Note on a Selection Theorem of Mas–Colell

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In [1] Mas–Colell proved a selection theorem for certain open-graph multifunctions with homeomorphically convex values. He asked if the theorem could be generalized to contractible valued multifunctions. The purpose of this note is to provide the desired generalization. The main ingredient is a lifting theorem of Wong [3].

Let $X$ and $Y$ be topological spaces. A set valued function $m: X \rightarrow Y$ will be called a multifunction (= correspondence in [1]) if $m(x) \neq \emptyset$, all $x \in X$. $G(m) = \{(x, y) \mid y \in m(x)\} \subseteq X \times Y$ is the graph of $m$. $m$ will be called an open-graph multifunction if $G(m)$ is an open subset of $X \times Y$. A single valued (continuous) function $f: X \rightarrow Y$ is a (continuous) selection for $m$ if $f(x) \in m(x)$, all $x \in X$. If $A \subseteq X$ and $g: A \rightarrow Y$ is a single valued function such that $g(x) \in m(x)$, all $x \in A$, then $g$ is called a partial selection for $m$.

**Theorem.** Let $(K, L)$ be a locally finite simplicial pair and $Y$ an AR(metric). Let $m: K \rightarrow Y$ be an open-graph multifunction with $m(x)$ contractible all $x \in K$. Then any continuous partial selection $g: L \rightarrow Y$ for $m$ can be extended to a continuous selection $f: K \rightarrow Y$ for $m$.

Mas–Colell’s result [1] assumed $m(x)$ heomeomorphically convex (so contractible), $Y = \mathbb{R}^n$, $K$ finite, and $L = \emptyset$. As in [1] we can deduce a fixed-point corollary.

**Corollary.** Let $X$ be a compact contractible polyhedron and $m: X \rightarrow X$ an open-graph multifunction with contractible values. Then $m$ has a fixed point (i.e., $x \in X$ with $x \in m(x)$).

Note. Let $Q$ be the Hilbert cube and define $m: Q \rightarrow Q$ by $m(x) = Q \setminus \{x\}$. Then $m$ is an open-graph multifunction with contractible values and no fixed point. So “compact contractible polyhedron” can not simply be replaced by “compact AR” in the corollary.

**Proof of the Corollary.** The theorem gives a continuous selection $f: X \rightarrow X$ and the generalized Brouwer fixed-point theorem [2, p. 196] gives the fixed point $x = f(x) \in m(x)$.
The theorem will be deduced from the following result of Wong [3]: Suppose \((K, L)\) a locally finite simplicial pair, \(p: E \to B\) a fiber bundle with \(E, B\) metric and fiber an AR(metric). Let \(A\) be a closed subset of \(E\), \(E_b = p^{-1}(b)\), \(A_b = A \cap E_b\). Suppose \(A_b\) is homotopy negligible in \(E_b\) all \(b \in B\). Let \(u: |K| \to B\), \(v: |L| \to E \setminus A\) be continuous functions satisfying \(pv = ui\), where \(i: |L| \subset |K|\). Then there is a continuous function \(F: |K| \to E \setminus A\) with \(pF = u\) and \(Fi = v\). (In general if \(P \subset Q\), \(P\) is said to be homotopy negligible in \(Q\) if \(j: (Q \setminus P) \subset P\) is a homotopy equivalence.)

**Proof of the Theorem.** Define \(p = \text{projection}: E = |K| \times Y \to |K| = B\) so that \(E \to B\) is a fiber bundle with AR fiber \(Y\). Let \(A = E \setminus G(m)\). Then \(A_b = A \cap (b \times Y) = (b \times Y) \setminus (G(m) \cap b \times Y) = (b \times Y) \setminus m(b)\).

\[ j: E_b \setminus A_b \to E_b \text{ is } m(b) \to b \times Y \]

and both are contractible \((m(b)\) by hypotheses, \(b \times Y\) because \(Y\) is AR\) so \(j\) is a homotopy equivalence and \(A_b\) is homotopy negligible in \(E_b\).

Let \(v = (i, g): |L| \to G(m) \subset |K| \times Y\) and \(u = \text{identity}: |K| \to |K|\). Wong’s result gives \(F: |K| \to G(m)\) with \(pF = u\) and \(Fi = v\).

Define \(f: |K| \to Y\) by \(f = \Pi_2F\). Then \(pF = u\) gives \(F(b) \in \pi^{-1}(b) = b \times m(b)\) so \(f(b) \in m(b)\). If \(b \in |L|\) then \(F(b) = v(b) = (b, g(b))\) so \(f(b) = g(b)\). Thus \(f\) is the desired extension of \(g\) to a selection for \(m\).

**References**