On Characterizing *N*-Matrices Using Linear Complementarity

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ABSTRACT

We show that a square matrix A with at least one positive entry and all principal minors negative can be characterized in terms of the number of solutions the linear complementarity problem (q, A) with the matrix A has for different vectors q. Such a matrix can also be characterized in terms of a sign nonreversal property. These results complement the known results for a square matrix $A < 0$ with all its principal minors negative.

1. INTRODUCTION

We call a square matrix A of order n a *P-matrix* if all the principal minors of A are positive. A is called an *N-matrix* if all its principal minors are negative. An N-matrix A is said to be of the *first category* if it has at least one positive element. Otherwise it is said to be of the *second category*. In fact, it is known that if A is an N-matrix of the first category, then each row and column of A has a positive entry. See [11, p. 58]. N-matrices arise in the theory of global univalence of functions, [2, 10], in multivariate analysis [7], and in linear complementarity problems [11, 3].

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Given a square matrix A of order n, and a vector $q \in R^n$, the *linear complementarity problem* is to find vectors w, $z \in R^n$ such that

$$w - Az = q, \tag{1}$$

$$w \geqslant 0, \qquad z \geqslant 0, \tag{2}$$

$$w^t z = 0. \tag{3}$$

We denote this problem by (q, A). A pair (w, z) of vectors satisfying (1) to (3) is called a *solution* to (q, A).

A square matrix B of order n whose jth column $B_{.j}$ is either $-A_{.j}$ or $I_{.j}$ for $1 \leqslant j \leqslant n$ is called a *complementary matrix* of $[I:-A]$. The cone generated by a complementary matrix B,

$$\text{pos}(B) = \{z : z = By,\ y \geqslant 0\}, \tag{4}$$

is called a complementary cone. A well-known result in the theory of the linear complementarity problem is the following characterization of P-matrices due to Samelson, Thrall, and Wesler [12].

THEOREM 1.1. *A is a P-matrix if and only if the complementary cones of $[I:-A]$ partition R^n, or equivalently, A is a P-matrix if and only if (q, A) has a unique solution for each $q \in R^n$.*

The linear complementarity problem with A as an N-matrix has earlier been studied by Saigal [11] and Kojima and Saigal [3]. They prove in [3] the following theorem.

THEOREM 1.2. *If A is an N-matrix of the second category, then (q, A) has exactly two solutions for any $q > 0$ and no solution for any $q \ngeqslant 0$. If A is an N-matrix of the first category, then for any $q \ngeqslant 0$, (q, A) has a unique solution, and for any $q > 0$ which is nondegenerate with respect to A, (q, A) has exactly three solutions.*

However, until recently there has been no published proof of the converse, viz., a characterization of N-matrices using the number of solutions to (q, A). Recently, Parthasarathy and Ravindran [9] proved the following.

THEOREM 1.3. *Let $A < 0$. Then A is an N-matrix if and only if (q, A) has exactly two solutions for each $q > 0$.*

Another characterization of N-matrices of the second category is given by Maybee [4].

A main result proved in this paper is the converse of the Kojima-Saigal result for N-matrices of the first category. This is posed as an open problem in [9]. As a consequence, we also obtain a characterization of N-matrices of the first category in terms of a sign nonreversal property (such a characterization of N-matrices of the second category has been proved in [9]; see Theorem 2.2 in Section 2 below).

2. PRELIMINARIES

The (i,j)th entry of a matrix A is denoted by a_{ij}. Let J and K be subsets of $\{1,2,\dots,n\}$. The matrix A_{JK} denotes the submatrix of A with row and column indices in J and K respectively, arranged in the natural order. If $J=\{1,2,\dots,n\}$, then we write A_{JK} as $A_{\cdot K}$.

We say that a square matrix A of order n *reverses the sign* of a vector $x \in R^n$ if $x_i(Ax)_i \leqslant 0$ $\forall i=1,\dots,n$. We say that x is *unisigned* if $x_i \leqslant 0$, $1 \leqslant i \leqslant n$, or $x_i \geqslant 0$, $1 \leqslant i \leqslant n$.

