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A finite dimensional extension of Lyusternik theorem with applications to multiobjective optimization

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Abstract

We consider a multiobjective program with inequality and equality constraints and a set constraint. The equality constraints are Fréchet differentiable and the objective function and the inequality constraints are locally Lipschitz. Within this context, a Lyusternik type theorem is extended, establishing afterwards both Fritz–John and Kuhn–Tucker necessary conditions for Pareto optimality. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

In this work we consider the next multiobjective program:

$$\text{(P) Min } f(x) \quad \text{subject to } g(x) \leqslant 0,\ h(x) = 0,\ x \in Q,$$

where f, g, h are functions from $\mathbb{R}^n$ to $\mathbb{R}^p$, $\mathbb{R}^m$ and $\mathbb{R}^r$, respectively, and Q is

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a nonempty subset of $\mathbb{R}^n$. We suppose that f and g are locally Lipschitz, h is Fréchet differentiable and Min is meant in Pareto's sense.

During the last two decades, a lot of works have been dedicated to the study of several constraint qualifications that allow establishing necessary conditions of Kuhn–Tucker type. For the case of differentiable programs, see for instance, the book of Bazaraa and Shetty [1] or the papers of Singh [2], Di [3] or Jiménez and Novo [4].

At the same time, since the introduction by F.H. Clarke, in 1972, of the concept of generalized gradient for locally Lipschitz functions and the subsequent development of this theory (see Clarke [5]), constraint qualifications for programs with Lipschitz conditions has been an important subject of study, as is shown in the works of Clarke [6], Hiriart-Urruty [7,8], Minami [9], Ishizuka and Shimizu [10], Craven [11–13], Jourani [14], Wang, Dong and Liu [15] or Mititelu [16,17].

However, in most of these papers, equality constraints are not considered (Ishizuka and Shimizu, Wang, Dong and Liu) and, in those in which they are considered, the constraint qualifications are mostly restrictive (Clarke, Mititelu).

In this paper we consider equality constraints defined by Fréchet differentiable functions (not necessary C^1 nor locally Lipschitz), together with inequality constraints and a set constraint. The work is structured as follows: Section 2 is devoted to notations, definitions and some of the previous results we are going to use. In Section 3 we obtain a result that can be considered as an extension of Lyusternik theorem within this context (see, for instance, Jahn [18]). This classical theorem establishes, under suitable conditions, different equality or content relationships between the contingent cone (or Bouligand cone) to a set defined by equality constraints and the linearized cone to the feasible set, a result that is basic for obtaining optimality conditions. In Section 4 we obtain, from this extension, both Fritz–John and Kuhn–Tucker optimality conditions for program (P) when Q is a convex set and, in Section 5, when Q is an arbitrary set. Finally, in Section 6 we make some final remarks that allow us to extend the class of functions to which the obtained results are applicable and also to use other generalized derivatives such as the Michel–Penot or the deconvolution of the upper Hadamard derivative and its corresponding (small convex-valued) subdifferentials instead of the Clarke's derivative and subdifferential.

2. Notations and preliminaries

Let S be a subset of $\mathbb{R}^n$. As usual, $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{ri} S$, $\operatorname{cone} S$ and $\operatorname{lin} S$, denote closure, interior, relative interior, generated cone and subspace generated by S, respectively. Let x and y be two points of $\mathbb{R}^n$, then we will write $x \leqslant y$ if $x_i \leqslant y_i$, $i = 1, \ldots, n$ and $x < y$ if $x_i < y_i$, $i = 1, \ldots, n$.

We are going to use the following tangent cones to S at $x_0 \in \operatorname{cl} S$:

(a) The contingent cone is

$$T(S, x_0) = \{v \in \mathbb{R}^n \colon \exists t_k > 0,\ \exists x_k \in S,\ x_k \to x_0 \text{ such that } t_k(x_k - x_0) \to v\}.$$

(b) The cone of attainable directions is

$$A(S, x_0) = \left\{v \in \mathbb{R}^n \colon \exists \delta > 0,\ \exists \gamma : [0, \delta] \to \mathbb{R}^n, \text{ such that } \gamma(0) = x_0,\ \gamma(t) \in S\ \forall t \in (0, \delta],\ \gamma'(0) = \lim_{t \to 0^+} \frac{\gamma(t) - \gamma(0)}{t} = v\right\}.$$

(c) The cone of linear directions is

$$Z(S, x_0) = \{v \in \mathbb{R}^n \colon \exists \delta > 0 \text{ such that } x_0 + tv \in S\ \forall t \in (0, \delta]\}.$$

The polar cone to $D \subset \mathbb{R}^n$ is $D^* = \{v \in \mathbb{R}^n \colon \langle v, d \rangle \leqslant 0\ \forall d \in D\}$, and the normal cone to S at x_0 is the polar of the contingent cone: $N(S, x_0) = T(S, x_0)^*$. When S is a convex set we have $T(S, x_0) = \operatorname{cl} \operatorname{cone}(S - x_0)$ and $N(S, x_0) = \{v \in \mathbb{R}^n \colon \langle v, x - x_0 \rangle \leqslant 0\ \forall x \in S\}$.

Let $f : \mathbb{R}^n \to \mathbb{R}$, $x_0, v \in \mathbb{R}^n$. We consider the next generalized directional derivatives of f at x_0 in the direction v:

$$\overline{D} f(x_0, v) = \limsup_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t},$$

$$\bar{d} f(x_0, v) = \limsup_{(t,u) \to (0^+, v)} \frac{f(x_0 + tu) - f(x_0)}{t},$$

$$d^0 f(x_0, v) = \limsup_{(x,t) \to (x_0, 0^+)} \frac{f(x + tv) - f(x)}{t}.$$

The first one is the upper Dini derivative, the second one is the upper Hadamard derivative and the third one is the Clarke derivative.

If f is locally Lipschitz, the Clarke subdifferential of f at x_0 is the set

$$\partial_{Cl} f(x_0) = \{\xi \in \mathbb{R}^n \colon \langle \xi, v \rangle \leqslant d^0 f(x_0, v)\ \forall v \in \mathbb{R}^n\}.$$

If f is convex, then we denote by $\partial f(x_0)$ the subdifferential of Convex Analysis and, if f is Fréchet differentiable, $\nabla f(x_0)$ denotes the differential of f at x_0.

