Hamiltonian colorings of graphs

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Abstract

For vertices \( u \) and \( v \) in a connected graph \( G \) of order \( n \), the length of a longest \( u-v \) path in \( G \) is denoted by \( D(u, v) \). A hamiltonian coloring \( c \) of \( G \) is an assignment \( c \) of colors (positive integers) to the vertices of \( G \) such that \( D(u, v) + |c(u) - c(v)| \geq n - 1 \) for every two distinct vertices \( u \) and \( v \) of \( G \). The value \( hc(c) \) of a hamiltonian coloring \( c \) of \( G \) is the maximum color assigned to a vertex of \( G \). The hamiltonian chromatic number \( hc(G) \) of \( G \) is \( \min \{hc(c)\} \) over all hamiltonian colorings \( c \) of \( G \). Hamiltonian chromatic numbers of some special classes of graphs are determined. It is shown that for every two integers \( k \) and \( n \) with \( k \geq 1 \) and \( n \geq 3 \), there exists a hamiltonian graph of order \( n \) with hamiltonian chromatic number \( k \) if and only if \( 1 \leq k \leq n - 2 \). Also, a sharp upper bound for the hamiltonian chromatic number of a connected graph in terms of its order is established.

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1. Introduction

For a connected graph \( G \) of order \( n \) and diameter \( d \) and an integer \( k \) with \( 1 \leq k \leq d \), a \textit{radio \( k \)-coloring} of \( G \) is defined in [1] as an assignment \( c \) of colors (positive integers) to the...
vertices of $G$ such that
\[ d(u, v) + |c(u) - c(v)| \geq 1 + k \]
for every two distinct vertices $u$ and $v$ of $G$. The value $rc_k(c)$ of a radio $k$-coloring $c$ of $G$ is the maximum color assigned to a vertex of $G$; while the radio $k$-chromatic number $rc_k(G)$ of $G$ is $\min\{rc_k(c)\}$ over all radio $k$-colorings $c$ of $G$. A radio $k$-coloring $c$ of $G$ is a minimum radio $k$-coloring if $rc_k(c) = rc_k(G)$. These concepts were inspired by the so-called channel assignment problem, where channels are assigned to FM radio stations according to the distances between the stations (and some other factors as well).

Since $rc_1(G)$ is the chromatic number $\chi(G)$, radio $k$-colorings provide a generalization of ordinary colorings of graphs. The radio $d$-chromatic number was studied in [1,2] and was also called the radio number. Radio $d$-colorings are also referred to as radio labelings since no two vertices can be colored the same in a radio $d$-coloring. Thus, in a radio labeling of a connected graph of diameter $d$, the labels (colors) assigned to adjacent vertices must differ by at least $d$, the labels assigned to two vertices whose distance is 2 must differ by at least $d - 1$, and so on, up to the vertices whose distance is $d$, that is, antipodal vertices, whose labels are only required to be different. A radio $(d - 1)$-coloring is less restrictive in that colors assigned to two vertices whose distance is $i$, where $1 \leq i \leq d$, are only required to differ by at least $d - i$. In particular, antipodal vertices can be colored the same. For this reason, radio $(d - 1)$-colorings are also called radio antipodal colorings or, more simply, antipodal colorings. Antipodal colorings of graphs were studied in [3,4], where $rc_{d-1}(G)$ was written as $ac(G)$.

Radio $k$-coloring of paths were studied in [5] for all possible values of $k$. In the case of an antipodal coloring of the path $P_n$ of order $n \geq 2$ and diameter $n - 1$, only the end-vertices of $P_n$ are permitted to be colored the same since the only pair of antipodal vertices in $P_n$ are its two end-vertices. Of course, the two end-vertices of $P_n$ are connected by a hamiltonian path. As mentioned earlier, if $u$ and $v$ are any two distinct vertices of $P_n$, then $|c(u) - c(v)| \geq n - 1 - i$. Since $P_n$ is a tree, not only is $i$ the length of a shortest $u-v$ path in $P_n$, it is, in fact, the length of every $u-v$ path in $P_n$ since every two vertices are connected by a unique path. Furthermore, the length of a longest $u-v$ path in $P_n$ is $i$ as well.

For vertices $u$ and $v$ in a connected graph $G$, let $D(u, v)$ denote the length of a longest $u-v$ path in $G$. Thus for every connected graph $G$ of order $n$ and diameter $d$, both $d(u, v)$ and $D(u, v)$ are metrics on $V(G)$. Radio $k$-colorings of $G$ are inspired by radio antipodal colorings $c$ which are defined by the inequality
\[ d(u, v) + |c(u) - c(v)| \geq d. \]

If $G$ is a path, then (1) is equivalent to
\[ D(u, v) + |c(u) - c(v)| \geq n - 1, \]
which suggests an extension of the coloring $c$ that satisfies (2) for an arbitrary connected graph $G$. A hamiltonian coloring $c$ of $G$ is an assignment of colors (positive integers) to the vertices of $G$ such that $D(u, v) + |c(u) - c(v)| \geq n - 1$ for every two distinct vertices $u$ and $v$ of $G$. In a hamiltonian coloring of $G$, two vertices $u$ and $v$ can be assigned the same color only if $G$ contains a hamiltonian $u-v$ path. The value $hc(c)$ of a hamiltonian coloring
c of $G$ is the maximum color assigned to a vertex of $G$. The hamiltonian chromatic number $\text{hc}(G)$ of $G$ is $\min\{\text{hc}(c)\}$ over all hamiltonian colorings $c$ of $G$. A hamiltonian coloring $c$ of $G$ is a minimum hamiltonian coloring if $\text{hc}(c) = \text{hc}(G)$.

A graph $G$ is hamiltonian-connected if for every pair $u, v$ of distinct vertices of $G$, there is a hamiltonian $u-v$ path. Consequently, we have the following fact.

