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On the global existence of solutions to a class of fractional differential equations

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ABSTRACT

We present two global existence results for an initial value problem associated to a large class of fractional differential equations. Our approach differs substantially from the techniques employed in the recent literature. By introducing an easily verifiable hypothesis, we allow for immediate applications of a general comparison type result from [V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. TMA 69 (2008), 2677–2682].

1. Introduction

Fractional calculus is a powerful tool which plays an important role in the modeling of multi-scale problems. Fractional calculus has been found appropriate to describe the dynamics of complex systems in several branches of science and engineering.

The generalization of differential calculus to non-integer orders of derivatives can be traced back to Leibniz [1–3]. However, the initial data (given also in a fractional order frame) in the initial value problems involving fractional differential operators are of a more delicate nature and their physical meaning is not yet fully understood [4, p. 230], [3, p. 80]. Therefore, the incorporation of classical derivatives (of integer order) of the initial data in the fractional differential operator was suggested by many authors [5–7]. In this spirit, consider the initial value problem

$$\begin{cases} D_0^a(x-x_0)(t) = f(t,x(t)), \quad t>0, \\ x(0)=x_0, \end{cases} \tag{1}$$

where the nonlinearity $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is assumed continuous. Here, $\mathbb{R}_+$ has its usual meaning of nonnegative semi-axis. Important recent investigations regarding these types of problems, its applications and various generalizations can be read in [8–13].

The differential operator D_0^a in (1) is the Riemann–Liouville differential operator of order $0 < a < 1$, namely

$$D_0^a x(t) = \frac{1}{\Gamma(1-a)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t \frac{x(s)}{(t-s)^a}\mathrm{d}s\right],$$

where $\Gamma(1-a) = \int_0^{+\infty} \mathrm{e}^{-t} t^{-a} \mathrm{d}t$ is the Gamma function.

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Assuming that the initial value problem has a solution $x(t)$, the formulas $\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin \pi a}$ and

$$\int_0^t f(s,x(s))\mathrm{d}s=\frac{\sin \pi a}{\pi}\int_0^t \frac{1}{(t-s)^a}\int_0^s \frac{f(\tau,x(\tau))}{(s-\tau)^{1-a}}\mathrm{d}\tau\mathrm{d}s,$$

see [14, p. 196], allow us to rewrite (1) via an integration as

$$\int_0^t \frac{1}{\Gamma(1-a)(t-s)^a}\left[x(s)-x_0-\frac{1}{\Gamma(a)}\int_0^s \frac{f(\tau,x(\tau))}{(s-\tau)^{1-a}}\mathrm{d}\tau\right]\mathrm{d}s=0, \tag{2}$$

where $t>0$.

In obvious accordance with formula (2), we are interested here in the *solutions of* (1) *in the sense of* [5–7], that is, the continuous functions $x(t)$ which satisfy the singular integral equation

$$x(t)=x_0+\frac{1}{\Gamma(a)}\int_0^t \frac{f(s,x(s))}{(t-s)^{1-a}}\mathrm{d}s, \quad t\ge 0. \tag{3}$$

The fractional calculus mathematical models appear in connection with the self-similar dynamics in complex systems. In the paper [15] a detailed discussion of this topic is made. Various applications, like in the reaction kinetics of proteins, the anomalous electron transport in amorphous materials, the dielectrical or mechanical relaxation of polymers, the modeling of glass-forming liquids and others, are successfully performed in numerous papers. See the presentations from [16,3].

Several recent advancements in the theory and applications of non-integer differentiation and integration are described in [17]. For instance, fractional Lagrangian and Hamiltonian treatments of the field and mechanical systems are proposed by Băleanu and Muslih in [17, p. 115 and following]. Other results concerning the promising new theory of fractional variational principles can be found in the contributions by [18–30].

