Level crossings of the empirical process

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The asymptotics for the number of times the empirical distribution function crosses the true distribution function are well-known (see Dwass, 1961; or Shorack and Wellner, 1986). We give a process version of this limit theorem and we identify the limiting process to be the local time of Brownian bridge. This substantially strengthens the usual central limit theorem for linear empirical processes. As a by-product of these results, we answer an open problem cited in Shorack and Wellner (1986).

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1. Introduction

The number of times the graph of an empirical cumulative distribution function crosses that of the underlying distribution (i.e., zero-crossings of the corresponding empirical process) has been the subject of much study (see, for example, Gaenssler and Gutjahr [6] and its references; see also the references in Shorack and Wellner [15].) As pointed out in Gaenssler and Gutjahr [6], the level crossings — and more generally line crossings — can be used to get an in-depth understanding of many branches of non-parametrics related to goodness-of-fit tests.

This article is concerned with the asymptotics of the level crossings of the uniform empirical process. It is enough to consider the uniform case, due to the distribution-free property of the level crossings. Hence our results presented here have immediate generalizations to the empirical process corresponding to any strictly increasing, continuous distribution function. To state our problem precisely, we first need some notation.

Let \( \{U_n(t); 0 \leq t \leq 1\} \) be the empirical process based on i.i.d. \( U(0, 1) \) data \( X_1, X_2, \ldots, X_n \). Then the crossing process for \( U_n \) is defined as

\[
C_x(U_n) = \#\{s \leq t: U_n(s) = x\}.
\]

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More generally, for any function, $f$ (random or not) we define the crossing process of $f$ as:

$$C^x_t(f) = \#\{s \leq t: f(s) = x\}, \quad (1.1)$$

when the definition makes sense. It is helpful to think of $t$ as time.

Dwass [5] uses a neat combination of a well-known Poissonization technique together with Karamata's Tauberian theorem to show that

$$\lim_{n \to \infty} P\{n^{-1/2} C^0_t(U_n) \geq x\} = e^{-x^2/2}. \quad (1.2)$$

The main result of this paper extends equation (1.2). We prove that the doubly-indexed stochastic process,

$$\{n^{-1/2} C^x_t(U_n); (x, t) \in \mathbb{R} \times [0, 1]\}$$

converges in distribution (in the appropriate sense) and the limiting stochastic process is identified as the Brownian bridge local time (for definition and properties of local times see Revuz and Yor [14]). Denote local time of a function, $f$, at $x$ up to time $t$ by $L^x_t(f)$. There is a natural correspondence between the processes $C(U_n)$ and $L(U_n)$, which we state as a lemma. The proof is easy and will be omitted.

**Lemma 1.** $n^{-1/2} C^x_t(U_n) = L^x_t(U_n)$. \(\square\)

In fact, we present a stronger result than the promised extension of (1.2). We shall prove the following strong approximation theorem based on an embedding scheme developed in Khoshnevisan [8]:

**Theorem 2.** On a probability space, $(\Omega, \mathcal{F}, P)$, there exists a sequence of standard Brownian bridges, $\{w_n(t); 0 \leq t \leq 1\}_{n \geq 1}$, and a sequence of uniform empirical processes, $\{U_n(t); 0 \leq t \leq 1\}_{n \geq 1}$, such that for all $\varepsilon > 0$,

$$\sup_{x \in \mathbb{R}} \sup_{0 \leq t \leq 1} |L^x_t(U_n) - L^x_t(w_n)| = o(n^{-1/4} \log n^{3/4+\varepsilon}) \quad a.s.$$

Our interest in this problem began when we recognized the limiting probability distribution in equation (1.2) (also known as the Rayleigh distribution) as the law of the local time of Brownian bridge at zero up to time one. In addition, the problem of showing that $L^x_t(U_n)$ converges uniformly to $L^x_t(w_n)$ is cited as being open in the recent book of Shorack and Wellner [15]. This had earlier been recognized by Lévy [11] and probably also by Smirnov [16]. We would also like to mention that Dwass's relation (1.2) has also been discovered by Révész [13], using a simple combinatorial argument together with a Skorohod embedding argument. The idea here is not unlike that of Theorem 4 of Révész [13]: find an embedding for scaled Poisson process, and somehow condition using path decomposition. Our approach uses a theorem on Poisson process embedding that we developed in [8] that is more precise than that quoted by Révész [13] and hence we manage to get an approximation uniformly at all levels.
In the next section, we state some facts that are relevant to the basic ideas in Section 3, where the proof of Theorem 2 is given. The derivation presented in Section 3 is quite technical and can be skipped if a mere application of the theorem is desired. The fourth section of this article hinges upon the clever discovery of Dwass [5] that intersections of two independent empirical processes are analogous to the level crossing problem discussed in the above paragraphs. Also some possible open directions for further research are pointed out. We have included a fifth section that essentially translates an extension of Theorem 2 into the language of weak convergence, hence giving a generalization of the central limit theorem for empirical processes. Finally in an appendix, we discuss some facts about local time of the Brownian bridge that have been used in earlier sections; the results here are basically known and there are no claims to their originality on the part of the author.