**Observation 1.1.** Let $G$ be a connected graph. Then $\text{hc}(G) = 1$ if and only if $G$ is hamiltonian-connected.

In a certain sense, the hamiltonian chromatic number of a connected graph $G$ measures how close $G$ is to being hamiltonian-connected, the nearer the hamiltonian chromatic number of a connected graph $G$ is to 1, the closer $G$ is to being hamiltonian-connected.

**2. Graphs with equal hamiltonian chromatic number and antipodal chromatic number**

Since the path $P_n$ is the only graph $G$ of order $n$ for which $\text{diam} G = n - 1$, we have the following.

**Observation 2.1.** If $G$ is a path, then $\text{hc}(G) = \text{ac}(G)$.

In [4] it was shown that $\text{ac}(P_n) \leq \left(\frac{n-1}{2}\right) + 1$ for every positive integer $n$. Moreover, it was shown in [5] that $\text{ac}(P_n) \leq \left(\frac{n-1}{2}\right) - \frac{n-1}{2} + 4$ for odd integers $n \geq 7$. Therefore, we have the following.

**Corollary 2.2.** For every positive integer $n$,

$$\text{hc}(P_n) \leq \left(\frac{n-1}{2}\right) + 1.$$

Furthermore, for all odd integers $n \geq 7$,

$$\text{hc}(P_n) \leq \left(\frac{n-1}{2}\right) - \frac{n-1}{2} + 4.$$

In order to see that the converse of Observation 2.1 is false, we first consider the following lemmas.

**Lemma 2.3.** Let $H$ be a hamiltonian graph of order $n - 1 \geq 3$. If $G$ is a graph obtained from $H$ by adding a pendant edge, then $\text{hc}(G) = n - 1$.

**Proof.** Let $C : v_1, v_2, \ldots, v_{n-1}, v_1$ be a hamiltonian cycle of $H$ and let $v_1v_n$ be the pendant edge of $G$. Let $c$ be a hamiltonian coloring of $G$. Since $D(u, v) \leq n - 2$ for all $u, v \in V(C)$, there is no pair of vertices in $C$ that are colored the same by $c$. This implies that $\text{hc}(c) \geq n - 1$ and so $\text{hc}(G) \geq n - 1$. 


Define a coloring $c_0$ of $G$ by $c_0(v_i) = i$ for $1 \leq i \leq n - 1$ and $c_0(v_n) = n - 1$ (see Fig. 1). We show that $c_0$ is a hamiltonian coloring of $G$.

First consider two vertices $v_i$ and $v_j$, where $1 \leq i < j \leq n - 1$. Then $|c_0(v_i) - c_0(v_j)| = j - i$, while $D(v_i, v_j) \geq n - 1 + j - i$. Thus $|c_0(v_i) - c_0(v_j)| + D(v_i, v_j) \geq n - 1$. Now consider the two vertices $v_i$ and $v_n$, where $1 \leq i \leq n - 1$. Then $|c_0(v_i) - c_0(v_n)| = n - 1 - i$, while $D(v_i, v_n) \geq i$. Hence $|c_0(v_i) - c_0(v_n)| + D(v_i, v_n) \geq n - 1$. Therefore, $c_0$ is a hamiltonian coloring of $G$ and so $hc(G) \leq hc(c_0) = n - 1$.

For $n \geq 4$, let $G_n$ be the graph obtained from the complete graph $K_{n-1}$ by adding a pendant edge. Then $G_n$ has order $n$ and diameter 2. Let $V(G_n) = \{v_1, v_2, \ldots, v_n\}$, where $\deg v_n = 1$ and $v_{n-1}v_n \in E(G)$. By Lemma 2.3, $hc(G_n) = n - 1$. We now show that $ac(G_n) = hc(G_n) = n - 1$. Let $c$ be an antipodal coloring of $G_n$. Since $diam G_n = 2$, it follows that the colors $c(v_1), c(v_2), \ldots, c(v_{n-1})$ are distinct and so $ac(G_n) \geq n - 1$. Moreover, the coloring $c'$ of $G_n$ defined by $c'(v_i) = i$ for $1 \leq i \leq n - 1$, $c'(v_n) = 1$ is an antipodal coloring of $G_n$ (see Fig. 2) and so $ac(G_n) = n - 1$. Hence there is an infinite class of graphs $G$ of diameter 2 such that $hc(G) = ac(G)$.

We now show that there exists an infinite class of graphs $G$ of diameter 3 such that $hc(G) = ac(G)$.

**Lemma 2.4.** For $n \geq 5$, let $H_n$ be the graph obtained from the complete graph $K_{n-2}$, where $V(K_{n-2}) = \{v_1, v_2, \ldots, v_{n-2}\}$, by adding the two pendant edges $v_1v_{n-1}$ and $v_{n-2}v_n$. Then $H_n$ is a graph of order $n$ and diameter 3 such that $hc(H_n) = ac(H_n) = 2n - 5$.
Furthermore, $D(v_i, v_j)$. Similarly, $|c(v_i) - c(v_j)| + D(v_i, v_j) = 2(j - i) + n - 3 \geq 2 + n - 3 = n - 1$.

Next, we consider two vertices $v_i$ and $v_{n-1}$, where $1 \leq i \leq n - 2$. In this case, $|c(v_i) - c(v_{n-1})| = (2n - 6) - (2i - 1) = 2n - 2i - 5$ if $1 \leq i \leq n - 3$, while $|c(v_{n-2}) - c(v_{n-1})| = 1$. Moreover, $D(v_i, v_{n-1}) = 1$ and $D(v_i, v_{n-1}) = n - 2$ for $2 \leq i \leq n - 2$. Thus, for $1 \leq i \leq n - 3$,

$$|c(v_i) - c(v_{n-1})| + D(v_i, v_{n-1}) \geq (2n - 2i - 5) + (n - 2) = 3n - 2i - 7 \geq n - 1;$$

while

$$|c(v_{n-2}) - c(v_{n-1})| + D(v_{n-2}, v_{n-1}) = 1 + (n - 2) = n - 1.$$

Similarly, $|c(v_i) - c(v_n)| + D(v_i, v_n) \geq n - 1$ for $1 \leq i \leq n - 1$. Hence $c_1$ is a hamiltonian coloring of $H_n$ and so $hc(H_n) \leq hc(c_1) = 2n - 5$. Therefore, $hc(H_n) = 2n - 5$.