Given the program

$$\operatorname{Min}\{f(x) \colon x \in M\},$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$ and $M \subset \mathbb{R}^n$, a point $x_0 \in M$ is said to be a weak Pareto minimum, denoted by $x_0 \in \operatorname{WMin}(f, M)$, if there exists no $x \in M$ such that

$f(x) < f(x_0)$. The usual concept of weak local Pareto minimum, for which the previous condition is required on a neighborhood of the point is also used. It is denoted by $x_0 \in \mathrm{WLMin}(f, M)$.

In order to simplify the notations, the following sets are defined for the initial program (P):

$$G = \{x \in \mathbb{R}^n\colon\ g(x) \leqslant 0\}, \qquad H = \{x \in \mathbb{R}^n\colon\ h(x) = 0\},$$
$$S = \{x \in \mathbb{R}^n\colon\ g(x) \leqslant 0,\ h(x) = 0\},$$

so that $S = G \cap H$ and the feasible set of (P) is $M = S \cap Q$.

Let f_i, $i \in I = \{1, \ldots, p\}$, g_j, $j \in J = \{1, \ldots, m\}$, h_k, $k \in K = \{1, \ldots, r\}$ the components functions of f, g and h, respectively, and, given $x_0 \in S$, let $J_0 = \{j \in J\colon\ g_j(x_0) = 0\}$ the set of active indexes at x_0.

We will suppose that f and g are locally Lipschitz and h is continuous on a neighborhood of x_0 and Fréchet differentiable at x_0. The cones that will be used to approximate S at x_0 are the natural extension to this context of the linearized cones:

$$C_0(S, x_0) = \{v \in \mathbb{R}^n\colon\ d^0 g_j(x_0, v) < 0,\ \forall j \in J_0;\ \nabla h_k(x_0)v = 0,\ \forall k \in K\},$$
$$C(S, x_0) = \{v \in \mathbb{R}^n\colon\ d^0 g_j(x_0, v) \leqslant 0,\ \forall j \in J_0;\ \nabla h_k(x_0)v = 0,\ \forall k \in K\}.$$

We denote $F = \{x \in \mathbb{R}^n\colon\ f(x) \leqslant f(x_0)\}$ and $K(H) = \mathrm{Ker}\,\nabla h(x_0)$. The sets G and F are defined by inequality constraints, and the cones $C_0(G)$, $C(G)$, $C_0(F)$ and $C(F)$ can be defined analogously, obtaining that $C_0(S) = C_0(G) \cap K(H)$ and $C(S) = C(G) \cap K(H)$. (We omit the point x_0 in the notation for shortness reasons.)

Let $Q \subset \mathbb{R}^n$ be a convex set. Let us denote $\mathrm{cone}_+ Q = \{v \in \mathbb{R}^n\colon\ \exists\lambda > 0, \exists x \in Q,\ v = \lambda x\}$. Obviously, if $0 \in Q$ then $\mathrm{cone}_+ Q = \mathrm{cone}\, Q$ and if $0 \notin Q$ then $\mathrm{cone}_+ Q = \mathrm{cone}\, Q \backslash \{0\}$. This cone allows to express the relative interior of the cone generated by a convex set as follows (see Rockafellar [19, Corollary 6.8.1]).

Lemma 2.1. *If Q is a nonempty convex set, then* $\mathrm{ri}\,\mathrm{cone}\, Q = \mathrm{cone}_+\, \mathrm{ri}\, Q$.

3. An extension of a Lyusternik type theorem

In this section we obtain our main result (Theorem 3.2) that can be considered as an extension of the classical Lyusternik theorem within this context. To prove it, two previous Lemmas 3.1 and 3.3 are required.

Lemma 3.1. *Let $Q \subset \mathbb{R}^n$ be a convex set, $x_0 \in S \cap Q$, $h : \mathbb{R}^n \to \mathbb{R}^r$ Fréchet differentiable at x_0 and $g : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz near x_0. If*

$$C_0(S) \cap \mathrm{ri}\,\mathrm{cone}(Q - x_0) \neq \emptyset \tag{1}$$

then

$$\mathrm{cl}\big[C_0(S)\cap \mathrm{ri}\,\mathrm{cone}(Q-x_0)\big]=C(S)\cap T(Q,x_0).$$

Proof. To prove this result, we just need to note that $C_0(S)$ is a convex cone, a relative open of $K(H)$, whose closure is $C(S)$ and apply Theorem 6.5 in Rockafellar [19] taking into account that $\mathrm{cl}\,\mathrm{cone}(Q-x_0)=T(Q,x_0)$, because of the convexity of Q. □

Note that if we apply Lemma 2.1, since $C_0(S)$ is a cone, condition (1) is equivalent to

$$C_0(S)\cap \mathrm{ri}(Q-x_0)\neq\emptyset.$$

Theorem 3.2. *Let us suppose the following*:

(a) $h:\mathbb{R}^n\to\mathbb{R}^r$ *is continuous on a neighborhood of* x_0 *and Fréchet differentiable at* x_0.
(b) $Q\subset\mathbb{R}^n$ *is a convex set and* $x_0\in H\cap Q$.
(c) *The regularity condition*

$$\text{(RC)}\ 0\in\sum_{k=1}^{r}v_k\nabla h_k(x_0)+N(Q,x_0)\ \Rightarrow\ v=0$$

holds.

Then

$$\begin{aligned}\mathrm{cl}\big[K(H)\cap \mathrm{cone}(Q-x_0)\big]&=A(H\cap Q,x_0)=T(H\cap Q,x_0)\\&=K(H)\cap T(Q,x_0).\end{aligned}$$

Lemma 3.3. *If* (RC) *holds, then* $K(H)\cap\mathrm{ri}(Q-x_0)\neq\emptyset$.

Proof. Let us suppose that $K(H)\cap\mathrm{ri}(Q-x_0)=\emptyset$. By the separation theorem, there exist $u\in\mathbb{R}^n\backslash\{0\}$ and $\alpha\in\mathbb{R}$ such that

$$\langle u,x-x_0\rangle\leqslant\alpha\leqslant\langle u,y\rangle,\quad \forall x\in Q,\ \forall y\in K(H). \tag{2}$$

As $x=x_0\in Q$ and $y=0\in K(H)$, we have that $\alpha=0$. Therefore $u\in N(Q,x_0)$ and,

$$-u\in K(H)^*=\mathrm{lin}\big\{\nabla h_k(x_0):\ k\in K\big\},$$

consequently $-u=\sum_{k=1}^{r}v_k\nabla h_k(x_0)$. From the hypothesis it follows that $v=0$, thus $u=0$, that is a contradiction. □

Proof of Theorem 3.2. Firstly, let us see that

$$K(H) \cap \operatorname{ri} \operatorname{cone}(Q - x_0) \subset A(H \cap Q, x_0). \tag{3}$$

For this, it is enough to prove that

$$K(H) \cap \operatorname{ri}(Q - x_0) \subset A(H \cap Q, x_0), \tag{4}$$

because if $w \in K(H) \cap \operatorname{ri} \operatorname{cone}(Q - x_0)$, using Lemma 2.1, we obtain that $w = \lambda v$ with $\lambda > 0$ and $v \in \operatorname{ri}(Q - x_0)$. If (4) is true, $v \in A(H \cap Q, x_0)$ and, consequently, $w \in A(H \cap Q, x_0)$.