We would like to acknowledge that ideas for a proof of this result using non-standard analysis have been given by G.R. Mendieta, at the 1981 Western Regional conference of the IMS. Also we point out (see Helmers [7]) that P. Révész has discovered an embedding of $L_0(U_n)$ in the local time of a Kiefer process that in particular implies that almost surely,

$$\limsup_{n \to \infty} \frac{L_0(U_n)}{\sqrt{\log \log n}} = 2^{1/2}.$$

Now a few words about the notation: Throughout this paper, we use a generic constant, $C$ (which may vary from line to line) when the constant in question is independent of anything interesting.

2. Preliminaries

In this section, we state a Poissonized version of Theorem 2. Having done so, we proceed — in the next section — to present the proof of our main result. The Poissonized version of Theorem 2 is as follows:

**Theorem 3.** On a probability space, $(\Omega, \bar{F}, P)$, there exists a sequence of Brownian motions, $\{B_n(t); t \geq 0\}_{n \geq 1}$, and a sequence of compensated Poisson processes, $\{Z_n(t); t \geq 0\}_{n \geq 1}$, with expected arrival rate of $1/n$, such that for all $\varepsilon, K > 0$, there exists a constant $C = C_{K, \varepsilon} > 0$, so that for all $n \geq 1$,

$$P\left\{ \sup_{(x,t) \in \mathbb{R} \times [0,1]} |L^x_t(Z_n) - L^x_t(B_n)| \geq n^{-1/4}(\log n)^{3/4+\varepsilon} \right\} \leq Cn^{-K}.$$

Indeed, in the above theorem, we have

$$Z_n(t) = Z(nt)/\sqrt{n},$$

$$B_n(t)B(nt)/\sqrt{n},$$

where

$\{Z(t); t \geq 0\}$ is a compensated Poisson process with rate 1,

$\{B(t); t \geq 0\}$ is a standard Brownian motion.
This is Theorem IV.6 of Khoshnevisan [8], specialized to the case of the normalized Poisson process. We further mention that the proof of the above requires the following embedding scheme, stated here— for convenience —as a proposition:

**Proposition 4.** Let $(\Omega, \mathcal{F}, P)$ be a probability space that carries a Brownian motion, \{B(t); t \geq 0\}. Then there exists a time-change, \{\sigma_t; t \geq 0\}, such that:

(i) \{\sigma_t; t \geq 0\} is a Lévy process, with finite moments of all orders.

(ii) The process, \{Z(t); t \geq 0\} = \{B(\sigma_t); t \geq 0\}, is a compensated Poisson process with rate 1. □

It should be now clear that the compensated Poisson process in Theorem 3 is that given by (ii) of Proposition 4. In particular, notice that certain properties of \sigma_t can be read easily enough from this time-change. For instance, it follows from a martingale argument that for any \(t\), \(E\sigma_t = t\), and hence by the law of the iterated logarithm \(|\sigma_n - n| = O(\sqrt{n \log \log n})\) a.s.

3. **Proof of Theorem 2**

First of all, by an elementary calculation (notation being standard)

\[
\{U_n(t); t \geq 0\} = \{Z_n(t); t \geq 0\mid Z_n(1) = 0\}. \tag{3.1}
\]

However for \(N(n)\) a Poisson random variable with mean \(n\),

\[
\{Z_n(1) = 0\} = \{N(n) = n\}.
\]