We now show that $ac(H_n) = 2n - 5$ as well. Let $c$ be an antipodal coloring of $H_n$. Since diam $H_n = 3$, it follows that the colors $c(v_1), c(v_2), \ldots, c(v_{n-2})$ differ by at least 2 and so $ac(H_n) \geq 2n - 5$. Since the coloring $c_1$ of $H_n$ shown in Fig. 3 is also an antipodal coloring of $H_n$, $ac(H_n) \leq 2n - 5$ and so $ac(H_n) = 2n - 5$. □

Whether there exists an infinite class of graphs $G$ that are not paths, whose diameter exceeds 3 and for which $hc(G) = ac(G)$, is not known. Indeed, it is not known if there is even one such graph that is not a path.

### 3. Hamiltonian chromatic numbers of some special classes of graphs

Since the complete graph $K_n$ is hamiltonian-connected, $hc(K_n) = 1$. We state this below for later reference.
Observation 3.1. For \( n \geq 1 \), \( \text{hc}(K_n) = 1 \).

We now consider the complete bipartite graphs \( K_{r,s} \), beginning with \( K_{r,r} \). The graph \( K_{r,r} \) has order \( n = 2r \) and is hamiltonian but is not hamiltonian-connected. For distinct vertices \( u \) and \( v \) of \( K_{r,r} \),

\[
D(u, v) = \begin{cases} 
  n - 1 & \text{if } uv \in E(K_{r,r}), \\
  n - 2 & \text{if } uv \notin E(K_{r,r}).
\end{cases}
\]

Therefore, for a hamiltonian coloring of \( K_{r,r} \), every two nonadjacent vertices must be colored differently (while adjacent vertices can be colored the same). This implies that \( \text{hc}(K_{r,r}) = \chi(K_{r,r}) = r \).

We now determine \( \text{hc}(K_{r,s}) \) with \( r < s \), beginning with \( r = 1 \).

Theorem 3.2. For \( n \geq 3 \), \( \text{hc}(K_{1,n-1}) = (n - 2)^2 + 1 \).

Proof. Since \( \text{hc}(K_{1,2}) = 2 \), the result holds for \( n = 3 \). So we may assume that \( n \geq 4 \). Let \( G = K_{1,n-1} \) with vertex set \( \{v_1, v_2, \ldots, v_n\} \), where \( v_n \) is the central vertex of \( G \). Define a coloring \( c \) of \( G \) by \( c(v_n) = 1 \) and \( c(v_i) = (n - 1) + (i - 1)(n - 3) \) for \( 1 \leq i \leq n - 1 \). Since \( c \) is a hamiltonian coloring,

\[
\text{hc}(G) \leq \text{hc}(c) = c(v_{n-1}) = (n - 1) + (n - 2)(n - 3) = (n - 2)^2 + 1.
\]

Next we show that \( \text{hc}(G) \geq (n - 2)^2 + 1 \). Let \( c \) be a minimum hamiltonian coloring of \( G \). Since \( G \) contains no hamiltonian path, no two vertices can be colored the same. We may assume that \( c(v_1) < c(v_2) < \cdots < c(v_{n-1}) \). We consider three cases.

Case 1: \( c(v_n) = 1 \). Since \( D(v_1, v_n) = 1 \) and \( D(v_i, v_{i+1}) = 2 \) for \( 1 \leq i \leq n - 2 \), it follows that

\[
c(v_1) \geq n - 1 \quad \text{and} \quad c(v_{i+1}) \geq c(v_i) + (n - 3) \quad \text{for all} \; 1 \leq i \leq n - 2.
\]

This implies that

\[
c(v_{n-1}) \geq (n - 1) + (n - 2)(n - 3) = (n - 2)^2 + 1.
\]

Therefore, \( \text{hc}(c) = \text{hc}(G) \geq (n - 2)^2 + 1 \).

Case 2: \( c(v_n) = c(v_{n-1}) \). Then \( 1 = c(v_1) < c(v_2) < \cdots < c(v_{n-1}) < c(v_n) \). For each \( i \) with \( 2 \leq i \leq n - 1 \), it follows that \( c(v_i) \geq (n - 2) + (i - 2)(n - 3) \). In particular, \( c(v_{n-1}) \geq (n - 2) + (n - 3)(n - 3) = n^2 - 5n + 7 \). Thus

\[
c(v_n) \geq c(v_{n-1}) + (n - 2) \geq (n^2 - 5n + 7) + (n - 2) = (n - 2)^2 + 1.
\]

Therefore, \( \text{hc}(c) = \text{hc}(G) \geq (n - 2)^2 + 1 \).

Case 3: \( c(v_j) < c(v_n) < c(v_{j+1}) \) for some \( j \) with \( 1 \leq j \leq n - 2 \). Thus

\[
c(v_j) \geq (n - 2) + (j - 2)(n - 3),
\]

\[
c(v_n) \geq c(v_j) + (n - 2) = 2(n - 2) + (j - 2)(n - 3),
\]

\[
c(v_{j+1}) \geq c(v_n) + (n - 2) \geq 3(n - 2) + (j - 2)(n - 3).
\]
This implies that
\[ c(v_{n-1}) \geq 3(n-2) + (n-4)(n-3) = n^2 - 4n + 6 > (n-2)^2 + 1. \]
Hence, \( hc(c) = hc(G) \geq (n-2)^2 + 1. \) □