Let us see that (4) is verified. In fact, let $v \in K(H) \cap \operatorname{ri}(Q - x_0)$ and let V be the smallest affine variety containing Q.

If $\dim V = n - l$, then V is intersection of l hyperplanes M_j, $j = 1, \ldots, l$, $V = \bigcap_{j=1}^{l} M_j$, defined by $M_j = \{x \in \mathbb{R}^n \colon \langle c_j, x - x_0 \rangle = 0\}$, being $c_1, \ldots, c_l$ linearly independent. Let us see that $\nabla h_1(x_0), \ldots, \nabla h_r(x_0), c_1, \ldots, c_l$ are linearly independent. Suppose that

$$\sum_{k=1}^{r} v_k \nabla h_k(x_0) + \sum_{j=1}^{l} \lambda_j c_j = 0.$$

Since $Q \subset V$ it follows that $N(V, x_0) \subset N(Q, x_0)$, but $N(V, x_0) = \operatorname{lin}\{c_1, \ldots, c_l\}$, hence

$$\sum_{j=1}^{l} \lambda_j c_j \in N(Q, x_0)$$

and, by (RC), we deduce that $v = 0$. Because of the linear independence of $c_1, \ldots, c_l$, we have that $\lambda = 0$.

Let $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^{r+l}$ defined by $\tilde{h}(x) = (h_1(x), \ldots, h_r(x), \langle c_1, x - x_0 \rangle, \ldots, \langle c_l, x - x_0 \rangle)$ and let us consider the system

$$\tilde{h}(x) = \tilde{h}(x_0) + t \nabla \tilde{h}(x_0) v. \tag{5}$$

Since $\tilde{h}$ is continuous on a neighborhood of x_0 and Fréchet differentiable at x_0 with maximal rank Jacobian, from Theorem 5.3, Chapter 3, in Hestenes [20], the system (5) has a solution

$$x = \gamma(t), \quad -\delta \leqslant t \leqslant \delta \text{ such that } \gamma(0) = x_0 \text{ and } \gamma'(0) = v. \tag{6}$$

Taking into account the first r components of (5), $\tilde{h}(x_0) = 0$ and $\nabla \tilde{h}(x_0) v = 0$ (since $v \in K(H)$, $v \in Q - x_0 \subset T(Q, x_0)$ and $\pm c_j \in N(Q, x_0) = T(Q, x_0)^*$), it follows $h(\gamma(t)) = 0$, $\forall t \in [-\delta, \delta]$. Considering the last l components,

$$\big\langle c_j, \gamma(t) - x_0 \big\rangle = 0, \quad \forall t \in [-\delta, \delta],\ j = 1, \ldots, l.$$

Therefore, $\gamma(t) \in V$. Let us see that $\gamma(t) \in Q$. Let $\alpha(t) = (\gamma(t) - x_0 - tv)/t$, hence $\gamma(t) = x_0 + t(v + \alpha(t))$, and by (6), $\lim_{t \to 0^+} \alpha(t) = 0$. Since $v \in \operatorname{ri}(Q - x_0)$, we have $v = q_0 - x_0$ with $q_0 \in \operatorname{ri} Q \subset V$ and, since $\gamma(t) \in V$, it follows that

$$\alpha(t) = t^{-1}\big(\gamma(t) - x_0\big) - (q_0 - x_0) \in V - x_0$$

because $V - x_0$ is a linear subspace. Since $q_0 \in \operatorname{ri} Q$, there exists $\varepsilon > 0$ such that $B(q_0, \varepsilon) \cap V \subset Q$. Since $\lim_{t \to 0^+}(x_0 + v + \alpha(t)) = q_0$ and $x_0 + v + \alpha(t) = q_0 + \alpha(t) \in V$, for t small enough, $x_0 + v + \alpha(t) \in B(q_0, \varepsilon) \cap V \subset Q$. Thus for $t > 0$ small enough, by convexity, $\gamma(t) = (1-t)x_0 + t(x_0 + v + \alpha(t)) \in Q$. Then, $\gamma(t) \in H \cap Q$, for all $t > 0$ small enough and, consequently, $v \in A(H \cap Q, x_0)$.

If $\dim V = n$, that is, if $\operatorname{int} Q \neq \emptyset$, then $V = \mathbb{R}^n$ and we do not need any hyperplane, because in this case it is enough to define $\tilde{h} = h$ and the solution $x = \gamma(t)$ verifies $\gamma(t) \in V$. The above deduction of $\gamma(t) \in Q$ is valid, if $\operatorname{ri} Q$ is replaced by $\operatorname{int} Q$.

Secondly, we prove the conclusion of theorem. From (RC), by Lemma 3.3 it follows that $K(H) \cap \operatorname{ri}(Q - x_0) \neq \emptyset$, and since $K(H)$ is a cone, by Lemma 2.1 the above condition is equivalent to $K(H) \cap \operatorname{ri} \operatorname{cone}(Q - x_0) \neq \emptyset$, obtaining, by Lemma 3.1, that

$$\begin{aligned} \operatorname{cl}\bigl[K(H) \cap \operatorname{cone}(Q - x_0)\bigr] &= \operatorname{cl}\bigl[K(H) \cap \operatorname{ri} \operatorname{cone}(Q - x_0)\bigr] \\ &= K(H) \cap T(Q, x_0). \end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned} A(H \cap Q, x_0) &\subset T(H \cap Q, x_0) \subset T(H, x_0) \cap T(Q, x_0) \\ &\subset K(H) \cap T(Q, x_0). \end{aligned} \tag{8}$$

Finally, taking closure in (3) and taking into account (7), (8) and that the cone of attainable directions is closed, we have the conclusion. □

Di [3] supposes that Q is a closed convex set and only obtains the expression $T(H \cap Q, x_0) = K(H) \cap T(Q, x_0)$ for the contingent cone. Our result is stronger.