If $J \subseteq \{1,2,\dots,n\}$, then $\bar{J}=\{1,2,\dots,n\} \setminus J = \{i : 1 \leqslant i \leqslant n,\ i \notin J\}$. For any two subsets J and K of $\{1,2,\dots,n\}$, $J \vartriangle K=(J \cap \bar{K}) \cup (\bar{J} \cap K)$ is the symmetric difference between J and K. For $J \subseteq \{1,2,\dots,n\}$, $x \in R^n$, $x=(x_J, x_{\bar{J}})^t$ denotes the partitioned form of x after a suitable permutation of its indices. R^n_+ denotes the nonnegative orthant, i.e., $R^n_+ = \{x \in R^n : x_i \geqslant 0,\ 1 \leqslant i \leqslant n\}$.

Given a square matrix A of order n, the pair of column vectors $\{-A_{\cdot j}, I_{\cdot j}\}$ for $1 \leqslant j \leqslant n$ is called a complementary pair. Let C be a matrix formed by taking one column each from any $n-1$ complementary pairs of columns of A. If rank $C=n-1$, then we call pos(C) an $(n-1)$-face. In what follows, we assume that the principal minors of A are all nonzero.

Let $F=\text{pos}(C)$ be an $(n-1)$-face. We say that a complementary cone pos(B) is *incident on* F if all the columns of C are columns of B also. Hence any $(n-1)$-face F has exactly two complementary cones incident on it. Under our assumption about the principal minors of A, it then follows that all the complementary matrices are nonsingular, and the subspaces generated by their $(n-1)$-faces are hyperplanes. We may then talk of the complementary cones incident on an $(n-1)$-face F, lying on the same side or opposite side of F. We say that the two cones incident on an $(n-1)$-face F are *properly situated* if they lie on opposite sides of F. We say that F is a *proper face* if either the two complementary cones incident on F are properly situated, or F is on the boundary of the set

$$D(A)=\{q \in R^n : (q,A) \text{ has a solution}\}. \tag{5}$$

These notions are due to Saigal [11]. Saigal proves the following theorem [11].

THEOREM 2.1. *Let A be an N-matrix. If it is of the second category, then all the $(n-1)$-faces are properly situated; if it is of the first category, then all the $(n-1)$-faces other than those of pos(I) are properly situated.*

DEFINITION (Principal pivot transform). Let B be a nonsingular complementary matrix. Note that we can write the n by $2n$ matrix $[I: -A]$ as $[B:\overline{B}]$, where $\overline{B}$ is the matrix of columns $[I: -A]$ not in B. We can transform the original problem (q, A) to an equivalent problem $(\overline{q}, \overline{A})$, where $\overline{A} = -B^{-1}\overline{B}$ and $\overline{q} = B^{-1}q$. The matrix $\overline{A}$ is then called a *principal pivot transform* of A with respect to the complementary matrix B.

Let $J = \{j: -A_{.j}$ is a column of $B\}$. We then rewrite B (if necessary with a rearrangement of rows and columns) as

$$B(J) = \begin{bmatrix} -A_{JJ} & 0 \\ -A_{\bar{J}J} & I_{\bar{J}\bar{J}} \end{bmatrix}, \tag{6}$$

where if $J = \emptyset$, then $B(J) = I$. We then have the following lemma which relates the determinant of a principal submatrix of A to the determinant of a principal submatrix of $\overline{A}$.

LEMMA 2.1. *Let $B(J)$ be a nonsingular complementary matrix with the index set as defined before. Let $\overline{A}$ be the principal pivot transform of A with respect to $B(J)$. Then for any $K \subseteq \{1,2,\ldots,n\} \neq J$,*

$$\det \overline{A}_{KK} = \det A_{K\vartriangle J, K\vartriangle J} / \det A_{JJ}. \tag{7}$$

See [1], [8], and [14].

We say that A is a Q-matrix if (q, A) has a solution for all $q \in R^n$. Notice that if A is a Q-matrix and $\overline{A}$ is a principal pivot transform of A, then $\overline{A}$ is also a Q-matrix.

The sign nonreversal property for N-matrices of the second category given by Parthasarathy and Ravindran [9] is as follows.