We now consider \( K_{r,s} \), where \( 2 \leq r < s \), with partite sets \( V_1 \) and \( V_2 \) such that \( |V_1| = r \) and \( |V_2| = s \). Then
\[
D(u, v) = \begin{cases} 
2r - 2 = n - s + r - 2 & \text{if } u, v \in V_1, \\
2r - 1 = n - s + r - 1 & \text{if } uv \in E(K_{r,s}), \\
2r = n - s + r & \text{if } u, v \in V_2.
\end{cases}
\]
Consequently, if \( c \) is a hamiltonian coloring of \( K_{r,s} \) \((r < s)\), then
\[
|c(u) - c(v)| \geq \begin{cases} 
s - r + 1 & \text{if } u, v \in V_1, \\
s - r & \text{if } uv \in E(K_{r,s}), \\
s - r - 1 & \text{if } u, v \in V_2.
\end{cases}
\]

**Theorem 3.3.** For integers \( r \) and \( s \) with \( 2 \leq r < s \)
\[
hc(K_{r,s}) = (s - 1)^2 - (r - 1)^2.
\]

**Proof.** Let \( V_1 = \{u_1, u_2, \ldots, u_r\} \) and \( V_2 = \{v_1, v_2, \ldots, v_s\} \) be the partite sets of \( K_{r,s} \). Define a coloring \( c \) of \( K_{r,s} \) by \( c(u_i) = 1 + (i-1)(s-r+1) \) for \( 1 \leq i \leq r-1 \), \( c(v_j) = c(u_{r-1}) + (s-r) + (j-1)(s-r-1) = (r-1)(s-r+1) + (j-1)(s-r-1) \) for \( 1 \leq j \leq s-1 \), and \( c(u_r) = c(v_s) + (s-r) = (s-1)^2 - (r-1)^2 \). Since \( c \) is a hamiltonian coloring of \( K_{r,s} \), it follows that \( hc(K_{r,s}) \leq hc(c) \leq (s - 1)^2 - (r - 1)^2 \).

It remains to show that \( hc(K_{r,s}) \geq (s - 1)^2 - (r - 1)^2 \). Let \( c \) be a hamiltonian coloring of \( K_{r,s} \) and let \( V(K_{r,s}) = \{w_1, w_2, \ldots, w_{r+s}\} \), where \( c(w_1) \leq c(w_2) \leq \cdots \leq c(w_{r+s}) \). By a \( V_1 \)-block of \( K_{r,s} \), we mean a set \( A = \{w_2, w_{2+1}, \ldots, w_\beta\} \), where \( 1 \leq \alpha \leq \beta \leq r + s \), such that \( A \subseteq V_1 \), \( w_{2+i} \in V_2 \) if \( \alpha > 1 \), and \( w_\beta+1 \in V_2 \) if \( \beta < r + s \). A \( V_2 \)-block of \( K_{r,s} \) is defined similarly. Let \( A_1, A_2, \ldots, A_p \) \((p \geq 1)\) be the distinct \( V_1 \)-blocks of \( K_{r,s} \) such that if \( w' \in A_i \) and \( w'' \in A_j \), then \( c(w') < c(w'') \). If \( p \geq 2 \), then \( K_{r,s} \) contains \( V_2 \)-blocks \( B_1, B_2, \ldots, B_{p-1} \) such that for each integer \( i \) \((1 \leq i \leq p - 1)\) and for \( w' \in A_i \), \( w'' \in A_{i+1} \), it follows that \( c(w') < c(w'') \).

The graph \( K_{r,s} \) may contain up to two additional \( V_2 \)-blocks, namely, \( B_0 \) and \( B_p \) such that if \( y \in B_0 \) and \( y' \in A_1 \), then \( c(y) < c(y') \); while if \( z \in A_p \) and \( z' \in B_p \), then \( c(z) < c(z') \). If \( p = 1 \), then at least one of \( B_0 \) and \( B_1 \) must exist. Hence \( K_{r,s} \) contains \( p V_1 \)-blocks and \( p - 1 + t \) \( V_2 \)-blocks, where \( t \in \{0, 1, 2\} \). Consequently, there are exactly (1) \( r - p \) distinct pairs \( \{w_i, w_{i+1}\} \) of vertices, both of which belong to \( V_1 \), (2) \( 2p - 2 + t \) distinct pairs \( \{w_i, w_{i+1}\} \) of vertices, exactly one of which belongs to \( V_1 \), and (3) \( s - (p - 1 + t) \) distinct pairs \( \{w_i, w_{i+1}\} \) of vertices, both of which belong to \( V_2 \).

Since (1) the colors of every two vertices \( w_i \) and \( w_{i+1} \), both of which belong to \( V_1 \), must differ by at least \( s - r + 1 \), (2) the colors of every two vertices \( w_i \) and \( w_{i+1} \), exactly one of which belongs to \( V_1 \), must differ by at least \( s - r \), and (3) the colors of every two vertices \( w_i \) and \( w_{i+1} \), both of which belong to \( V_2 \), must differ by at least \( s - r - 1 \), it
follows that
\[ c(w_{r+s}) \geq 1 + (r - p)(s - r + 1) + (2p - 2 + t)(s - r) + (p - 1 + t)(s - r - 1) = (s - 1)^2 - (r - 1)^2 + t. \]
Since \( hc(K_{r,s}) \leq (s - 1)^2 - (r - 1)^2 \) and \( t \geq 0 \), it follows that \( t = 0 \) and that \( hc(K_{r,s}) = (s - 1)^2 - (r - 1)^2 \).

We now determine the hamiltonian chromatic number of each cycle. Minimum hamiltonian colorings of the cycles \( C_n \) for \( n = 3, 4, 5 \) are shown in Fig. 4.

For a hamiltonian coloring \( c \) of a graph \( G \), a set \( S = \{u, v\} \) of distinct vertices of \( G \) is called a \( c \)-pair if \( c(u) = c(v) \). We also write \( c(S) = c(u) = c(v) \).

**Lemma 3.4.** Let \( c \) be a minimum hamiltonian coloring of \( C_n \), where \( n \geq 4 \).

(a) If \( \{u, v\} \) is a \( c \)-pair, then \( u \) and \( v \) are adjacent.