Remark 3.4.

(1) Note that if $Q = \mathbb{R}^n$ and h is of class C^1 on a neighborhood of x_0 with maximal rank Jacobian, this theorem becomes the Lyusternik theorem (see, for instance, Jahn [18, Theorems 4.21 and 4.22]), it expresses:

$$K(H) = T(H, x_0).$$

(2) If (RC) is not verified, the conclusion of theorem may be false as the next simple example shows: in $\mathbb{R}^2$, taking $h(x, y) = y - x^2$ and $Q = \{(x, y): y = 0\}$.

(3) According to the proof of this theorem, (RC) implies that the gradients $\nabla h_1(x_0), \ldots, \nabla h_r(x_0)$ are linearly independent.

(4) In the particular case that the convex Q has nonempty interior, we get the next sufficient condition for (RC).

If $K(H) \cap \operatorname{int}(Q - x_0) \neq \emptyset$ and $\nabla h_1(x_0), \ldots, \nabla h_r(x_0)$ are linearly independent, then (RC) holds.

In fact, let

$$0 = \sum_{k=1}^{r} v_k \nabla h_k(x_0) + d,$$

with $d \in N(Q, x_0)$ and let us take

$$u \in K(H) \cap \operatorname{int}(Q - x_0).$$

Multiplying the above equality by u, it follows that $\langle d, u\rangle = 0$. Now, for $\lambda > 0$ small enough we have that $u \pm \lambda d \in Q - x_0$ and consequently, $\langle d, u \pm \lambda d\rangle \leqslant 0$, hence $\pm\lambda\langle d, d\rangle \leqslant 0$, following $\langle d, d\rangle = 0$, it means $d = 0$. Therefore,

$$0 = \sum_{k=1}^{r} v_k \nabla h_k(x_0),$$

and because of the linear independence of gradients, we have $v = 0$.

(5) The regularity condition (RC) has been used by many authors, see for example Rockafellar [21, p. 198].

4. Necessary optimality conditions with a convex set constraint

Necessary optimality conditions, both Fritz–John and Kuhn–Tucker type, for the problem (P) are obtained in this section. First of all, two theorems analyzing different relationship among the used conical approximation are given. We will suppose throughout the section that Q is a convex set.

Theorem 4.1. *Let $x_0 \in S \cap Q$ and assume the following*:

(a) $h : \mathbb{R}^n \to \mathbb{R}^r$ *is continuous on a neighborhood of x_0 and Fréchet differentiable at x_0.*
(b) *The regularity condition* (RC) *holds.*
(c) $g : \mathbb{R}^n \to \mathbb{R}^m$ *is Lipschitz near x_0.*

Then

$$C_0(S) \cap \operatorname{cone}(Q - x_0) \subset A(S \cap Q, x_0).$$

Proof. Let $v \in C_0(S) \cap \operatorname{cone}(Q - x_0) = C_0(G) \cap K(H) \cap \operatorname{cone}(Q - x_0)$. Then $v \in K(H) \cap \operatorname{cone}(Q - x_0)$ and by Theorem 3.2, $v \in A(H \cap Q, x_0)$. Hence, there exist $\delta > 0$ and a function $\gamma : [0, \delta] \to \mathbb{R}^n$ such that $\gamma(0) = x_0$, $\gamma'(0) = v$ and $\gamma(t) \in H \cap Q$, $\forall t \in [0, \delta]$. Let us see that $\gamma(t) \in G$ for t small enough, following that $\gamma(t) \in S \cap Q$ and, consequently, $v \in A(S \cap Q, x_0)$.

Because g_j is Lipschitz near x_0, we obtain that $\bar{d} g_j(x_0, v) = \overline{D} g_j(x_0, v)$ (Glover and Jeyakumar [22, Proposition 2.1]) and because upper Dini derivative is less than Clarke derivative or equal to it, we have that $\bar{d}g_j(x_0, v) \leqslant d^0 g_j(x_0, v) < 0$, $\forall j \in J_0$, $v \in C_0(G)$. Therefore, taking into account that $g_j(x_0) = 0$, by Proposition 4.2.10 in Flett [23],

$$\limsup_{t\to 0^+} \frac{g_j(\gamma(t))}{t} \leqslant \bar{d}g_j(x_0, v) < 0, \quad \forall j \in J_0.$$

Then, there exist $\delta_j \in (0, \delta]$ for $j \in J_0$ such that $g_j(\gamma(t)) < 0$, $\forall t \in (0, \delta_j)$. For each $j \in J \backslash J_0$, due to the continuity of g_j at x_0 and of γ at $t = 0$, there exists $\delta_j > 0$ such that $g_j(\gamma(t)) < 0$, $\forall t \in [0, \delta_j)$. Taking $\delta_0 = \mathrm{Min}\{\delta_j\colon\ j \in J\}$, we obtain $g_j(\gamma(t)) < 0$, $\forall t \in (0, \delta_0)$ and $\forall j \in J$; subsequently, $\gamma(t) \in G$. □

If there is no equality constraint, the proof of Theorem 4.1 is not valid, because it is based on the existence of solution of system (5). Therefore, next we give an analogous theorem and a straightforward proof in this case.

Theorem 4.2. *Let $x_0 \in G \cap Q$ and let us suppose that $g : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz near x_0, then*

(i) $C_0(G) \cap \mathrm{cone}(Q - x_0) \subset Z(G \cap Q, x_0) \subset A(G \cap Q, x_0)$.

(ii) *Besides, if the constraint qualification*

$$\text{(CQ1): } C_0(G) \cap (Q - x_0) \neq \emptyset$$

is true, then we have that

$$\mathrm{cl}\big[C_0(G) \cap \mathrm{cone}(Q - x_0)\big] = C(G) \cap T(Q, x_0) \subset \mathrm{cl}\, Z(G \cap Q, x_0) \subset A(G \cap Q, x_0).$$

Proof. (i) First of all, let us see that

$$C_0(G) \subset Z(G, x_0). \tag{9}$$

Let $v \in C_0(G)$. Then $d^0 g_j(x_0, v) < 0$, $\forall j \in J_0$, and hence, $\overline{D}g_j(x_0, v) < 0$. This means that

$$\limsup_{t\to 0^+} \frac{g_j(x_0 + tv) - g_j(x_0)}{t} < 0, \quad \forall j \in J_0,$$

therefore exists $\varepsilon_j > 0$ such that

$$\sup_{t\in(0,\varepsilon_j)} \frac{g_j(x_0 + tv) - g_j(x_0)}{t} < 0.$$

Taking into account that $g_j(x_0) = 0$, it follows that $g_j(x_0 + tv) < 0$, $\forall t \in (0, \varepsilon_j)$ and $\forall j \in J_0$. If $j \in J \backslash J_0$, $g_j(x_0) < 0$, and by the continuity of g_j at x_0, there exists $\varepsilon_j > 0$ such that $g_j(x) < 0$, $\forall x \in B(x_0, \varepsilon_j)$. Taking $\varepsilon = \mathrm{Min}\{\varepsilon_j\colon\ j \in J\}$, then we have that $g_j(x_0 + tv) < 0$, $\forall t \in (0, \varepsilon)$ and $\forall j \in J$. Consequently, $v \in Z(G, x_0)$.