Therefore by Theorem 3, for all \(n, K \geq 1\),

\[
P\left\{ \sup_{(x,t) \in \mathbb{R} \times [0, 1]} |L^x(Z_n) - L^x(B_n)| \geq an^{-1/4}(\log n)^{3/4 + \epsilon} |Z_n(1) = 0\right\},
\]

\[
\leq \frac{C_K n^{-K-1/2}}{P\{N(n) = n\}} \leq C_K \left(\frac{n^n e^{-n}}{n!}\right)^{-1} n^{-K-1/2} \leq c_K n^{-K} \tag{3.2}
\]

where \(c_K\) is a positive constant depending only on \(K\), and the last inequality follows by Stirling's formula. At this point, inequality (3.2), combined with (3.1), might lead one to think the proof is complete. However, we also need to know that \(B_n\), conditioned on \(\{Z_n(1) = 0\}\) is a Brownian bridge. Unfortunately, this is not so, even though it is almost the case. The rest of the proof makes the preceding statement more precise.

Define

\[
a(n, \delta) = \sup\{t \leq \sigma_n; |B(t) - B(\sigma_n)| = \delta\}.
\]

Then \(a(n, 0) = \sigma_n\); as a result, by continuity of Brownian motion paths

\[
\lim_{\delta \to 0} a(n, \delta) = \sigma_n.
\]
So for each fixed \( n \),
\[
\lim_{\delta \to 0} P[|a(n, \delta)/\sigma_n - 1| > n^{-1}\varepsilon] = 0,
\]
uniformly in \( 1 > \varepsilon > 0 \). Hence, there exists a sequence \( \delta_n \downarrow 0 \), such that
\[
P[|a_n/\sigma_n - 1| < n^{-1}\varepsilon] \leq 2^{-n},
\]
where
\[
a_n = a(n, \delta_n). \tag{3.4}
\]
Embed an independent Brownian motion, \( \{\tilde{B}(t); t \geq 0\} \), in our (possibly enlarged) probability space. Define
\[
\tau(n, \delta) = \inf\{t: \tilde{B}(t) = B(a(n, \delta))\}. \tag{3.5}
\]
It then follows that
\[
\lim_{\delta \to 0} \tau(n, \delta) = \inf\{t: \tilde{B}(t) = B(\sigma_n)\}. \tag{3.6}
\]
Letting \( \delta \to 0 \),
\[
P_n[\tau(n, \delta) > \varepsilon] = P[\tau(n, \delta) > \varepsilon | B(\sigma_n) = 0] \to P[\inf\{t: \tilde{B}(t) = 0\} > \varepsilon] = 0,
\]
uniformly in \( 1 > \varepsilon > 0 \). Therefore, there exists a sequence which we shall continue to call \( \delta_n \), such that the following holds, as well as equation (3.3):
\[
P_n[\tau_n > 1] \leq 2^{-n}, \tag{3.7}
\]
where \( \tau_n = \tau(n, \delta_n) \).

Define a new process,
\[
\beta_n(t) = B(t)1\{t \leq a_n\} + \tilde{B}(\tau_n + a_n - t)1\{a_n < t \leq a_n + \tau_n\}.
\]
It can be checked that the process
\[
w_n(t) = (a_n + \tau_n)^{-1/2}\beta_n(t(a_n + \tau_n)), \quad 0 \leq t \leq 1,
\]
is a standard Brownian bridge on \([0, 1]\), for every fixed integer, \( n \). Furthermore, this Brownian bridge is independent of \( B(\sigma_n) \), by well-known last exit decomposition results for Brownian motion (for example, see Revuz and Yor [14] and this and relations to Brownian excursions). Notice that the occupation density interpretation of local times imply that
\[
L^\alpha_t(w_n) = (a_n + \tau_n)^{-1/2}L^\alpha_t(a_n + \tau_n)(\beta_n).
\]
For the sake of convenience let
\[
\Delta_n = a_n + \tau_n, \quad D_n = L_t^\alpha(w_n) - L_t^\alpha(B_n), \quad \phi_n = n^{-1/4} \log n^{3/4+\varepsilon},
\]
\[
A_n = \{\omega \in \Omega: t(a_n(\omega) + \tau_n(\omega)) \leq a_n(\omega)\}, \quad P_n[\cdot] = P[\cdot | B(\sigma_n) = 0],
\]
where \( x, \varepsilon \) and \( t < 1 \) are fixed for the time being. Then,
\[
P_n[|D_n| \geq \alpha \phi_n]
\leq P_n[|\Delta_n^{-1/2}L_{i\Delta_n}^{\alpha\Delta_n}t(B) - n^{-1/2}L_{i\Delta_n}^{\alpha\Delta_n}(B)| > \alpha \phi_n] + P_n\{A_n^C\}. \tag{3.8}
\]
It can be checked, using (3.3), (3.7), and Proposition 4(i), that (recall that we have assumed $t < 1$) for any $K > 0$ there exists $C = C_K$ so that for all $n \geq 1$,