(b) If \( S \) and \( S' \) are distinct \( c \)-pairs, then \( S \cap S' = \emptyset \) and \( c(S) \neq c(S') \).

**Proof.** If \( u \) and \( v \) are nonadjacent vertices of \( C_n \), then \( D(u, v) < n - 1 \), implying that \( c(u) \neq c(v) \) and so (a) holds.

To verify (b), let \( S = \{u, v\} \) and \( S' = \{u', v'\} \) be distinct \( c \)-pairs. Assume that \( S \cap S' \neq \emptyset \) or \( c(S) = c(S') \). If \( S \cap S' \neq \emptyset \), then we may assume that \( u \neq u' \) and \( v = v' \). This implies that \( c(u) = c(v) = c(u') = c(v') \) and therefore, \( \{u, u'\} \) is a \( c \)-pair as well. If \( c(S) = c(S') \), then \( \{u, u'\} \) is also a \( c \)-pair. By (a), \( \{\{u, u', v\}\} = C_3 \), which is a contradiction.

**Theorem 3.5.** For \( n \geq 3 \), \( hc(C_n) = n - 2 \).

**Proof.** Let \( C_n : v_1, v_2, \ldots, v_n, v_1 \). Since \( hc(C_n) = n - 2 \) for \( n = 3, 4, 5 \), we may assume that \( n \geq 6 \). Define a coloring \( c \) of \( C_n \) by \( c(v_1) = n - 2 \), \( c(v_2) = 1 \), and \( c(v_i) = i - 2 \) for \( 3 \leq i \leq n \) (see Fig. 5). Since \( c \) is a hamiltonian coloring, \( hc(C_n) \leq n - 2 \).

Next we show that \( hc(C_n) \geq n - 2 \). Let \( c \) be a minimum hamiltonian coloring of \( C_n \) and let \( q \) be the number of distinct \( c \)-pairs. Since \( hc(C_n) \leq n - 2 \), it follows that \( q \geq 2 \).

Denote these \( q \) \( c \)-pairs by \( S_1, S_2, \ldots, S_q \). By Lemma 3.4(b), for all \( i, j \) with \( 1 \leq i \neq j \leq q \), we have \( S_i \cap S_j = \emptyset \) and \( c(S_i) \neq c(S_j) \). If \( q = 2 \), then \( hc(c) \geq n - 2 \); so we assume that \( q \geq 3 \). Without loss of generality, we may assume that \( c(S_1) < c(S_2) < \cdots < c(S_q) \).
Fig. 5. A hamiltonian coloring of $C_n$ for $n \geq 6$.

For each $i$ with $1 \leq i \leq q - 1$, let
$$A_i = \{ u \in V(G) : c(S_i) < c(u) < c(S_{i+1}) \}.$$  

There exist nonnegative integers $a_1, a_2, \ldots, a_{q-1}$ such that $|A_i| = c(S_{i+1}) - c(S_i) - 1 - a_i$ for each integer $i$ ($1 \leq i \leq q - 1$). Define
$$a = a_1 + a_2 + \cdots + a_{q-1}$$

and
$$I = \{ i : a_i = 0, \text{ where } 1 \leq i \leq q - 1 \}.$$

At most $c(S_1) - 1$ vertices of $C_n$ are assigned a color less than $c(S_1)$ and at most $hc(c) - c(S_q)$ vertices of $C_n$ are assigned a color exceeding $c(S_q)$. Since all $n$ vertices of $C_n$ are assigned a color by $c$, it follows that

$$n \leq (c(S_1) - 1) + |S_1| + |A_1| + |S_2| + |A_2| + \cdots + |A_{q-1}| + |S_q| + (hc(c) - c(S_q))$$

$$= (c(S_1) - 1) + (hc(c) - c(S_q)) + \sum_{i=1}^{q-1} |S_i| + \sum_{i=1}^{q-1} |A_i|$$

$$= (c(S_1) - 1) + (hc(c) - c(S_q)) + 2q + \sum_{i=1}^{q-1} (c(S_{i+1}) - c(S_i) - 1 - a_i)$$

$$= hc(c) + q - a.$$  

Since $hc(c) \leq n - 2$, we get $a \leq q - 2$. This implies that $I \neq \emptyset$ and so $a_j = 0$ for some $j$ with $1 \leq j \leq q - 1$ and $j \in I$. Let $S_j = \{ x, x' \}$ and $S_{j+1} = \{ y, y' \}$. By Lemma 3.4(a), $xx', yy' \in E(C_n)$. Then $C_n - xx' - yy'$ consists of two nontrivial paths $P_1$ and $P_2$. Assume, without loss of generality, that $x$ and $y$ are the end-vertices of $P_1$ and thus $x'$ and $y'$ are the end-vertices of $P_2$. Since $j \in I$, there exists a vertex $x_1$ of $C_n$ such that $c(x_1) = c(S_j) + 1$. Since $|c(x_1) - c(x)| = 1 = |c(x_1) - c(x')|$, we have $D(x_1, x) \geq n - 2$ and $D(x_1, x') \geq n - 2$. It follows that $x_1$ is adjacent to either $x$ or $x'$, say $x$.

Now, let $n = 2k$ or $n = 2k + 1$ for some $k \geq 3$, according to whether $n$ is even or $n$ is odd. We claim that $c(S_{j+1}) - c(S_j) \geq k - 1$, for suppose that $c(S_{j+1}) - c(S_j) \leq k - 2$.  


We claim that $P$ namely edges, which is impossible. Thus $I$.

If $c(S_j) + 1 = c(S_{j+1})$, then $y = x_1$. If $c(S_j) + 1 < c(S_{j+1})$, then let $x_2$ be a vertex of $C_n$ such that $c(x_2) = c(S_j) + 2$. Then $D(x_2, x_1) \geq n - 2$; while $D(x_2, x) \geq n - 3$, and $D(x_2, x') \geq n - 3$. This implies that $x_2$ is adjacent to $x_1$. Continuing in this manner, we see that $P_1$ has length $c(S_{j+1}) - c(S_j)$ and its vertices are colored by $c$ as shown in Fig. 6.