Moreover, from (9) it follows that

$$C_0(G) \cap \operatorname{cone}(Q - x_0) \subset Z(G, x_0) \cap Z(Q, x_0)$$
$$= Z(G \cap Q, x_0) \subset A(G \cap Q, x_0),$$

since $\operatorname{cone}(Q - x_0) = Z(Q, x_0)$ and the last inclusion is valid for any set.

(ii) (CQ1) is equivalent to $C_0(G) \cap \operatorname{ri}(Q - x_0) \neq \emptyset$. In fact, if $C_0(G) \cap \operatorname{ri}(Q - x_0) = \emptyset$, then $\operatorname{int} C_0(G) \cap \operatorname{cl} \operatorname{ri}(Q - x_0) = \emptyset$, but $C_0(G)$ is open and $\operatorname{cl} \operatorname{ri}(Q - x_0) = \operatorname{cl}(Q - x_0)$, following that $C_0(G) \cap \operatorname{cl}(Q - x_0) = \emptyset$, contradicting the hypothesis.

Using Lemmas 3.1 and 2.1, we have that

$$\operatorname{cl}\bigl[C_0(G) \cap \operatorname{cone}(Q - x_0)\bigr] = C(G) \cap T(Q, x_0). \tag{10}$$

Therefore, taking closure in (i), considering (10) and that the cone of attainable directions is closed, the conclusion follows. □

In the next theorem, necessary conditions for a weak local minimum for program (P) are shown. The first one is a Fritz–John type condition, the second one is a primal form and another Fritz–John condition and the third one, requiring additional hypotheses, is a Kuhn–Tucker type condition.

Theorem 4.3. *Let $Q \subset \mathbb{R}^n$ be a convex set, $x_0 \in S \cap Q$ and assume the following*:

(a) $h : \mathbb{R}^n \to \mathbb{R}^r$ *is continuous on a neighborhood of* x_0 *and Fréchet differentiable at* x_0.
(b) $f : \mathbb{R}^n \to \mathbb{R}^p$ *and* $g : \mathbb{R}^n \to \mathbb{R}^m$ *are Lipschitz near* x_0.
(c) $x_0 \in \operatorname{WLMin}(f, S \cap Q)$.

Then

(i) *There exists* $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^{J_0} \times \mathbb{R}^r$, $(\lambda, \mu, \nu) \neq 0$ *such that*

$$\left.\begin{aligned} &(\lambda, \mu) \geqslant 0, \\ &0 \in \sum_{i=1}^{p} \lambda_i \partial_{Cl} f_i(x_0) + \sum_{j \in J_0} \mu_j \partial_{Cl} g_j(x_0) \\ &\quad + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N(Q, x_0). \end{aligned}\right\} \tag{11}$$

(ii) *If, moreover,* (RC) *holds, then*

$C_0(S) \cap \operatorname{cone}(Q - x_0) \cap C_0(F) = \emptyset$ *and*

(11) *is true with* $(\lambda, \mu) \neq 0$.

(iii) *If, in addition to* (ii), *the constraint qualification*

$$\text{(CQ2)}: C_0(S) \cap (Q - x_0) \neq \emptyset$$

holds, then (11) *is true with* $\lambda \neq 0$.

Proof. First, we prove (ii). Let us suppose that there exists $v \in C_0(S) \cap \text{cone}(Q - x_0) \cap C_0(F)$. Using Theorem 4.1, or Theorem 4.2(i) if there is no equality constraint, $v \in A(S \cap Q, x_0)$ and, therefore, there exist a number $\delta > 0$ and a function $\gamma : [0, \delta] \to \mathbb{R}^n$ such that $\gamma(0) = x_0$, $\gamma(0) = v$ and $\gamma(t) \in S \cap Q$, $\forall t \in [0, \delta]$. Since f_i is Lipschitz near x_0 and $d^0 f_i(x_0, v) < 0$, reasoning as for the proof of Theorem 4.1 (now with f_i instead of g_j), we have

$$\limsup_{t \to 0^+} \frac{f_i(\gamma(t)) - f_i(x_0)}{t} \leqslant \bar{d} f_i(x_0, v) \leqslant d^0 f_i(x_0, v) < 0, \quad \forall i = 1, \ldots, p.$$

Then, we have $f_i(\gamma(t)) < f_i(x_0)$ for all $t > 0$ small enough and for each $i = 1, \ldots, p$, contradicting the weak minimality of x_0.

Next, we prove second part of (ii). We have established that there exists no $v \in \mathbb{R}^n$ such that

$$\begin{cases} d^0 f_i(x_0, v) < 0, & \forall i = 1, \ldots, p, \\ d^0 g_j(x_0, v) < 0, & \forall j \in J_0, \\ \nabla h_k(x_0) v = 0, & \forall k = 1, \ldots, r, \\ v \in Q - x_0. \end{cases}$$

By Theorem 21.2 in Rockafellar [19], which can be used since $K(H) \cap \text{ri}(Q - x_0) \neq \emptyset$ by Lemma 3.3, there exists $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^{J_0} \times \mathbb{R}^r$, $(\lambda, \mu) \geqslant 0$, $(\lambda, \mu) \neq 0$, such that

$$\sum_{i=1}^{p} \lambda_i d^0 f_i(x_0, v) + \sum_{j \in J_0} \mu_j d^0 g_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) v \geqslant 0, \quad \forall v \in Q - x_0. \tag{12}$$

Therefore, $v = x_0 - x_0 = 0 \in Q - x_0$ is a minimum on the convex set $Q - x_0$ of the convex function

$$\varphi(v) = \sum_{i=1}^{p} \lambda_i d^0 f_i(x_0, v) + \sum_{j \in J_0} \mu_j d^0 g_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) v.$$

Hence,

$$0 \in \partial\varphi(0) + N(Q, x_0) = \sum_{i=1}^{p} \lambda_i \partial d^0 f_i(x_0, \cdot)(0) + \sum_{j \in J_0} \mu_j \partial d^0 g_j(x_0, \cdot)(0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N(Q, x_0),$$

which is equivalent to (11).