$$P_n\{A_n^\top\} \leq Cn^{-K}.$$  

Therefore (3.8) and the above together imply that for any $K > 0$ there exists $C = C_K$ so that for all $n \geq 1$,

$$P_n\{|D_n| \geq \alpha \phi_n\} \leq P_n\{|\Delta_n^{-1/2}L_{x,\lambda_n}(B) - n^{-1/2}L_{x,n}(B)| > \alpha \phi_n\} + Cn^{-K}.$$

We now make the following:

**Claim 1.** \(\sup_{|x| = 2(\log n)^{1/2+\varepsilon/2}} P_n\{|\Delta_n^{-1/2}L_{x,\lambda_n}(B) - n^{-1/2}L_{x,n}(B)| > \alpha \phi_n\} \leq Cn^{-K}\).

If so, then the above estimate implies that for all $\alpha > 0$, and for all $K$, $n \geq 1$,

$$\sup_{|x| = 2(\log n)^{1/2+\varepsilon/2}} P_n\{|L_i^x(w_n) - L_i^x(B_n)| > \alpha \phi_n\} \leq Cn^{-K}.$$

Therefore, for any set $S_n$ of polynomial cardinality (in $n$),

$$P_n \left\{ \sup_{x \in S_n} |L_i^x(w_n) - L_i^x(B_n)| > \alpha \phi_n \right\} \leq Cn^{-K}.$$

Hence for all $K > 0$ there exists $C = C_K$ such that for all $n \geq 1$ and all $\alpha > 0$,

$$P_n \left\{ \sup_{x \in S_n} |L_i^x(w_n) - L_i^x(U_n)| > 2\alpha \phi_n \right\}$$

$$= P_n \left\{ \sup_{x \in S_n} |L_i^x(w_n) - L_i^x(Z_n)| > 2\alpha \phi_n \right\}$$

$$\leq P_n \left\{ \sup_{x \in S_n} |L_i^x(Z_n) - L_i^x(B_n)| > \alpha \phi_n \right\} + P_n \left\{ \sup_{x \in S_n} |L_i^x(B_n) - L_i^x(w_n)| > \alpha \phi_n \right\}$$

$$\leq Cn^{-K},$$

since it is easy to check that

$$P_n \left\{ \sup_{t \leq 1} |w_n(t)| > 2(\log n)^{1/2+\varepsilon/2} \right\} \leq Cn^{-K}$$

and

$$P_n \left\{ \sup_{t \leq 1} |B_n(t)| > 2(\log n)^{1/2+\varepsilon/2} \right\} \leq Cn^{-K}.$$
By a well-known argument, the above holds uniformly for \( t \in [0, 1 - \eta] \), for any \( \eta > 0 \). As is customary in studying the Brownian bridge, there is a 'discontinuity' at time 1. So in order to finish the proof of the theorem, we need an extra argument handling the convergence at time one. Namely we need to prove:

**Claim 2.** \( \sup_x |L(t)^\gamma(U_n) - L(t)^\gamma(w_n)| = o(\phi_n) \).

The second claim holds, as can be checked from the definition of \( w_n \) (and hence \( \beta_n \)). The proof involves equations (3.3) and (3.7) very much like the previous argument and is omitted.

At this point, the only statement that still needs a proof is Claim 1.

**Proof of Claim 1.** Throughout this proof, fix \( \varepsilon > 0 \). Notice that for

\[
\psi_n = n^{-1/2}(\log n)^\varepsilon,
\]

by an application of Bernstein's inequality, and truncation one can show that for all \( K > 0 \) there exists a \( C = C_K \) so that for all \( n \geq 1 \),

\[
P\{|\sigma_n / n - 1| > \psi_n\} \leq Cn^{-K}.
\]

From this, (3.3) and (3.7) together it follows that

\[
P_n\{|\Delta_n / n - 1| > \psi_n\} \leq Cn^{-K},
\]

which, by a simple argument, shows that

\[
P_n\{|\sqrt{n/\Delta_n - 1}| > \psi_n\} \leq Cn^{-K}.
\]

(3.9)