It is clear that $P_2$ has length $n - 2 - (c(S_{j+1}) - c(S_j))$. Since $c(S_{j+1}) - c(S_j) \leq k - 2$, we get $D(x', y') = n - 2 - (c(S_{j+1}) - c(S_j))$. Thus $|c(x') - c(y')| + D(x', y') = n - 2$, which contradicts the fact that $c$ is a hamiltonian coloring of $C_n$. Thus we have $c(S_{j+1}) - c(S_j) \geq k - 1$, as claimed.

Let $y_1$ be a vertex such that $c(y_1) = c(S_{j+1}) - 1$. We see that $y_1$ is adjacent to either $y'$ or $y$. We claim that $y_1$ is adjacent to $y$, for suppose that $y_1$ is adjacent to $y'$. Then $y_1$ belongs to $P_2$. Recall that $a_j \neq 0$. Since $x_2$ belongs to $P_1$ and the paths $P_1$ and $P_2$ have no common vertex, there exist vertices $x^*$ and $y^*$ of $C_n$ such that $c(x_1) \leq c(x^*), c(x^*) + 1 = c(y^*) \leq c(y_1)$ and that $x^*$ or $y^*$ belongs to $P_1$ and $y^*$ or $P_2$. Hence $D(x^*, y^*) \leq n - 3$; a contradiction. Thus $y_1$ is adjacent to $y$.

We can therefore find vertices $x_2, \ldots, x_{k-2}$ of $C_n$ such that $c(x_2) = c(S_j) + i$ for $i = 2, \ldots, k - 2$ and

$$P_x : x, x_1, x_2, \ldots, x_{k-2}$$

is a subpath of $P_1$. Similarly, we can find vertices $y_2, \ldots, y_{k-2}$ of $C_n$ such that $c(y_2) = c(S_{j+1}) - i$ for $i = 2, \ldots, k - 2$ and that

$$P_y : y_{k-2}, \ldots, y_2, y_1, y$$

is a subpath of $P_1$. We claim that $P_x$ and $P_y$ are not vertex-disjoint, for suppose that they are. Then since $q \geq 3$, it follows that $n \geq 2k + 2$; a contradiction. Thus $P_x$ and $P_y$ have a common vertex. This implies that the path $P_1$ contains exactly one vertex colored $i$ for each $i$ with $c(S_j) \leq i \leq c(S_{j+1})$ and has no vertices of any other color. (See Fig. 6 for the coloring of the vertices of $P_1$.) Therefore, the length of $P_1$ is $c(S_{j+1}) - c(S_j)$ and the length of $P_2$ is $n - 2 - (c(S_{j+1}) - c(S_j))$.

Recall that $c(S_{j+1}) - c(S_j) \geq k - 1$. If, in addition, $\ell \in I$, where $\ell \neq j$, then, as above, $c(S_{\ell+1}) - c(S_\ell) \geq k - 1$ and there is a path $Q_1$ of length $c(S_{\ell+1}) - c(S_\ell)$ whose vertices are colored by $c(S_\ell), c(S_\ell) + 1, \ldots, c(S_{\ell+1})$ and where, necessarily, $Q_1$ is a proper subpath of $P_2$. Assume, without loss of generality, that $\ell > j$. Let $S_\ell = \{z, z'\}$ and $S_{\ell+1} = \{w, w'\}$. Then the sets $S_j, S_{j+1}, S_\ell, S_{\ell+1}$ are distinct except possibly $S_{j+1} = S_\ell$. Since $C_n = xx' - yy' - zz' - ww'$ consists of at least three nontrivial paths, the lengths of at least two of which, namely $P_1$ and $Q_1$, are at least $k - 1$, it follows that $C_n$ has at least $2(k - 1) + 4 = 2k + 2$ edges, which is impossible. Thus $I = \{j\}$. Recall that $a \leq q - 2$. Since $|I| = 1$, it follows that $a \geq q - 2$. Hence $a = q - 2$. Since $n \leq hc(C_n) + q - a$, we have $n \leq hc(c) + 2$. So $hc(c) \geq n - 2$, as desired. \(\square\)
4. On the hamiltonian chromatic number of graphs having a given order

In this section we shall assume that we are considering connected graphs of order \( n \) for some fixed integer \( n \geq 3 \). We have already mentioned that a graph \( G \) has hamiltonian chromatic number 1 if and only if \( G \) is hamiltonian-connected. We now show that it is possible for a graph \( G \) to have hamiltonian chromatic number 2. All the graphs (of orders 3–5) shown in Fig. 7 have hamiltonian chromatic number 2.

The graphs \( G_2 \) and \( G_4 \) (and \( G_3 \) and \( G_5 \)) are actually special cases of a more general class of graphs. For \( n \geq 4 \), let \( G_{2n-6} \) be the graph of order \( n \) obtained by joining two vertices \( u \) and \( v \) of \( K_{n-1} \) to a new vertex \( w \) and let \( G_{2n-5} = G_{2n-6} - uv \). Then \( \text{hc}(G_{2n-6}) = \text{hc}(G_{2n-5}) = 2 \) for all \( n \geq 4 \).

We also have other graphs of order \( n \) with hamiltonian chromatic number 2 if \( n \) is sufficiently large. The graphs \( H_1 \) and \( H_2 \) of Fig. 8 have hamiltonian chromatic number 2.