THEOREM 2.2 (Theorem 2 of [9]). *Let A be a square matrix of order n with $a_{ij} < 0$ for all i, j. Then the following are equivalent:*

(i) *A is an N-matrix.*
(ii) *A does not reverse the sign of any non-unisigned vector.*

3. SIGN PATTERN OF N-MATRICES

We note that if A is an N-matrix, then no entry of A can be zero. The following lemma is due to Ravindran [10]; see also [9].

LEMMA 3.1. *Let A be a square matrix of order n, whose principal minors of order 3 or less are negative. Then there is a diagonal matrix S whose diagonal entries are either $+1$ or -1 such that*

$$SAS < 0. \tag{8}$$

This lemma determines the sign pattern of entries of an N-matrix. To obtain an explicit form, we define the following.

DEFINITION. Let $x, y \in R^n$ have nonzero coordinates; we say that x and y have the same sign pattern if $x_i y_i > 0$ $\forall i = 1, \dots, n$. If x and y have the same sign pattern, they are said to be *sign equivalent*.

We have the following lemma on the sign equivalence of columns of a matrix A.

LEMMA 3.2. *Let A be a square matrix of order n, $n \geqslant 3$, with all of its principal minors of order 3 or less negative. Then sign equivalence is an equivalence relation on the set of columns of A which partitions the columns of A into two classes, J and $\bar{J}$, where $J = \{i : a_{11} a_{1i} > 0,\ 1 \leqslant i \leqslant n\}$.*

Proof. Let $A_{\cdot i}$ and $A_{\cdot k}$ be two columns of A. Suppose $a_{ii} a_{ik} > 0$. Then clearly, $a_{ik} < 0$, and considering the 2 by 2 principal submatrix

$$\begin{bmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{bmatrix}, \tag{9}$$

we see that $a_{ki} < 0$. Thus $a_{ki} a_{kk} > 0$. We now claim that $a_{ji} a_{jk} > 0$ $\forall j$.

Suppose for some $r \neq i$ or k, $a_{ri}a_{rk} < 0$. Consider the 3 by 3 principal submatrix

$$\begin{bmatrix} a_{ii} & a_{ir} & a_{ik} \\ a_{ri} & a_{rr} & a_{rk} \\ a_{ki} & a_{kr} & a_{kk} \end{bmatrix}. \tag{10}$$

The sign pattern of this matrix is either

$$\begin{bmatrix} - & + & - \\ + & - & - \\ - & - & - \end{bmatrix} \text{ or } \begin{bmatrix} - & - & - \\ - & - & + \\ - & + & - \end{bmatrix}, \tag{11}$$

according as $a_{ri} > 0$ or $a_{ri} < 0$. But these are not the sign patterns of an N-matrix of order 3: see Parthasarathy and Ravindran [9]. Hence if $a_{ii}a_{ik} > 0$, then $a_{ji}a_{jk} > 0\ \forall j$. Similarly, we can show that if $a_{ii}a_{kk} < 0$, then $a_{ji}a_{jk} < 0$ $\forall j$, for any $i, k \in \{1, 2, \ldots, n\}$.

Now consider the index set

$$J = \{i : a_{11}a_{1i} > 0,\ 1 \leqslant i \leqslant n\}. \tag{12}$$

J is nonempty, and it follows that all the columns of A whose indices are in J are sign equivalent. The index sets J and $\bar{J}$ induce the desired partition of the columns of A. If J is empty, then $A < 0$. ∎

REMARK 3.1. It follows from the above lemma that if A is a square matrix of order $n > 3$, all of whose order 3 or less principal minors are negative, then A can be written in the partitioned form

$$A = \begin{bmatrix} A_{JJ} & A_{J\bar{J}} \\ A_{\bar{J}J} & A_{\bar{J}\bar{J}} \end{bmatrix}, \tag{13}$$

where the index set J is as given in (12), $A_{JJ} < 0$, $A_{\bar{J}\bar{J}} < 0$, and $A_{J\bar{J}}$ and $A_{\bar{J}J}$ are matrices with all positive entries. This representation also specifies immediately the matrix S in (8).