Now, we prove (i). If (RC) does not hold, i.e., if there exists $\nu \in \mathbb{R}^r$, $\nu \neq 0$, such that

$$0 \in \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N(Q, x_0),$$

then the conclusion is obviously obtained with $(\lambda, \mu) = (0, 0)$. So, we can assume that (RC) holds, and part (ii) allows us to conclude.

Finally, let us prove (iii). Suppose that $\lambda = 0$ and take $u \in C_0(S) \cap (Q - x_0)$. If some $\mu_j > 0$, then we have that

$$\sum_{j \in J_0} \mu_j d^0 g_j(x_0, u) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) u < 0,$$

which contradicts the result obtained in (12) taking $v = u$ (with $\lambda = 0$).

Let us note that the constraint qualification (CQ2) is transformed into (CQ1) in absence of equality constraints.

5. Necessary optimality conditions with an arbitrary set constraint

In this section several necessary optimality conditions are provided when the problem (P) involves an arbitrary constraint set. These conditions are expressed in terms of the sequential interior tangent cone.

Let us recall that the sequential interior tangent cone (or cone of quasi-interior directions, [24, Definition 6]) to $Q \subset \mathbb{R}^n$ at x_0, denoted $\mathrm{IT}_s(Q, x_0)$, is the cone defined by the following expression:

Let $v \in \mathbb{R}^n$, $v \in \mathrm{IT}_s(Q, x_0)$ if and only if there exist a number $\varepsilon > 0$ and a sequence $t_n \to 0^+$ such that

$$x_0 + t_n u \in Q, \quad \forall u \in B(v, \varepsilon),\ \forall n \in \mathbb{N}. \tag{13}$$

Theorem 5.1. *Let $Q \subset \mathbb{R}^n$ be an arbitrary set, $x_0 \in S \cap Q$ and suppose the following*:

(a) *$h : \mathbb{R}^n \to \mathbb{R}^r$ is continuous on a neighborhood of x_0 and Fréchet differentiable at x_0.*
(b) *$g : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz near x_0.*
(c) *$f : \mathbb{R}^n \to \mathbb{R}^p$ is Lipschitz near x_0 and $x_0 \in \mathrm{WLMin}(f, S \cap Q)$.*

Then

(i) *If $\nabla h(x_0)$ has full rank, then $C_0(S) \cap \mathrm{IT}_s(Q, x_0) \cap C_0(F) = \emptyset$.*
(ii) *If $\mathrm{IT}_s(Q, x_0)$ is a convex cone, then there exists $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^{J_0} \times \mathbb{R}^r$, $(\lambda, \mu) \geqslant 0$, $(\lambda, \mu, \nu) \neq 0$ such that*

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_{Cl} f_i(x_0) + \sum_{j \in J_0} \mu_j \partial_{Cl} g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + \mathrm{IT}_s(Q, x_0)^*. \tag{14}$$

(iii) *If* $\mathrm{IT}_s(Q, x_0)$ *is a convex cone,* $C_0(S) \cap \mathrm{IT}_s(Q, x_0) \neq \emptyset$ *and* $\nabla h(x_0)$ *has full rank, then* (14) *is true with* $\lambda \neq 0$.

Before giving the proof we need a lemma.

Lemma 5.2. *Suppose that the above conditions* (a) *and* (b) *are verified and that* $\nabla h(x_0)$ *has full rank, then* $C_0(S) \cap \mathrm{IT}_s(Q, x_0) \subset T(S \cap Q, x_0)$.

Proof. Let $v \in C_0(S) \cap \mathrm{IT}_s(Q, x_0)$. From the definition of the sequential interior tangent cone there exist $\varepsilon > 0$ and $t_n \to 0^+$ such that (13) holds.

Let $\Gamma = \{v \in \mathbb{R}^n\colon\ x = x_0 + tu$ with $u \in B(v, \varepsilon),\ t \in [0, 1]\}$. We have that Γ is a convex set and $v \in K(H) \cap \mathrm{int}(\Gamma - x_0)$. This last condition implies that the regularity condition (RC) holds for the convex set Γ because $\nabla h(x_0)$ has full rank and we can apply Remark 3.4(6). Then, by Theorem 3.2, $v \in A(H \cap \Gamma, x_0)$ and consequently, there exist $\delta > 0$ and $\gamma : [0, \delta] \to \mathbb{R}^n$ such that $\gamma(0) = x_0$, $\gamma(t) \in H \cap \Gamma\ \forall t \in [0, \delta]$ and $\gamma'(0) = v$. Let $\alpha(t) = (\gamma(t) - x_0 - tv)/t$. Since $\lim_{t \to 0+} \alpha(t) = 0$, for the above ε there exists $\delta_0 \in (0, \delta]$ such that $v + \alpha(t) \in B(v, \varepsilon)\ \forall t \in (0, \delta_0)$, and for δ_0 there exists $n_0 \in \mathbb{N}$ such that $t_n \in (0, \delta_0)$ for every $n \geqslant n_0$. Hence, by (13)

$$x_n = \gamma(t_n) = x_0 + t_n(v + \alpha(t_n)) \in Q \quad \forall n \geqslant n_0.$$

Therefore, $v \in T(H \cap Q, x_0)$. From here it is continued as in the proof of Theorem 4.1 (the sequence $x_n = \gamma(t_n)$ is considered instead of the curve $\gamma(t)$ and we obtain that $x_n \in G$ and then $v \in T(S \cap Q, x_0)$). □

Proof of Theorem 5.1. (i) If we suppose that there exists $v \in C_0(S) \cap \mathrm{IT_s}(Q, x_0) \cap C_0(F)$, then $d^0 f_i(x_0, v) < 0$, $i = 1, \ldots, p$, and, by Lemma 5.2, there exist $x_n \in S \cap Q$ and $t_n \to 0^+$ such that $\lim_{n \to \infty} t_n^{-1}(x_n - x_0) = v$. From here we proceed as in the proof of Theorem 4.3(i) (the sequence x_n is used instead of the curve $\gamma(t)$).