In general, for \( n = 3k \geq 6 \), let \( H_{k-1} \) be the graph obtained from \( K_{2k} \), where \( V(K_{2k}) = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\} \), by adding the \( k \) new vertices \( w_1, w_2, \ldots, w_k \) and joining \( w_i \) to \( u_i \) and \( v_i \) for \( 1 \leq i \leq k \). Then \( \text{hc}(H_{k-1}) = 2 \) for all \( k \geq 2 \). A hamiltonian coloring of \( H_{k-1} \) assigns 1 to \( u_i \) and \( w_i \) and 2 to \( v_i \) for all \( i (1 \leq i \leq k) \). This class of examples shows that there exist graphs \( G \) with \( \text{hc}(G) = 2 \) such that each of the two colors is used an arbitrarily large number of times in a minimum hamiltonian coloring of \( G \). Other graphs with hamiltonian chromatic number 2 are shown in Fig. 8.

Fig. 7. Graphs of order \( n \) (3 ≤ \( n \) ≤ 5) having hamiltonian chromatic number 2.

Fig. 8. Other graphs with hamiltonian chromatic number 2.
Theorem 4.3. Let $j$ and $n$ be integers with $1 \leq i \leq k$.

The constructions described above for producing classes of graphs with hamiltonian chromatic number 2 can be altered to produce graphs (indeed, hamiltonian graphs) with larger hamiltonian chromatic numbers. Let $k$ and $n$ be integers with $n \geq 2k \geq 4$ and let $F_k$ be the graph of order $n$ obtained by identifying an edge of $K_{n-k+1}$ and an edge of $K_{k+1}$. Denote the identified edge by $uv$. Since $n \geq 2k$, it follows that $n-k+1 \geq k+1$. Furthermore, $D(u, v) = n-k$. The coloring $c$ that assigns 1 to every vertex of $F_k$ except $v$ and assigns $k$ to $v$ is a hamiltonian coloring of $F_k$. Since

$$|c(u) - c(v)| + D(u, v) = (k - 1) + (n - k) = n - 1,$$

it follows that $c$ is, in fact, a minimum hamiltonian coloring of $F_k$ and so $hc(F_k) = k$. Of course, $hc(F_k - uv) = k$ as well. This gives us the following result.

**Proposition 4.1.** For every two integers $k$ and $n$, where $1 \leq k \leq \lfloor n/2 \rfloor$, there exists a hamiltonian graph $G$ of order $n$ with $hc(G) = k$.

Proposition 4.1 can be extended however. First, the following lemma will be useful.

**Lemma 4.2.** Let $G$ be a connected graph of order $n$ and $H$ an induced subgraph of order $k$ in $G$. If $D_H(u, v) \geq D_G(u, v) - (n - k)$ for every two distinct vertices $u$ and $v$ of $H$, then $hc(H) \leq hc(G)$.

**Proof.** Let $c$ be a minimum hamiltonian coloring of $G$ and $c'$ the restriction of $c$ to $H$. Let $u, v \in V(H)$. Since $D_H(u, v) \geq D_G(u, v) - (n - k)$, it follows that

$$|c'(u) - c'(v)| + D_H(u, v) \geq |c(u) - c(v)| + D_G(u, v) - (n - k) \geq (n - 1) - (n - k) = k - 1.$$

Thus $c'$ is a hamiltonian coloring of $H$ and so $hc(H) \leq hc(c') \leq hc(c) = hc(G)$. □

**Theorem 4.3.** Let $j$ and $n$ be integers with $2 \leq j \leq (n + 1)/2$ and $n \geq 6$. Then there is a hamiltonian graph of order $n$ with hamiltonian chromatic number $n - j$.

**Proof.** Let $G = G(n, j)$ be the graph consisting of the cycle $C_n : v_1, v_2, \ldots, v_n, v_1$ together with all edges joining vertices in $\{v_1, v_2, \ldots, v_{j-1}, v_n\}$. If $j = 2$, then $G = G(n, 2) = C_n$. Since $hc(C_n) = n - 2$, we can assume that $j \geq 3$. Define a coloring $c^*$ of $V(G)$ by

$$c^*(v_i) = \begin{cases} 
1 & \text{if } 1 \leq i \leq j, \\
 p + 1 & \text{if } i = j + p \text{ and } 1 \leq p \leq n - 2 - j, \\
 n - j & \text{if } i = n - 1, n.
\end{cases}$$

The graph $G$ together with the coloring $c^*$ is shown in Fig. 9. It is straightforward to show that $c^*$ is a hamiltonian coloring. Thus $hc(G) \leq hc(c^*) = n - j$.

Next we show that $hc(G) \geq n - j$. Let $H = \{v_{j-1}, v_j, v_{j+1}, \ldots, v_n\} = C_{n-j+2}$. Thus $hc(H) = n - j$ by Theorem 3.5. With $k = n - j + 2$, we see that $G$ and $H$ satisfy the conditions in Lemma 4.2. It then follows that $n - j = hc(H) \leq hc(G)$, completing the proof. □
Combining Proposition 4.1 and Theorem 4.3, we have the following.

**Corollary 4.4.** For every two integers \( k \) and \( n \) with \( 1 \leq k \leq n - 2 \), there is a hamiltonian graph of order \( n \) with hamiltonian chromatic number \( k \).

We now know that for every integer \( n \geq 3 \), there exists a graph \( G \) of order \( n \) with a certain specified hamiltonian chromatic number. But how large can the hamiltonian chromatic number of a graph of order \( n \) be? In order to answer this question, we establish an upper bound for the hamiltonian chromatic number of a graph in terms of its order. We begin with an observation. Let \( G \) be a connected graph containing an edge \( e \) such that \( G - e \) is connected. For every two distinct vertices \( u \) and \( v \) in \( G - e \), the length of a longest \( u-v \) path in \( G - e \) does not exceed the length of a longest \( u-v \) path in \( G \). Thus every hamiltonian coloring of \( G - e \) is a hamiltonian coloring of \( G \). This observation yields the following lemma.

**Lemma 4.5.** If \( e \) is an edge of a connected graph \( G \) such that \( G - e \) is connected, then \( \text{hc}(G) \leq \text{hc}(G - e) \).

Combining Theorem 3.5 and Lemma 4.5, we have the following.