The following theorem is about the number of solutions to (q, A), when A is an N-matrix of the first category and $q \in R^n_+$. This result has been observed by Kojima and Saigal [3]. But here, we give a different proof.

THEOREM 3.1. *Let A be an N-matrix of the first category. Then for each $q > 0$, (q, A) has exactly three solutions. If q is on a face of* pos(I), *then (q, A) has at most two solutions.*

Proof. Let J be as defined in (12). Let the matrix A be partitioned as in Remark 3.1.

Consider any $q > 0$, and let $q = (q_J, q_{\bar{J}})^t$. As A_{JJ} is an N-matrix of the second category, from [3] and [9] we see that there is exactly one solution (w_J^*, z_J^*) to the subproblem (q_J, A_{JJ}) in which $z_J^* \neq 0$. Now define $\bar{w} \in R^n$, $\bar{z} \in R^n$ by

$$\begin{aligned} \bar{w}_J &= w_J^*, & \bar{z}_J &= z_J^*, \\ \bar{w}_{\bar{J}} &= q_{\bar{J}} + A_{\bar{J}J} z_J^*, & \bar{z}_{\bar{J}} &= 0. \end{aligned} \tag{14}$$

It is easy to see that $(\bar{w}, \bar{z})$ solves (q, A), and $\bar{z}_J \neq 0$. Since (w_J^*, z_J^*) uniquely determines $\bar{w}_{\bar{J}}$, there is exactly one solution to (q, A) with $z_J \neq 0$, $z_{\bar{J}} = 0$.

By a similar argument, we can show that there is exactly one solution (u, v) in which $v_J = 0$ and $v_{\bar{J}} \neq 0$. In addition, we have the trivial solution $w = q$, $z = 0$, and hence we have three solutions to (q, A).

To show that (q, A) has exactly three solutions, we show that there is no solution (x, y) to (q, A) in which y_J and $y_{\bar{J}}$ are nonzero. Suppose on the contrary, there is a solution (x, y) to (q, A) with $y_J \neq 0$ and $y_{\bar{J}} \neq 0$. Let $L = \{s : y_s > 0,\ 1 \leqslant s \leqslant n\}$. By our hypothesis, $L \cap J \neq \varnothing$ and $L \cap \bar{J} \neq \varnothing$. Therefore the principal submatrix A_{LL} is an N-matrix of the first category. Further we have

$$- q_L = A_{LL} y_L, \tag{15}$$

which contradicts a well-known property of N-matrices of the first category due to Inada (see Nikaido [6, p. 362]).

If q is contained in a face of pos(I), then the above arguments show that there are at most two solutions to (q, A). This completes the proof. ■

REMARK 3.2. Let us define two classes of complementary cones of A. Let

$$\begin{aligned} \mathbf{C}_1 = \{&\mathrm{pos}(B) : B \text{ is a complementary matrix of } [I : -A] \\ &\text{with } B_{\cdot k} = I_{\cdot k}\ \forall k \in \bar{J}\}, \\ \mathbf{C}_2 = \{&\mathrm{pos}(B) : B \text{ is a complementary matrix of } [I : -A] \\ &\text{with } B_{\cdot k} = I_{\cdot k}\ \forall k \in J\}. \end{aligned} \tag{16}$$

Geometrically Theorem 3.1 shows that the complementary cones in $\mathbf{C}_1$ other than pos(I), intersected with R^n_+, make a partition of the positive orthant [if there is only one complementary cone in $\mathbf{C}_1$, it covers the whole of pos(I) when intersected with R^n_+]. So also for the cones in $\mathbf{C}_2$.

REMARK 3.3. The above theorem corrects a wrong assertion in the statement of Theorem 3.3 of Kojima and Saigal [3], which claims that the number of solutions to (q, A) when $q > 0$ is degenerate with respect to A is two. This mistake has also been pointed out by Stone [13].

REMARK 3.4. Theorem 3.4 in Kojima and Saigal [3], on the number of solutions to (q, A) when q is contained in a face of pos(I), is also wrong. It asserts that the number of solutions is exactly two when $q \geqslant 0$ with $q_i = 0$ for at least one i, $i = 1, \ldots, n$. The following example shows that this need not be so.