In two very recent contributions [31,32], Lakshmikantham and Vatsala investigated the existence theory of (3) and its delay integral equation counterpart by means of integral inequalities and perturbation techniques. A Peano type local existence theorem has been established and also a comparison principle for global existence was presented.

Theorem 1 (*Comparison Principle, [31]*). *Assume that there exists the function $g:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ continuous and nondecreasing with respect to the second argument such that*

$$|f(t,x)|\le g(t,|x|) \quad \text{for all } t\ge 0,\ x\in\mathbb{R}.$$

If the maximal solution of the initial value problem

$$\begin{cases} D_0^a(u-u_0)(t)=g(t,u(t)), & t>0,\\ u(0)=u_0, \end{cases} \tag{4}$$

exists in $\mathbb{R}_+$ then all the solutions of (1) *with $|x_0|\le u_0$ exist in $\mathbb{R}_+$.*

The existence and uniqueness theory of solutions to initial value problems for fractional differential operators of various orders is discussed in [3,4]. As the authors [4, p. 232] explain, there should be a great interest in studying such a qualitative theory in the case $0<a<1$. To emphasize this, recall that the solution $x(t)$ to the initial value problem for a general fractional differential equation (with Miller–Ross sequential fractional derivatives)

$${}_0\mathcal{D}_t^{\sigma_n}u=f(t,u)$$

and

$$\left[{}_0\mathcal{D}_t^{\alpha_n-1}\,{}_0\mathcal{D}_t^{\alpha_{n-1}}\cdots\,{}_0\mathcal{D}_t^{\alpha_1}u(t)\right]_{t=0}=a_k, \quad k=\overline{1,n},$$

where $\sigma_k=\sum_{i=1}^k \alpha_i$, $k\le n$ and $\alpha_i\in(0,1]$, satisfies the singular integral equation

$$x(t)=\sum_{i=1}^n \frac{a_i t^{\sigma_i-1}}{\Gamma(\sigma_i)}+\frac{1}{\Gamma(\sigma_n)}\int_0^t (t-\tau)^{\sigma_n-1}f(s,x(s))\mathrm{d}s,$$

see [3, p. 127–128].

For a connection with some applications see, for instance, [16, p. 7180]. In [16], the fractional orders have the significance of introducing *memory* in the physical processes (memory of stress and strain). It is interesting to note that the Peano type result for fractional differential equations from [31, Theorem 3.1] has been proved by means of the memory-like procedure of Tonelli [33, p. 23].

The asymptotic behavior of solutions to fractional differential equations has been studied in [15] in the case of the linear fractional differential equation

$$\tau_0^{-a}D_0^a[\Phi(t)]+\Phi(t)-\Phi(0)=0.$$

The solutions were expressed using a Mittag–Leffler transcendental function of order a

$$\Phi(t) = \Phi(0)E_a\left(-\left(\frac{t}{\tau_0}\right)^a\right),$$

see [15, p. 49]. In this way, $\Phi(t) \sim t^{-a}$ as the time t increases indefinitely. Several improvements in integrating linear fractional differential equations of various orders are proposed by Bonilla, Rivero and Trujillo in [17, pp. 77 and following].

The special formula of the Riemann–Liouville differential operator produces a lot of complications when one tries to mimic the proof of the fundamental results from the existence, uniqueness and asymptotic integration theory of ordinary differential equations. They have been bypassed only recently using delicate machineries like the generalized Banach Fixed Point Theorem due to Weissinger [34], the Mittag–Leffler transcendental functions theory [35] or the exponentially weighted Chebyshev norms theory due to Bielecki [36].

Our aim here is to complete the conclusions of Theorem 1 by establishing two results of global existence for (3).

2. Main results

Theorem 2. *Assume that there exists the continuous function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t, x) - f(t, y)| \le F(t)|x - y|$$

for all $t \ge 0$, $x, y \in \mathbb{R}$. Then the integral equation (3) has a unique solution defined in $\mathbb{R}_+$.