Therefore, writing \( L(x, t) \) for \( L(t)^\gamma(B) \) to simplify the notation,

\[
P_n\{|L(x, t) - L(x, n^{-1}\Delta_n)| > \alpha \phi_n\} \leq P_n\{L(0, 1) > \alpha \log n^{1/2+3\varepsilon/2}\} + Cn^{-K}
\]

\[
\leq Cn^{-K},
\]

(3.10)

by additivity and by scaling properties of local time. (See Revuz and Yor [14].) Hence

\[
P_n\{|\Delta_n^{-1/2}L(x/\sqrt{\Delta_n}, t\Delta_n) - L(x/\sqrt{n}, tn)| > \alpha \phi_n\}
\]

\[
\leq Cn^{-K} + P_n\{|\Delta_n^{-1/2}L(x/\sqrt{\Delta_n}, t\Delta_n) - n^{-1/2}L(x/\sqrt{n}, t\Delta_n)| > \alpha \phi_n\}.
\]

Let \( II \) denote the second term in the above inequality. Then

\[
II = P_n[\Delta_n^{-1/2}L(x/\sqrt{\Delta_n}, t\Delta_n) - n^{-1/2}L(x/\sqrt{n}, t\Delta_n) > \alpha \phi_n; +] + P_n[\Delta_n^{-1/2}L(x/\sqrt{\Delta_n}, t\Delta_n) + n^{-1/2}L(x/\sqrt{n}, t\Delta_n) > \alpha \phi_n; +]
\]

\[
= P_n(+) + P_n(-).
\]

Here we use the notation that for random variable \( V \),

\[
P_n[V \geq \alpha \phi_n; +] = P_n[V \geq \alpha \phi_n; V \geq 0].
\]
We now proceed to bound the first term in the above last inequality, the second term being more or less the same.

\[ P_n(+) = P_n[n^{-1/2} (L(x, t_{\Delta_n}) - L(x, t_{\Delta_n}/n)) > \alpha \phi_n; +] \]

\[ \leq P_n[n^{-1/2} L(x, t_{\Delta_n}) - L(x, t_{\Delta_n}/n)] > \frac{1}{2} \alpha \phi_n \]

\[ + P_n[n^{-1/2} L(x, t_{\Delta_n}/n) - 1] > \frac{1}{2} \alpha \phi_n. \]

(3.11)

We bound the two terms above separately. By (3.9), the second term is bounded above by

\[ Cn^{-K} + P_n[n^{-1/2} L(x, t_{\Delta_n}/n) > \frac{1}{2} \alpha \phi_n \psi_n^{-1}] \]

\[ \leq C_1 n^{-K} + C_2 n^{1/2} P\left\{ L(x, t_{\Delta_n}/n, t_{\Delta_n}/n) > \frac{1}{2} n^{1/4} \log n^{3/4} \right\} \]

\[ \leq C_1 n^{-K} + C_2 n^{1/2} P\left\{ \sup_{x \in \mathbb{R}} L(x, 2t) > \frac{1}{2} n^{1/4} \log n^{3/4} \right\} \]

\[ \leq C_3 n^{-K} + C_4 n \log n^{1/2} \exp\left\{-\frac{1}{16} n^{1/2} \log n^{3/2}\right\} \]

\[ \leq Cn^{-K}. \]

where the second to the last line holds by Theorem 1.7 of Borodin [2]. This bounds the second term of (3.11). We shall now find a similar upper bound for the first term. Notice that if \( x = 0 \), then the claim is now obvious. So from now on, assume \( x \neq 0 \). Then the first term in (3.11) is bounded above by

\[ C_1 n^{1/2} P\left\{ \sup_{x \neq 0} \sup_{|v-n| < |x|} |L(u, t) - L(v, t)| > \frac{1}{2} \alpha \phi_n \right\} + C_2 n^{-K} \]

\[ = C_1 n^{1/2} |x|^{-1} \psi_n^{-1} \exp\left\{-\frac{1}{2} \alpha |x|^{-1/2} \psi_n^{-1/2} \phi_n \right\} + C_2 n^{-K} \]

\[ = C_3 n (\log n)^4 |x|^{-1} \exp\left\{-\frac{1}{2} \alpha |x|^{-1/2} (\log n)^{3/4+\varepsilon/2} \right\} + C_2 n^{-K}. \]

(3.12)