**Proposition 4.6.** If \( G \) is a hamiltonian graph of order \( n \geq 3 \), then \( \text{hc}(G) \leq n - 2 \).

The length of a longest cycle in a connected graph is called the *circumference* of \( G \) and is denoted by \( \text{cir}(G) \).

**Proposition 4.7.** If \( G \) is a connected graph of order \( n \geq 4 \) with \( \text{cir}(G) = n - 1 \), then \( \text{hc}(G) \leq n - 1 \).

**Proof.** Since \( G \) is connected and \( \text{cir}(G) = n - 1 \), it follows that \( G \) contains a spanning subgraph \( H \) obtained by adding a pendant edge to a cycle of length \( n - 1 \). By Lemma 2.3, \( \text{hc}(H) = n - 1 \), and by Lemma 4.5, \( \text{hc}(G) \leq n - 1 \). \( \square \)
Indeed, by Corollary 4.4, every pair $k, n$ of integers with $1 \leq k \leq n - 2$ can be realized as the Hamiltonian chromatic number and the order of some Hamiltonian graph. Consequently, this result cannot be improved. Lemma 4.5 also provides us with the following result.

**Proposition 4.8.** If $T$ is a spanning tree of a connected graph $G$, then
\[ \text{hc}(G) \leq \text{hc}(T). \]

The following lemma will also be useful to us. The complement $\overline{G}$ of a graph $G$ is the graph with vertex set $V(G)$ such that two vertices are adjacent in $G$ if and only if they are not adjacent in $G$.

**Lemma 4.9.** If $T$ is a tree of order at least 4 that is not a star, then $\overline{T}$ contains a Hamiltonian path.

**Proof.** We proceed by induction on the order $n$ of $T$. For $n = 4$, the path $P_4$ of order 4 is the only tree of order 4 that is not a star. Since $\overline{P_4} = P_4$, the result holds for $n = 4$. Assume that for every tree of order $k - 1 \geq 4$ that is not a star, its complement contains a Hamiltonian path.

Now let $T$ be a tree of order $k$ that is not a star. Then $T$ contains an end-vertex $v$ such that $T - v$ is not a star. By the induction hypothesis, $\overline{T - v}$ contains a Hamiltonian path, say $v_1, v_2, \ldots, v_{k-1}$. Since $v$ is an end-vertex of $T$, it follows that $v$ is adjacent to at most one of $v_1$ and $v_{k-1}$. Without loss of generality, assume that $v_1$ and $v$ are not adjacent in $T$. Then $v$ and $v_1$ are adjacent in $T$ and so $v, v_1, v_2, \ldots, v_{k-1}$ is a Hamiltonian path in $T$. □

**Theorem 4.10.** If $T$ is a tree of order $n \geq 2$, then
\[ \text{hc}(T) \leq (n - 2)^2 + 1. \]

**Proof.** If $T$ is a star, then by Theorem 3.2, $\text{hc}(T) = (n - 2)^2 + 1$ and the result holds. So we may assume that $T$ is a tree of order $n \geq 4$ that is not a star. By Lemma 4.9, the complement $\overline{T}$ of $T$ contains a Hamiltonian path, say $v_1, v_2, \ldots, v_n$ is a Hamiltonian path in $\overline{T}$. This implies that for each $i$ with $1 \leq i \leq n$, the vertices $v_i$ and $v_{i+1}$ are nonadjacent in $T$. Thus $D(v_i, v_{i+1}) \geq 2$ for all $i$ with $1 \leq i \leq n - 1$. Define a labeling $c$ of $T$ by $c(v_i) = (n - 2) + (i - 2)(n - 3)$ for each $i$ with $1 \leq i \leq n$. Let $1 \leq i < j \leq n$. Then $|c(v_i) - c(v_j)| = (j - i)(n - 3)$. If $j = i + 1$, then $|c(v_i) - c(v_j)| + D(v_i, v_j) \geq (n - 3) + 2 = n - 1$. If $j \geq i + 2$, then $|c(v_i) - c(v_j)| + D(v_i, v_j) \geq 2(n - 3) + 1 = 2n - 5 \geq n - 1$ for $n \geq 4$. Thus $c$ is a Hamiltonian coloring of $T$. Therefore,
\[ \text{hc}(T) \leq \text{hc}(c) = c(v_n) = (n - 2)^2 < (n - 2)^2 + 1, \]
as desired. □

As a consequence of Proposition 4.8 and Theorem 4.10, we obtain a sharp upper bound for the Hamiltonian chromatic number of a nontrivial connected graph in terms of its order.

**Corollary 4.11.** If $G$ is a nontrivial connected graph of order $n$, then
\[ \text{hc}(G) \leq (n - 2)^2 + 1. \]
The preceding results suggest defining the following set and parameter for each integer $n \geq 2$,

$$
HC(n) = \{ k : \text{there exists a graph } G \text{ of order } n \text{ with } \text{hc}(G) = k \}.
$$

Therefore, $\min\{HC(n)\} = 1$ and $\max\{HC(n)\} = (n - 2)^2 + 1$. Also,

$$
\text{hc}(n) = \max\{k : p \in HC(n) \text{ for all } p, \ 1 \leq p \leq k\}.
$$

By Proposition 4.7, Theorem 3.2, Corollaries 4.4, and 4.11, it follows that

$$
n - 1 \leq \text{hc}(n) \leq (n - 2)^2 + 1.
$$

That $HC(4) = \{1, 2, 3, 4, 5\}$ and $HC(5) = \{1, 2, \ldots, 10\} - \{9\}$ is illustrated in Figs. 10 and 11. Consequently, $hc(4) = 5$ and $hc(5) = 8$.

Among the many unsolved problems is to determine those integers $n \geq 2$ for which $n \in HC(n)$.

Fig. 10. Graphs $I_i$ of order 4 with $hc(I_i) = i$ ($1 \leq i \leq 5$).

Fig. 11. Graphs $J_i$ of order 5 with $hc(J_i) = i$ ($1 \leq i \leq 10$, $i \neq 9$).
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References