This theorem is a fundamental in the case of ordinary differential equations ($a = 1$). The book by Kartsatos [37], as well as many other monographs devoted to the *qualitative theory* of ordinary differential equations, contains a detailed presentation of the typical proof. This consists of the introduction of a complete metric space endowed with an exponentially weighted metric (the Bielecki metric), followed by a verification of the claim that a certain integral operator acting on the metric space is a contraction. The metric space is given by families of continuous functions defined on a bounded interval of *arbitrary length*. This is the key feature which allows one to say that *a solution of (3) whose existence can be established on any compact subinterval of $\mathbb{R}_+$ exists, naturally, throughout $\mathbb{R}_+$.*

In his rapid demonstration of the Picard–Lindelöf theorem of existence and uniqueness of the solution to a Cauchy problem, Brezis [38] shows how *a prospective study of the growth* of a solution to (3) leads to establishing the existence of solutions throughout $\mathbb{R}_+$ *at once*, that is, without the unpleasant artifice of bringing into the study the arbitrarily long compact subintervals of $\mathbb{R}_+$. His proof, however, regarded a particular case of the nonlinearity $f(t, x)$. The trick needed in the general circumstances is more involved and it will be given in the following in the unifying context of the integral equation (3).

The second result deals with prescribed growth of solutions to the fractional differential equations.

Theorem 3. *Assume that the function g from Theorem 1 verifies throughout $\mathbb{R}_+$ the inequality*

$$g(t, u(t)) \le K\mathrm{e}^t - \varepsilon_0,$$

where $K \ge \varepsilon_0 > 0$ are fixed, for all the continuous functions $u : \mathbb{R}_+ \to \mathbb{R}_+$ subjected to

$$u(t) \le L\mathrm{e}^t, \quad t \ge 0,$$

for a certain L such that

$$L \ge |x_0| + \frac{K}{a\Gamma(a)}\left(1 + \int_0^{+\infty} \frac{s^a}{\mathrm{e}^s}\mathrm{d}s\right).$$

Then the maximal solution of (4) is defined throughout $\mathbb{R}_+$ and the integral equation (3) has a solution in $\mathbb{R}_+$ which behaves as $O(\mathrm{e}^t)$ when t increases indefinitely.

3. Proofs

Proof of Theorem 2. Introduce the continuous functions

$$h(t) = 1 + |x_0| + \frac{1}{\Gamma(a)}\int_0^t \frac{|f(s, x_0)|}{(t - s)^{1-a}}\mathrm{d}s$$

and

$$H_\lambda(t) = h(t)\exp\left(t + \frac{\lambda}{q}\int_0^t [h(s)F(s)]^q \mathrm{d}s\right), \quad t \ge 0,$$

for a fixed $\lambda > 0$. Here, q is taken such that $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \min\left\{\frac{1}{a}, \frac{1}{1-a}\right\}$.

Introduce the complete metric space $\mathcal{X} = (X, d_\lambda)$, where X is the set of all elements of $C(\mathbb{R}_+, \mathbb{R})$ that behave as $O(H_\lambda(t))$ when t goes to $+\infty$ and $d_\lambda(x, y) = \sup_{t\geq 0}\left\{\frac{|x(t)-y(t)|}{H_\lambda(t)}\right\}$ for any $x, y \in X$. Since $H_\lambda(t) \geq 1$, all the constant functions belong to X.

Given the operator $T : \mathcal{X} \to C(\mathbb{R}_+, \mathbb{R})$ with the formula