The second inequality used above, follows from Trotter [17] and a simple argument. Now it is a simple calculus exercises to find an upper bound for the right-hand side of (3.12); it is found to be maximized when setting

\[ |x| = 2^{3/2} \left(\frac{1}{2} \alpha \log n^{3/4+\varepsilon/2}\right)^{-1/3}. \]

So (3.12) is bounded above by

\[ C_3 n (\log n)^{1/2-2\varepsilon/3} \exp\{-2^{-5/4} \alpha^{4/3} (\log n)^{1+2\varepsilon/3}\} + C_2 n^{-K} \leq C n^{-K}. \]

This concludes the proof of the claim and hence that of the theorem. \( \square \)

Remarks. (1) With a little more work, we can prove that for all \( \varepsilon, \alpha > 0 \), there are constants \( c_i = c_i(\alpha, \varepsilon), i = 1, 2 \), such that for all \( n \),

\[ P\left\{ \sup_{(x, t)} |L^i_n(w_n) - L^i_n(U_n)| > \alpha \phi_n \right\} \leq c_i \exp\{-c_2 (\log n)^{1+\varepsilon}\}. \]

It would be of some technical interest to see whether or not the \( \varepsilon \) can be dropped, for \( \alpha \) large enough. In the case of Poisson process (i.e., the unconditioned case), the answer is in the affirmative (see Khoshnevisan [8, Theorem IV.8]).
(2) In this particular embedding, the term \( n^{1/4} \) cannot be improved upon. One can show that the stochastic process \( \{n^{1/4}(L_n^x(w_n) - L_n^x(U_n))\}_{1 \geq t > 0} \) (x being held fixed,) converges in \( D(0, 1) \) to another process \( \{w_\infty(L^x_t)\}_{1 \geq t > 0} \) where \( L^x_t \) is the local time of a Brownian bridge, \( w_0 \), at \( x \) up to time \( t \), and \( w_\infty \) is a Brownian bridge independent of \( w_0 \). For the analogous result for random walks and Brownian motion, see Borodin [2]. It would be interesting to find out whether our embedding is optimal; this would be the case only if all other embeddings give rise to approximations that are not more accurate.

(3) Without the supremum in the space variable (i.e., \( x \)) the exact rate of growth in Theorem 2 is \( n^{-1/4}(\log \log n)^{1/4}(\log n)^{1/2} \) (see Khoshnevisan [9, Chapter 5]).

4. Intersection of empirical processes

Dwass [5] has posed and solved the following interesting question:

"How often do the paths of two independent empirical processes from the same distribution cross?"

Here we have to precisely describe what we mean by a 'crossings,' since the set of times when two independent empirical processes cross, is not only not discrete but also has positive Lebesgue measure. We shall call it a crossing (see below) essentially every time the flat parts of the two empirical distribution functions meet.

This question is closely related to the problem dealt with in Theorem 2, namely the level crossings of the uniform empirical process. In order to state the main result of the section, we need some further notation.

Let \( \xi_1, \xi_2, \ldots \) and \( \xi_1, \xi_2, \ldots \) be two totally independent copies of an infinite sequence of i.i.d. \( U(0, 1) \) random variables. Define the empirical c.d.f.'s

\[
F_n(t) = \frac{1}{n} \sum_{j=1}^{n} I\{\xi_i \leq t\}, \quad G_n(t) = \frac{1}{n} \sum_{j=1}^{n} I\{\xi_i \leq t\}.
\]

Define the corresponding empirical processes

\[
U_n(t) = n^{1/2}[F_n(t) - t], \quad V_n(t) = n^{1/2}[G_n(t) - t].
\]

Notice that \( U_n(t) - V_n(t) = x \) if and only if \( F_n(t) - G_n(t) = n^{-1/2}x \). So we can either look at level crossings of \( U_n - V_n \) or those of \( F_n - G_n \). Define the counting process, \( C_n \), as

\[
C_n(x, t) = \sum_{j=1}^{n} I\{F_n(\xi_j) - G_n(\xi_j) = n^{-1/2}x\} + \sum_{j=1}^{n} I\{F_n(\xi_j) - G_n(\xi_j) = n^{-1/2}x\} = \sum_{j=1}^{n} I\{U_n(\xi_j) - V_n(\xi_j) = x\} + \sum_{j=1}^{n} I\{U_n(\xi_j) - V_n(\xi_j) = x\}.
\]

In other words, for every time a flat part of \( U_n - V_n \) hits \( x \), we count that as an \( x \)-crossing. We now state and prove the result of this section on path intersections.
of the uniform empirical process. The extension of this result to empirical process from other continuous one-dimensional distributions via distribution-free methods is standard.