EXAMPLE.

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

is an N-matrix of the first category. Here, for $q = (0, 0, 1)^t$, (q, A) has a unique solution.

4. SOME CHARACTERIZATION THEOREMS FOR N-MATRICES

In this section we prove some theorems characterizing N-matrices. The first theorem is a converse of Kojima and Saigal's result [3] on the number of solutions to (q, A) when A is an N-matrix of the first category. We start with two lemmas.

LEMMA 4.1. *Let F be an $(n-1)$-face on which* pos(B) *and* pos(B^1) *are incident. Let* $\det B \neq 0$. *Then the complementary cones* pos(B) *and* pos(B^1) *are properly situated on F if and only if*

$$\det B^1 / \det B \leqslant 0. \tag{17}$$

This is from Lemma 5.1 of Saigal [11].

LEMMA 4.2. *Suppose X is a square matrix of order n with nonzero principal minors. Let the two complementary cones incident on any* $(n-1)$-*face which is not a face of* $\mathrm{pos}(-X)$ *be properly situated. Then all the proper principal minors of X are positive.*

Proof. The proof is by induction on the order of the principal minors of X. We first show that all the principal minors of order 1 of X are positive. To show that $x_{jj}>0$, $1\leqslant j\leqslant n$, consider

$$\mathrm{pos}(B^1)=\mathrm{pos}(I_{\cdot 1},\dots,I_{\cdot j-1},-X_{\cdot j},I_{\cdot j+1},\dots,I_{\cdot n})$$

and

$$\mathrm{pos}(B)=\mathrm{pos}(I).$$

Since these two cones are properly situated on the $(n-1)$-face,

$$F=\mathrm{pos}(I_{\cdot 1},\dots,I_{\cdot j-1},I_{\cdot j+1},\dots,I_{\cdot n});$$

using Lemma 4.1, it follows that

$$\det B^1/\det B<0,$$

which implies

$$\det B^1=-x_{jj}<0,\quad \text{or}\quad x_{jj}>0,\qquad 1\leqslant j\leqslant n. \tag{18}$$

Let us assume that all the principal minors of order r $(r\leqslant n-1)$ of X are positive; consider a submatrix X_{JJ} of order $r+1$. Let $s\in J$ and $L=J\setminus\{s\}$; consider the two cones

$$\mathrm{pos}(B^1)=\mathrm{pos}\{-X_{\cdot j},\ j\in J;\ I_{\cdot j},\ j\notin J\},$$

$$\mathrm{pos}(B)=\mathrm{pos}\{-X_{\cdot j},\ j\in L;\ I_{\cdot j},\ j\notin L\}$$

and the face

$$F=\mathrm{pos}\{-X_{\cdot j},\ j\in L;\ I_{\cdot j},\ j\notin L \text{ and } j\neq s\}.$$

Since pos(B) and pos(B^1) are properly situated on F, it follows from (17) that

$$\det B^1/\det B < 0.$$

Now $\det B = (-1)^r \det X_{LL}$, and by induction, $\det X_{LL} > 0$. Hence $\det B < 0$ if r is odd, and $\det B > 0$ if r is even. It follows that $\det B^1 > 0$ if r is odd, and $\det B^1 < 0$ if r is even. Since $\det B^1 = (-1)^{r+1} \det X_{JJ}$, it is clear that $\det X_{JJ} > 0$ in either case. The proof is complete. ∎

The following theorem characterizes N-matrices of the first category in terms of the number of solutions to (q, A), for $q \in R^n$.

THEOREM 4.1. *Let A be a square matrix of order n, each column of which contains a positive entry. Suppose (q, A) has a unique solution whenever $q \not\geqslant 0$ and a finite number of solutions whenever $q \geqslant 0$, with more than one solution for at least one $q > 0$. Then A is an N-matrix of the first category.*

Proof. Since (q, A) has a finite number of solutions for any $q \in R^n$, it follows that none of the principal minors of A are zero. See K. G. Murty [5].

