(y_n), y_0 \in U(x_0, r) \) \( (r \in B) \) converges to \( x^* \). Moreover, the following error bounds hold, for all \( n \geq 0 \),
\[
\|x^* - y_n\| \leq \frac{c^n(r)}{1 - c(r)} \|y_1 - y_0\|
\]
and
\[
\|x^* - y_n\| \leq c^n(r) \|x^* - y_0\|.
\]

We refer [14] for more details.

In order to show the performance of the introduced method, we have tested it on the results obtained for the zeros of some equations. We compare the classical Steffensen’s method and method (2), that we denote \( M(\alpha) \), taking \( g_1(x) = x \) and \( g_2(x) = x - \alpha f(x) \).

On the other hand, using Taylor expansion, the \( M(\alpha) \) methods can be written as
\[
x_{n+1} = x_n - \left( 1 + \frac{1}{2} L_f(x_n) + O\left( L_f(x_n)^2 \right) \right) \frac{f(x_n)}{f'(x_n)},
\]
where
\[
L_f(x) = f'(x)^{-1} f''(x) f'(x)^{-1} f(x).
\]
Thus, they are third-order methods for simple roots [15].

In Table 1, we study the equation \( x^2 - x = 0 \), for different initial data, we compute the error until obtain convergence, the best results are obtained with the method \( M(10^{-8}) \). We can see numerically the order three.

<table>
<thead>
<tr>
<th>Iter.</th>
<th>Steff.</th>
<th>( M(1) )</th>
<th>( M(10^{-8}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.05e-04</td>
<td>6.37e-06</td>
<td>2.09e-06</td>
</tr>
<tr>
<td>2</td>
<td>8.41e-08</td>
<td>1.55e-15</td>
<td>1.83e-17</td>
</tr>
<tr>
<td>3</td>
<td>1.41e-14</td>
<td>2.17e-44</td>
<td>6.84e-49</td>
</tr>
<tr>
<td>4</td>
<td>3.99e-28</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Error, \( f(x) = x^2 - x, x_0 = 0.01 \).
In Tables 2 and 3, we consider $x^2 + 20x = 0$ and $x^2 + 200x = 0$, respectively. We obtain the same good results, for $\alpha = 10^{-8}$, but we observe some difficulty with the first iterations for the other methods in 3, where $|f(x_0)|$ is not small enough. In the second case, only the proposed iterative method (2) verifies the convergence hypothesis and obtain the desired accuracy in all the iterations.

Before giving examples in several dimensions, we present a divided difference $[u, v; F]$ of a nonlinear operator $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$.

$$[u, v; F]_{ij} = \frac{F_i (u^{(1)}, \ldots, u^{(j)}, v^{(j-1)}, \ldots, v^{(n)}) - F_i (u^{(1)}, \ldots, u^{(j-1)}, v^{(j)}, \ldots, v^{(n)})}{u^{(j)} - v^{(j)}}.$$ 

Next, we consider quadratic equations of the type

$$f(x) = x'Ax + Bx + C = 0,$$

where

\[ \dim(A) = (n \times n) \times n, \]
\[ \dim(B) = n \times n, \]

and

\[ \dim(C) = \dim(x) = n. \]

The above kind of equations may come from the discretization of equilibrium problems, where interacting forces between particles determine the output. However, the actual case we are going to analyze is prepared to get an exact solution in order to make it easy the evaluation of the errors. We randomly generate $A$ and $B$, and then, we determine $C$, such that

\[ x^*(i) = 2, \quad i = 1, 2, \ldots, n, \]

is a solution of (32).
Table 4. \(x_0 = 2.2, x^* = 2, l_{\infty} \text{ error}\), nonlinear system (32).

<table>
<thead>
<tr>
<th>Iterations</th>
<th>(M(10^{-8}))</th>
<th>Halley</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.15e - 03</td>
<td>1.25e - 03</td>
</tr>
<tr>
<td>2</td>
<td>4.84e - 09</td>
<td>4.31e - 10</td>
</tr>
<tr>
<td>3</td>
<td>7.89e - 13</td>
<td>2.00e - 13</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that the second Fréchet derivative is constant,

\[ f''(x) = A + A'. \]

In Table 4, we consider \(n = 100\) for different initial data. We compute classical Halley's method,

\[ x_{n+1} = x_n - \left( I + \frac{1}{2} L_f(x_n) \left( I - \frac{1}{2} L_f(x_n) \right)^{-1} \right) f^{-1}(x_n) f(x_n). \]  

Accuracy is similar for both schemes.

Finally, let us consider the Hammerstein equation,

\[ x(s) = 1 - \frac{1}{4} \int_0^1 \frac{s}{t+s} x(t) \, dt, \quad s \in [0,1], \]

studied in [16].

Using the trapezoidal rule of integration with step \(h = 1/m\), we obtain the following system of nonlinear equations, for \(i = 0, 1, \ldots, m\),

\[ 0 = x^i - 1 + \frac{1}{4m} \left( \frac{1}{2} t_i + t_0 \sum_{k=0}^n \frac{t_i}{t_i + t_k} x^k + \frac{1}{2} \frac{t_i}{t_i + t_m} x^m \right), \]

where \(t_j = j/m\).

In this case, the second Fréchet derivative is diagonal by blocks.

We consider \(m = 20\) in the quadrature trapezoidal formula. In Table 5, we compare the obtained results with the iterative scheme (2) and classical Halley's iterative scheme. With both schemes, we obtain third-order convergence.

Table 5. \(x_0 = 1.5, l_{\infty} \text{ error}\), \(m = 20\), nonlinear system (35).

<table>
<thead>
<tr>
<th>Iterations</th>
<th>(M(10^{-8}))</th>
<th>Halley</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.30e - 03</td>
<td>1.75e - 02</td>
</tr>
<tr>
<td>2</td>
<td>4.06e - 09</td>
<td>1.03e - 06</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the two last examples, we remark that the classical third-order methods, like Halley's schemes (33), use the evaluation of the second Fréchet derivative. The methods evaluating second derivative, in general, only are competitive when the order of the system is small or the evaluation of the second derivative (bilinear operator) is not more expensive than the evaluation of several values of the original function. Thus, in general, it is preferred the introduced method (2).

REFERENCES