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(a)}\int_0^t \frac{f(s, x(s))}{(t-s)^{1-a}}\mathrm{d}s,$$

we have the following estimates

$$\begin{aligned}|(Tx)(t) - (Ty)(t)| &\leq \frac{1}{\Gamma(a)}\int_0^t \frac{F(s)}{(t-s)^{1-a}}|x(s) - y(s)|\mathrm{d}s\\ &= \frac{1}{\Gamma(a)}\int_0^t \frac{\mathrm{e}^s}{(t-s)^{1-a}}\cdot F(s)\frac{|x(s)-y(s)|}{\mathrm{e}^s}\mathrm{d}s\\ &\leq I(t)\cdot J(x, y)(t),\end{aligned}$$

where (notice that $1 - a - \frac{1}{p} = \frac{1}{q} - a$)

$$\begin{aligned}I(t) &= \frac{1}{\Gamma(a)}\left(\int_0^t (t-s)^{p(a-1)}\mathrm{e}^{ps}\mathrm{d}s\right)^{\frac{1}{p}} = \frac{1}{\Gamma(a)}\left(\int_0^t s^{p(a-1)}\mathrm{e}^{p(t-s)}\mathrm{d}s\right)^{\frac{1}{p}}\\ &= \frac{\mathrm{e}^t}{\Gamma(a)}\left(\int_0^{pt}\left(\frac{s}{p}\right)^{p(a-1)}\mathrm{e}^{-s}\cdot\frac{\mathrm{d}s}{p}\right)^{\frac{1}{p}} \leq \frac{\mathrm{e}^t}{\Gamma(a)}\cdot p^{\frac{1}{q}-a}\Gamma(1 - p(1-a))\\ &= c(a, p)\cdot \mathrm{e}^t\end{aligned}$$

and (recall that $h(t) \geq 1$)

$$\begin{aligned}J(x, y)(t) &= \left(\int_0^t [F(s)]^q\cdot\frac{|x(s)-y(s)|^q}{\mathrm{e}^{sq}}\mathrm{d}s\right)^{\frac{1}{q}}\\ &= \left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\exp\left(\lambda\int_0^s [h(\tau)F(\tau)]^q\mathrm{d}\tau\right)}{\lambda}\right)\left(\frac{|x(s)-y(s)|}{H_\lambda(s)}\right)^q\mathrm{d}s\right)^{\frac{1}{q}}\\ &\leq \left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\exp\left(\lambda\int_0^s [h(\tau)F(\tau)]^q\mathrm{d}\tau\right)}{\lambda}\right)\mathrm{d}s\right)^{\frac{1}{q}} d_\lambda(x, y)\\ &\leq \lambda^{-\frac{1}{q}}\cdot\frac{H_\lambda(t)}{\mathrm{e}^t}\cdot d_\lambda(x, y).\end{aligned}$$

By combining these estimates, we infer that

$$d_\lambda(Tx, Ty) \leq c(a, p)\lambda^{-\frac{1}{q}}\cdot d_\lambda(x, y)\quad \text{for all } x, y \in X.$$

The formula is valid only if we establish that $Tx \in X$ whenever $x \in X$. This follows from the next estimates

$$\begin{aligned}|(Tx)(t)| &\leq |(Tx)(t) - (Tx_0)(t)| + |(Tx_0)(t)|\\ &\leq c(a, p)\lambda^{-\frac{1}{q}}H_\lambda(t)\cdot d_\lambda(x, x_0) + h(t)\\ &\leq H_\lambda(t)\cdot(c(a, p)\lambda^{-\frac{1}{q}}d_\lambda(x, x_0) + 1)\\ &= O(H_\lambda(t))\quad \text{as } t \to +\infty.\end{aligned}$$

In conclusion, the operator $T : \mathcal{X} \to \mathcal{X}$ is a contraction for every $\lambda > [c(a, p)]^q$. Its unique fixed point is the solution of (3) with (global) existence in the future. □

Proof of Theorem 3. We start by showing that the integral equation

$$u(t) = |x_0| + \varepsilon + \frac{1}{\Gamma(a)}\int_0^t \frac{g(s, u(s)) + \varepsilon}{(t-s)^{1-a}}\mathrm{d}s,\quad \varepsilon \in (0, \varepsilon_0],$$

has a solution $u(t; \varepsilon)$ defined in $\mathbb{R}_+$.