We shall show that, if F is an $(n-1)$-face which is not a face of pos(I), then the two complementary cones of A incident on F are properly situated. Suppose not. Let F be an $(n-1)$-face generated by k columns of I and $n-k-1$ columns of $-A$, $1 \leqslant k \leqslant n-2$, such that the two complementary cones pos(B) and pos(B^1) incident on it lie on the same side of F.

If $F \subseteq \text{pos}(I)$, then for some r, $-A_{.r}$, which is in the set of columns generating F, is in pos(I), contrary to the hypothesis. Hence $F \not\subseteq \text{pos}(I)$. Suppose the complementary pair of vectors left out in generating F are $-A_{.s}$ and $I_{.s}$. Since pos(B) and pos(B^1) lie on the same side of F and $F \not\subseteq \text{pos}(I)$, we can find a $q \in F$, $q \notin \text{pos}(I)$, and an $\varepsilon > 0$ such that

$$q + \varepsilon(-A_{.s}) \in \text{pos}(B) \cap \text{pos}(B^1).$$

But $q + \varepsilon(-A_{.s}) \not\geqslant 0$ and $(q + \varepsilon(-A_{.s}), A)$ has at least two solutions, which contradicts our hypothesis. Hence our assertion follows.

Let $X = A^{-1}$. By Lemma 6.4 of Saigal [11], it follows that if F is an $(n-1)$-face of $[I : -X]$ other than pos($-X$), then the two complementary cones incident on it are properly situated. We note from Lemma 4.2 that all the proper principal minors of X are positive.

Now if $\det X > 0$, then X, and hence $X^{-1} = A$, is a P-matrix, which contradicts our hypothesis about the number of solutions to (q, A) for $q > 0$.

Hence $\det X < 0$. From Lemma 2.4 of Kojima and Saigal [3], it follows that A is an N-matrix, and it is of the first category. ∎

THEOREM 4.2. *Let A be a square matrix of order n, with each column of A having at least one positive entry. A is an N-matrix of the first category if and only if (q, A) has a unique solution for all $q \ngeq 0$, exactly three solutions for all $q > 0$, and at most two solutions for any other $q \in R^n_+$.*

Proof. This follows from the results of Kojima and Saigal [3] and our Theorem 4.1. ∎

The next theorem gives a characterization of N-matrices of the first category in terms of a sign nonreversal property.

THEOREM 4.3. *A square matrix A of order n is an N-matrix of the first category if and only if*

(i) *A can be written in the partitioned form (after a principal rearrangement of its rows and columns, if necessary)*

$$\begin{bmatrix} A_{JJ} & A_{J\bar{J}} \\ A_{\bar{J}J} & A_{\bar{J}\bar{J}} \end{bmatrix}$$

with $A_{JJ} < 0$, $A_{\bar{J}\bar{J}} < 0$, $A_{J\bar{J}} > 0$, and $A_{\bar{J}J} > 0$ where $\varnothing \neq J \subseteq \{1, 2, \dots, n\}$; and

(ii) *A (as partitioned in (i)) reverses the sign of only the vectors of the form $(x_J, x_{\bar{J}})^t$ with either $x_J \leqslant 0$ and $x_{\bar{J}} \geqslant 0$ or $x_J \geqslant 0$ and $x_{\bar{J}} \leqslant 0$.*

Proof. This theorem easily follows from the sign nonreversal property proved in Theorem 2 of [9] by observing that A reverses the sign of a vector $x \in R^n$ if and only if SAS reverses the sign of Sx, where S is a diagonal matrix with diagonal entries as $+1$ or -1. However, our proof of this theorem is based on linear complementarity.

"Only if": Suppose A is an N-matrix of the first category; then by Lemma 3.2, A has the partition specified above (after a principal rearrangement of rows and columns, if necessary) where J is as defined in (12). Thus condition (i) follows. It is clear that the partitioned form of A reverses the sign of all vectors $(x_J, x_{\bar{J}})^t$ if either $x_J \leqslant 0$ and $x_{\bar{J}} \geqslant 0$ or $x_J \geqslant 0$ and $x_{\bar{J}} \leqslant 0$. To show that A does not reverse the sign of any other vector, we proceed as follows.