**Theorem 5.** There exists a suitable probability space, carrying $U_n$, $V_n$, and a sequence of Brownian bridges, $II_n$, such that for all $\varepsilon > 0$, the following holds with probability one:

$$\lim_{n \to \infty} n^{1/4}(\log n)^{-3/4-\varepsilon} \sup_{x} \sup_{0 \leq t \leq 1} |n^{-1/2} C_n(x, t) - \sqrt{2} L_{\varepsilon}^{x} (II_n)| = 0.$$  

**Proof.** Pick a probability space rich enough to carry two independent Brownian motions, $\{B(t); t \geq 0\}$, and $\{p(t); t \geq 0\}$. The construction of Khoshnevisan [8] gives us two independent compensated Poisson processes, $\{Z_n(t); t \geq 0\}$ and $\{P_n(t); t \geq 0\}$, of expected rate $1/n$. Define the crossing process associated with the Poissonized intersection process, $Z_n - P_n$, in analogy to the definition of $C_n$, and denote it by $Q_n(x, t)$. Going through all the steps in the proof of the mentioned construction, we see that the Poissonized version of our theorem holds, i.e.,

$$\lim_{n \to \infty} n^{1/4}(\log n)^{-3/4-\varepsilon} \sup_{x} \sup_{0 \leq t \leq 1} |n^{-1/2} Q_n(x, t) - L_{\varepsilon}^{x} (B_n - \beta_n)| = 0,$$  

where $\beta_n$ and $B_n$ are the corresponding scaled processes, $\{n^{-1/2} \beta(nt)\}_{t \geq 0}$ and $\{n^{-1/2} B(nt)\}_{t \geq 0}$. Observe that the stochastic process

$$\{W_n(t)\}_{t \geq 0} = \{2^{-1/2} [B_n(t) - \beta_n(t)]\}_{t \geq 0}$$

is a Brownian motion with local time

$$L_{\varepsilon}^{x} (W_n) = 1/\sqrt{2} L_{\varepsilon}^{x} (B_n - \beta_n),$$

as can be checked out from the definition of local times as occupation densities. Therefore (4.1) implies,

$$\lim_{n \to \infty} n^{1/4}(\log n)^{-3/4-\varepsilon} \sup_{x} \sup_{0 \leq t \leq 1} |n^{-1/2} Q_n(x, t) - \sqrt{2} L_{\varepsilon}^{x} (W_n)| = 0.$$  

At this point, a 'conditioning' argument very similar to the proof of our Theorem 2 furnishes the remainder of the proof. $\square$

In particular, we recover Theorem 2 of Dwass [5] as a simple consequence. Namely (see display (1.2))

$$\lim_{n \to \infty} P\{C_n(0, 1)/\sqrt{2n} \geq x\} = P\{L_{1}^{x} (W) \geq x\} = \exp\{-\frac{1}{2} x^2\}, \quad x \geq 0.$$  

(Recall that $W_1$ is a standard 1-dimensional Brownian bridge.)

**Remarks and open problems**

1. A satisfactory and thorough description of crossings of two independent empirical processes that arise from two different distributions remains an open problem. A partial solution to this can be found in Nair et al. [11], where they study the number of zero-crossings before time one, and some other related functionals of such empirical processes.
The following are two open problems of Shorack and Wellner [15] that still need to be addressed:

2. Find \( \limsup_{n \to \infty} a_n \sup_x L^*_n(U_n) \) for an appropriate choice of \( a_n \). In view of Révész's result mentioned in the introduction, \( a_n \) is probably of form \( (\log \log n)^{-1/2} \).

3. Find a functional iterated logarithm law for \( x \to a_n L^*_n(U_n) \).

We would like to mention a last open problem regarding the asymptotic behavior of crossings:

4. Find \( \lim \inf_{n \to \infty} b_n \sup_x L^*_n(U_n) \). In light of a theorem of Kesten [10], we expect \( b_n \) to be of form \( (\log \log n)^{1/2} \).

5. Further remarks

It can be shown that in our construction of \( U_n \) and \( w_n \) (recall Section 3) we also can get a strong central limit theorem:

\[
\sup_{0 \leq t \leq 1} \left| w_n(t) - U_n(t) \right| = O(n^{-1/4}(\log \log n)^{1/4}(\log n)^{1/2}).
\]  

This is the same rate of convergence to Brownian bridge as that of the so-called Brillinger process (see Shorack and Wellner [15]) with a different construction. As a result of this work, one gets a weak convergence theorem that strengthens the usual central limit theorem for the empirical process. To state this, however, we need some definitions which we mention rather informally.