Consider the set $U = \{u \in C(\mathbb{R}_+, \mathbb{R}) : 0 \leq u(t) \leq L\mathrm{e}^t \text{ for all } t \geq 0\}$. A partial order on U is given by the usual pointwise order "$\leq$", that is, we say that $u_1 \leq u_2$ if and only if $u_1(t) \leq u_2(t)$ for all $t \geq 0$, where $u_1, u_2 \in U$.

Introduce the operator $T : U \to C(\mathbb{R}_+, \mathbb{R})$ by the formula

$$(Tu)(t) = |x_0| + \varepsilon + \frac{1}{\Gamma(a)} \int_0^t \frac{g(s, u(s)) + \varepsilon}{(t-s)^{1-a}} \mathrm{d}s.$$

We have the next estimates

$$0 \le (Tu)(t) \le |x_0| + \frac{K}{\Gamma(a)} \int_0^t (t-s)^{a-1} \mathrm{e}^s \mathrm{d}s$$

and

$$\begin{aligned}\int_0^t (t-s)^{a-1} \mathrm{e}^s \mathrm{d}s &= \int_0^t s^{a-1} \mathrm{e}^{t-s} \mathrm{d}s = \frac{t^a}{a} + \frac{\mathrm{e}^t}{a} \int_0^t \frac{s^a}{\mathrm{e}^s} \mathrm{d}s \\ &\le (L - |x_0|) \frac{\Gamma(a)}{K} \cdot \mathrm{e}^t\end{aligned}$$

which lead to $Tu \in U$ whenever $u \in U$.

Since g is nondecreasing with respect to the second argument, the application T is isotone, that is, $Tu_1 \le Tu_2$ whenever $u_1 \le u_2$, and it satisfies $0 \le T(0)$. By application of the Knaster–Tarski fixed point theorem [39, p. 14], T has a fixed point in U, denoted $u(\cdot; \varepsilon)$.

Two properties of the family $(u(\cdot; \varepsilon))_{\varepsilon \in (0, \varepsilon_0]}$ must be established now. The first is that $u(\cdot; \varepsilon_1) < u(\cdot; \varepsilon_2)$ whenever $\varepsilon_1 < \varepsilon_2$. The second property claims that the family is relatively compact in $C(I, \mathbb{R})$ for any compact interval $I \subset \mathbb{R}_+$.

To prove the first property, introduce $\varepsilon_3 = \frac{\varepsilon_1 + \varepsilon_2}{2}$ and the integral operator $V_{\varepsilon_3} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ given by the formula

$$(V_{\varepsilon_3} u)(t) = |x_0| + \varepsilon_3 + \frac{1}{\Gamma(a)} \int_0^t \frac{g(s, u(s)) + \varepsilon_3}{(t-s)^{1-a}} \mathrm{d}s.$$

We have the next estimates

$$u(0; \varepsilon_1) = (Tu(\cdot; \varepsilon_1))(0) = |x_0| + \varepsilon_1 < u(0; \varepsilon_2)$$

and

$$u(t; \varepsilon_1) = (Tu(\cdot; \varepsilon_1))(t) < (V_{\varepsilon_3} u(\cdot; \varepsilon_1))(t), \quad t \ge t_0,$$

and

$$u(t; \varepsilon_2) > (V_{\varepsilon_3} u(\cdot; \varepsilon_2))(t), \quad t \ge t_0.$$

According to [31, Theorem 2.1], we conclude that $u(t; \varepsilon_1) < u(t; \varepsilon_2)$ throughout $\mathbb{R}_+$.