(ii) If $\nabla h(x_0)$ has not full rank the conclusion is evidently true. Otherwise, condition (i) holds. If $\mathrm{IT}_s(Q, x_0) = \emptyset$, then the conclusion is obviously verified because $\mathrm{IT}_s(Q, x_0)^* = \mathbb{R}^n$. If $\mathrm{IT}_s(Q, x_0) \neq \emptyset$ but $\mathrm{IT}_s(Q, x_0) \cap K(H) = \emptyset$, since $\mathrm{IT}_s(Q, x_0)$ is an open convex cone and $K(H)$ is a closed convex cone, then, applying the separation theorem [19, Theorems 11.3 and 11.7], there exists $u \in \mathbb{R}^n \setminus \{0\}$ such that

$$\langle u, x \rangle \leqslant 0 \leqslant \langle u, y \rangle, \quad \forall x \in \mathrm{IT}_s(Q, x_0),\ \forall y \in K(H).$$

Hence, $u \in \mathrm{IT}_s(Q,x_0)^*$ and $-u \in K(H)^* = \mathrm{lin}\{\nabla h_k(x_0)\colon\ k \in K\}$, therefore there exists $\nu \in \mathbb{R}^r$ such that $-u = \sum_{k=1}^r \nu_k \nabla h_k(x_0)$. Consequently, $\sum_{k=1}^r \nu_k \nabla h_k(x_0) + u = 0$, and (ii) holds with $\nu \neq 0$, otherwise it would be $u = 0$ which is a contradiction. Finally, if $\mathrm{IT}_s(Q,x_0) \cap K(H) \neq \emptyset$, since (i) holds, that is, there exists no $v \in \mathbb{R}^n$ such that

$$\begin{cases} d^0 f_i(x_0,v) < 0, & i = 1,\dots,p,\\ d^0 g_j(x_0,v) < 0, & \forall j \in J_0,\\ \nabla h_k(x_0)v = 0, & k = 1,\dots,r,\\ v \in \mathrm{IT}_s(Q,x_0), \end{cases}$$

we can follow as in the proof of Theorem 4.3(ii) (the role of $Q - x_0$ is now played by $\mathrm{IT}_s(Q,x_0) \cup \{0\}$).

(iii) In the first place, in (ii) $(\lambda,\mu) \neq 0$, otherwise it would be

$$0 \in \sum_{k=1}^r \nu_k \nabla h_k(x_0) + \mathrm{IT}_s(Q,x_0)^*$$

with $\nu \neq 0$, that is, (RC) does not hold for the convex $\mathrm{IT}_s(Q,x_0)$ and this is in contradiction with what is obtained by applying Remark 3.4(6). To prove that $\lambda \neq 0$ we argue as for the proof of Theorem 4.3(iii). □

6. Final remarks

In Theorems 4.1, 4.2, 4.3 and 5.1 we have supposed that the functions f and g are Lipschitz near x_0, but they are also valid if we suppose that these functions are Hadamard differentiable at x_0 with convex derivative, and even in the case that we only suppose the existence of upper Hadamard derivative at x_0 (upper stable functions) and that this be convex as function of the direction. In this last case, to define the cones $C_0(S)$, $C(S)$, $C_0(F)$ and $C(F)$ we have to use the upper Hadamard derivative instead of that of Clarke's and in the expressions in which the Clarke subdifferential ((11) and (14)) appears we have to substitute it by the upper Hadamard subdifferential:

$$\bar{\partial} f(x_0) = \left\{\xi \in \mathbb{R}^n\colon\ \langle \xi, v\rangle \leqslant \bar{d} f(x_0,v)\ \forall v \in \mathbb{R}^n\right\}.$$

Taking this remark into account, Theorem 5.1 is a generalization of Theorem 9 of Giorgi and Guerraggio [24] in which it is supposed that h is $C^1(x_0)$ with full rank Jacobian, f and g are differentiable Fréchet at x_0 and f is $\mathbb{R}$-valued.

If f and g are Lipschitz near x_0, to define the cones $C_0(S)$, $C(S)$, $C_0(F)$ and $C(F)$ we can use the Michel–Penot derivative or the deconvolution of the upper Hadamard derivative (which coincides, in this case, with the deconvolution of the upper Dini derivative) instead of the Clarke derivative. The resulting theorems after adapting Theorems 4.1, 4.2, 4.3 and 5.1 are still valid. Of course, we should use the corresponding subdifferential to the derivative that we are dealing with, instead of the Clarke subdifferential. The proof will not change, since all these derivatives are greater than upper Hadamard derivative or equal to it. See [25] for the definitions and properties of these derivatives. As an example we state Theorem 6.1, resulting from 4.3, by using the deconvolution of upper Hadamard derivative after the previous introduction of the necessary notations.

Let $f:\mathbb{R}^n \to \mathbb{R}$ be a real function. The deconvolution of the upper Hadamard derivative of f at x_0 is

$$\bar{d}^* f(x_0, v) = \mathrm{Sup}\big\{\bar{d} f(x_0, v + w) - \bar{d} f(x_0, w):\ w \in \mathbb{R}^n\big\}.$$

If f is Lipschitz near x_0, $\bar{d}^* f(x_0, v)$ is convex and finite for all v and we have:

$$\overline{D} f(x_0, v) = \bar{d} f(x_0, v) \leqslant \bar{d}^* f(x_0, v) \leqslant d^0 f(x_0, v). \tag{15}$$

The associate subdifferential to this derivative is

$$\bar{\partial}^* f(x_0) = \big\{\xi \in \mathbb{R}^n:\ \langle \xi, v\rangle \leqslant \bar{d}^* f(x_0, v)\ \forall v \in \mathbb{R}^n\big\} = \partial \bar{d}^* f(x_0, \cdot)(0),$$

and it is contained, by (15), in the Clarke subdifferential:

$$\bar{\partial}^* f(x_0) \subset \partial_{Cl} f(x_0). \tag{16}$$

We suppose that $f:\mathbb{R}^n \to \mathbb{R}^p$ and $g:\mathbb{R}^n \to \mathbb{R}^m$ are Lipschitz near x_0 and $h:\mathbb{R}^n \to \mathbb{R}^r$ is Fréchet differentiable at x_0. It is denoted

$$C_0\big(S, \bar{d}^*\big) = \big\{v \in \mathbb{R}^n:\ \bar{d}^* g_j(x_0, v) < 0, \forall j \in J_0;\ \nabla h_k(x_0)v = 0,\ \forall k \in K\big\},$$

$$C\big(S, \bar{d}^*\big) = \big\{v \in \mathbb{R}^n:\ \bar{d}^* g_j(x_0, v) \leqslant 0, \forall j \in J_0;\ \nabla h_k(x_0)v = 0,\ \forall k \in K\big\}.$$

And similarly, $C_0(F, \bar{d}^*)$ and $C(F, \bar{d}^*)$. Obviously

$$C_0\big(S, d^0\big) \subset C_0\big(S, \bar{d}^*\big) \quad \text{and} \quad C\big(S, d^0\big) \subset C\big(S, \bar{d}^*\big) \tag{17}$$

(to make it clearer we now denote $C_0(S, d^0)$ and $C(S, d^0)$ what we had previously denoted $C_0(S)$ and $C(S)$).