Let \( X, T \) be intervals which may or may not be bounded. Then define the space \( D_0(X \times T) \) as the space of real functions, \( f: X \times T \to \mathbb{R} \), such that the map \( x \to f(x, \cdot) \) is left-continuous with right limits, and the map \( t \to f(\cdot, t) \) is right-continuous with left limits. Endow this space with compact-open topology induced by the product topology of \( X \times T \), i.e., for \( f_n \in D_0(X \times T) \), \( n = 1, 2, \ldots, \infty \), we say \( f_n \) converges to \( f_\infty \) if and only if

\[
\lim_{n \to \infty} \sup_{x \in K} \sup_{t \in I} |f_n(x, t) - f_\infty(x, t)| = 0,
\]

where \( I \) is any bounded interval in \( T \), and \( K \) is any compact subset of \( X \). Let \( D_0 \) abbreviate \( D_0(\mathbb{R} \times [0, 1]) \). Then Theorem 2 and (5.1) together imply the following weak convergence statement:

\( U_n \) be an empirical process from a continuous distribution function that is strictly increasing. Then, in the sense of \( D([0, 1]) \times D_0 \) (see Billingsley [1] for a definition of \( D([0, 1]) \)), the random vector-valued process \( (U_n, L(U_n)) \) converges weakly to \( (w, L(w)) \), where \( w \) is a Brownian bridge, and \( L \) is the local time operator.

This can easily be seen by the fact that, as elements of \( D([0, 1]) \times D_0 \), the law of \( (w_n, L(w_n)) \) is independent of \( n \). It should also be noted that similar results hold using Theorem 5, regarding path intersections.
Appendix

In this appendix, we briefly present a few facts about the local time of a standard Brownian bridge. The main idea here is that a Brownian bridge is simply a Brownian motion that is rescaled in both space and time, by a random amount. Namely the following well-known fact (for part (a) see Revuz and Yor [14, chapter on excursion theory]; also Khoshnevisan [9] has a proof involving weak convergence):

Lemma A.1. (a) For a standard Brownian motion, \( \{B(t)\}_{t \geq 0} \), the stochastic process, \( \{w(t)\}_{t \geq 0} \), defined by

\[
w(t) = g^{-1/2}B(gt),
\]

is a standard Brownian bridge, independent of \( g \), where \( g \) is the last hitting time of zero before time 1, i.e.,

\[
g = \sup\{s \leq 1: B(s) = 0\}.
\]

(b) For \( w \) defined as above, it has a jointly continuous local time given by

\[
L^*_\gamma(w) = g^{-1/2}L^*_\gamma(B).
\]

It is clear that the above lemma implies the following proposition, which is otherwise hard to prove. This is a strong extension of a result in Billingsley [1].

Proposition A.2. Let \( P^0 \) be the joint law of the random vector process \( (w, L(w)) \) on the space \( \mathcal{D} \times \mathcal{D}_0 \), Then for all \( P^0 \)-continuity sets \( S \), as \( \delta \to 0 \),

\[
P\{(B, L(B)) \in S \mid B(1) \leq \delta\} \to P^0\{S\}. \quad \square
\]

This result states a fact that is intuitively clear, namely that a standard Brownian bridge jointly with its local time, is a standard Brownian motion conditioned to hit 0 at time 1, jointly with its local time. The reason this result is not immediate from standard weak convergence theory (it is, if we only look at \( B \) and \( w \), without their local times) is that local times, as functions of paths are not at all continuous.

Using the above proposition and a result of Trotter [17], one can then say something about the modulus of continuity of the local time of a standard Brownian bridge, in the space variable. For example (we do not include the strongest form) it follows that:

Proposition A.3. Let \( L^*_\gamma \) denote the local time of Brownian bridge \( w \) described above. Then as \( \delta \to 0 \),

\[
\sup_{0 \leq t \leq 1} \sup_{\|x-y\| = \delta} |L^*\gamma_t - L^*\gamma_{t'}| = O(\sqrt{\delta \log(1/\delta)}).
\]

Many other results about Brownian bridge local time follow trivially from Proposition A.2 and the corresponding result for Brownian motion. See Révész [13].
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