To prove the second property, set $I = [0, t_0]$ for a certain $t_0 > 0$. Fix also $t_1 \le t_2$ from I. We have the following estimates

$$\begin{aligned}|u(t_2; \varepsilon) - u(t_1; \varepsilon)| &\le \frac{1}{\Gamma(a)} \int_0^{t_1} \left[(t_1 - s)^{a-1} - (t_2 - s)^{a-1}\right] g(s, u(s; \varepsilon)) \mathrm{d}s \\ &\quad + \frac{\varepsilon}{\Gamma(a)} \int_0^{t_1} \left[(t_1 - s)^{a-1} - (t_2 - s)^{a-1}\right] \mathrm{d}s + \frac{1}{\Gamma(a)} \int_{t_1}^{t_2} (t_2 - s)^{a-1} [g(s, u(s; \varepsilon)) + \varepsilon] \mathrm{d}s \\ &\le \frac{1}{\Gamma(a)} \int_0^{t_1} \left[(t_1 - s)^{a-1} - (t_2 - s)^{a-1}\right] (K \mathrm{e}^s) \mathrm{d}s + \frac{1}{\Gamma(a)} \int_{t_1}^{t_2} (t_2 - s)^{a-1} (K \mathrm{e}^s) \mathrm{d}s \\ &\le \frac{K \mathrm{e}^{t_0}}{\Gamma(a+1)} [2(t_2 - t_1)^a + t_1^a - t_2^a] \\ &\le \frac{2K \mathrm{e}^{t_0}}{\Gamma(a+1)} (t_2 - t_1)^a.\end{aligned}$$

This establishes the equicontinuity of the family in $C(I, \mathbb{R})$. As a subset of U, the family is locally uniformly bounded which means that it satisfies all the requirements of the Ascoli–Arzelà theorem [37]. We thus conclude the proof of the second property.

Fix now the increasing sequence $(t_n)_{n \ge 0}$ from $\mathbb{R}_+$ such that $\lim_{n \to +\infty} t_n = +\infty$. The family $(u^n(\cdot; \varepsilon))_{\varepsilon \in (0, \varepsilon_0]}$, where

$$u^n(t; \varepsilon) = u(t; \varepsilon), \quad t \in I_n = [0, t_n],$$

is relatively compact in $C(I_n, \mathbb{R})$. This means that there exists a sequence of functions $u^n(\cdot; \varepsilon_m^n)_{m \ge 1}$ which converges uniformly to the function $u_n \in C(I_n, \mathbb{R})$ that verifies the integral equation

$$u_n(t) = |x_0| + \frac{1}{\Gamma(a)} \int_0^t \frac{g(s, u_n(s))}{(t-s)^{1-a}} \mathrm{d}s, \quad t \in I_n.$$

The first property of the family $(u(\cdot;\varepsilon))_{\varepsilon\in(0,\varepsilon_0]}$ helps improving this conclusion:

$$\lim_{\varepsilon\searrow 0} u(\cdot;\varepsilon) = u_n \quad \text{uniformly in } I_n. \tag{5}$$

Formula (5) implies that the function u_{n+1} is the extension to I_{n+1} of the function u_n. The function $u \in C(\mathbb{R}_+,\mathbb{R})$ given by the formula

$$u(t) = u_n(t), \quad t \in I_n,\ n \geq 1, \tag{6}$$

is thus a solution in $\mathbb{R}_+$ of problem (4) for $u_0 = |x_0|$.

We claim that *u from (6) is actually the maximal solution of (4)*. To see this, consider another solution, denoted v. Then, on any I_n, we have the estimates

$$v(0) = |x_0| < u(0;\varepsilon_m^n)$$

and

$$v(t) < (V_{\varepsilon_m^n} v)(t), \quad t \in I_n.$$

Since

$$u(t;\varepsilon_m^n) = (V_{\varepsilon_m^n} u(\cdot;\varepsilon_m^n))(t), \quad t \in I_n,$$

we conclude, via [31, Theorem 2.1], that

$$v(t) < u(t;\varepsilon_m^n), \quad t \in I_n,$$

and further, taking into account (5),

$$v(t) \leq u(t), \quad t \in I_n.$$

The claim is established.

The conclusion of Theorem 3 now follows from Theorem 1. □

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