Theorem 6.1. *Under the hypotheses of Theorem* 4.3 *we have*:

(i) *There exists* $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^{J_0} \times \mathbb{R}^r$, $(\lambda, \mu, \nu) \neq 0$ *such that*

$$\left.\begin{aligned} &(\lambda, \mu) \geqslant 0, \\ &0 \in \sum_{i=1}^{p} \lambda_i \bar{\partial}^* f_i(x_0) + \sum_{j \in J_0} \mu_j \bar{\partial}^* g_j(x_0) \\ &\quad + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N(Q, x_0). \end{aligned}\right\} \tag{18}$$

(ii) *If, moreover,* (RC) *holds, then*

$$C_0(S, \bar{d}^*) \cap \mathrm{cone}(Q - x_0) \cap C_0(F, \bar{d}^*) = \emptyset \quad \textit{and}$$

(18) *is true with* $(\lambda, \mu) \neq 0$.

(iii) *If, in addition to* (ii), *the constraint qualification*

$$(\mathrm{CQ2}^*)\text{: } C_0\big(S, \bar{d}^*\big) \cap (Q - x_0) \neq \emptyset$$

holds, then (18) *is true with* $\lambda \neq 0$.

Note that (i), (ii) and (iii) are finer than 4.3(i), 4.3(ii) and 4.3(iii) by (17) and (16). Even (iii) is of less restrictive application than 4.3(iii) (because (CQ2) $\Rightarrow$ (CQ2*) by (17)).

Many authors have obtained Fritz–John and Kuhn–Tucker conditions for locally Lipschitz programs. See, for example, Jourani [26, Theorems 4.2 and 4.6]. Treiman [27] considers scalars programs and uses the Mordukhovich and linear subdifferentials. In these papers, h is locally Lipschitz, so their results are not comparable with our results.

References

[1] M.S. Bazaraa, C.M. Shetty, Foundations of Optimization, Springer-Verlag, Berlin, 1976.

[2] C. Singh, Optimality conditions in multiobjective differentiable programming, J. Optim. Theory Appl. 53 (1) (1987) 115–123.

[3] S. Di, Classical optimality conditions under weaker assumptions, SIAM J. Optim. 6 (1) (1996) 178–197.

[4] B. Jiménez, V. Novo, Cualificaciones de restricciones en problemas de optimización vectorial diferenciables, in: Actas XVI C.E.D.Y.A./VI C.M.A., Vol. I, Universidad de Las Palmas de Gran Canaria, Spain, 1999, pp. 727–734.

[5] F.H. Clarke, Optimization and Nonsmooth Analysis, John Wiley & Sons, New York, 1983.

[6] F.H. Clarke, A new approach to Lagrange multipliers, Math. Oper. Res. 1 (2) (1976) 165–174.

[7] J.B. Hiriart-Urruty, On optimality conditions in nondifferentiable programming, Math. Programming 14 (1978) 73–86.

[8] J.B. Hiriart-Urruty, Refinements of necessary optimality conditions in nondifferentiable programming I, Appl. Math. Optim. 5 (1979) 63–82.

[9] M. Minami, Weak Pareto-optimal necessary conditions in a nondifferentiable multiobjective program on a Banach space, J. Optim. Theory Appl. 41 (1983) 451–461.

[10] Y. Ishizuka, K. Shimizu, Necessary and sufficient conditions for the efficient solutions of nondifferentiable multiobjective problems, IEEE Transactions on Systems, Man and Cybernetics SMC-14 (4) (1984) 625–629.

[11] B.D. Craven, Nondifferentiable optimization by smooth approximation, Optimization 17 (1986) 3–17.

[12] B.D. Craven, Nonsmooth multiobjective programming, Numer. Funct. Anal. Optim. 10 (1–2) (1989) 49–64.

[13] B.D. Craven, Control and Optimization, Chapman & Hall, London, 1995.

[14] A. Jourani, Constraint qualifications and Lagrange multipliers in nondifferentiable programming problems, J. Optim. Theory Appl. 81 (3) (1994) 533–548.

[15] L. Wang, J. Dong, Q. Liu, Optimality conditions in nonsmooth multiobjective programming, Systems Sci. Math. Sci. 7 (3) (1994) 250–255.
[16] S. Mititelu, A survey on optimality and duality in nonsmooth programming, in: Lecture Notes in Econom. and Math. Systems, Vol. 405, 1994, pp. 211–225.
[17] S. Mititelu, Nonsmooth programming with nonconvex functions, Rev. Roumaine Math. Pures Appl. 40 (7–8) (1995) 655–667.
[18] J. Jahn, Introduction to the Theory of Nonlinear Optimization, Springer, Berlin, 1996.
[19] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[20] M.R. Hestenes, Optimization Theory. The Finite Dimensional Case, Robert E. Krieger Publishing Company, New York, 1981.
[21] R.T. Rockafellar, Lagrange multipliers and optimality, SIAM Review 35 (1993) 183–238.
[22] B.M. Glover, V. Jeyakumar, Abstract nonsmooth nonconvex programming, in: Lecture Notes in Econom. and Math. Systems, Vol. 405, 1994, pp. 186–210.
[23] T.M. Flett, Differential Analysis, Cambridge University Press, Cambridge, 1980.
[24] G. Giorgi, A. Guerraggio, First order generalized optimality conditions for programming problems with a set constraint, in: Lecture Notes in Econom. and Math. Systems, Vol. 405, 1994, pp. 171–185.
[25] B.D. Craven, D. Ralph, B.M. Glover, Small convex-valued subdifferentials in mathematical programming, Optimization 32 (1) (1995) 1–21.
[26] A. Jourani, Qualification conditions for multivalued functions in Banach spaces with applications to nonsmooth vector optimization problems, Math. Programming 66 (1994) 1–23.
[27] J.S. Treiman, Lagrange multipliers for nonconvex generalized gradients equality, inequality, and set constraints, SIAM J. Control Optim. 37 (5) (1999) 1313–1329.