Suppose A reverses the sign of x, where x_J and $x_{\bar{J}}$ are nonnegative with at least one coordinate in x_J and one coordinate in $x_{\bar{J}}$ positive. Consider the index set $L=\{i: x_i>0,\ 1\leqslant i\leqslant n\}$. We have $L\cap J\neq\varnothing$, $L\cap\bar{J}\neq\varnothing$.

Let $(Ax)_L=q_L=A_{LL}x_L$. Note that $q_L\leqslant 0$. Note also that A_{LL} is an N-matrix of the first category. Thus we arrive at a contradiction to the result of Inada cited in connection with (15). See [6, p. 362].

The only other possibility to be considered is the possibility of A reversing the sign of a vector of mixed signs in x_J and $x_{\bar{J}}$. Let the sign of x where x_J has both a positive and a negative coordinate be reversed by A. Let

$$x_i^+=\begin{cases} x_i & \text{if}\quad x_i>0,\\ 0 & \text{otherwise}\end{cases}$$

$$x_i^-=\begin{cases} -x_i & \text{if}\quad x_i<0,\\ 0 & \text{otherwise}.\end{cases}$$

Now $x=x^+-x^-$, and we see that with $u=Ax$,

$$u^+-Ax^+=u^--Ax^-=\bar{q}.$$

Thus $(\bar{q},A)$ has two distinct solutions, (u^+,x^+) and (u^-,x^-), as $x^+\neq x^-\neq 0$. There are two cases:

Case (i): $\bar{q}\ngeqslant 0$. We have a contradiction to the result that for such a $\bar{q}$, $(\bar{q},A)$ has a unique solution.

Case (ii): $\bar{q}\geqslant 0$. If $x_{\bar{J}}=0$, then we have at least three solutions to $(\bar{q}_J,A_{JJ})$, a contradiction to Theorem 1.2. If $x_{\bar{J}}\neq 0$, then we have a contradiction to Theorem 3.1.

Similarly, we can show that A does not reverse the sign of a vector $(x_J,x_{\bar{J}})^t$ when $x_{\bar{J}}$ has both a positive and a negative coordinate. This completes the proof of the "only if" part.

"If": Suppose A can be partitioned as in (i) and A does not reverse the sign of any nonzero vector $x=(x_J,x_{\bar{J}})^t$ except when $x_J\geqslant 0$ and $x_{\bar{J}}\leqslant 0$ or when $x_J\leqslant 0$ and $x_{\bar{J}}\geqslant 0$. By taking either $x_J=0$ or $x_{\bar{J}}=0$ we see from Theorem 2 of [9] that this implies A_{JJ} and $A_{\bar{J}\bar{J}}$ are N-matrices of the second category. Let $\mathbf{C}_1,\mathbf{C}_2$ be the classes of complementary cones of $[I:-A]$, as defined in (16). Then by the proof of Theorem 3.1, any $q>0$ is contained in exactly one complementary cone from $\mathbf{C}_1$ other than pos(I). We now show that for such a $q>0$, there is no solution (w,z) in which $z_i>0$ for some

$i \in J$ and $z_k > 0$ for some $k \in \bar{J}$. Suppose this is not true. Let

$$L = \{k : z_k > 0,\ 1 \leqslant k \leqslant n\},$$

and note that $L \cap J \neq \varnothing$, $L \cap \bar{J} \neq \varnothing$. Note also that

$$q_L = -A_{LL} z_L.$$

Define y by taking $y_L = z_L$; $y_{\bar{L}} = 0$. Note that A reverses the sign of y, contradicting our hypothesis. This contradiction shows that under our hypothesis about A, for any $q > 0$, (q, A) has exactly three solutions.

We now show that no principal subdeterminant of A (including $\det A$) is zero. Suppose not. Suppose $\det A_{LL} = 0$ for some set $L \subseteq \{1,2,\dots,n\}$; then there is a $0 \neq x \in R^{|L|}$ such that $A_{LL}x = 0$. Without loss of generality we may assume that no coordinate of x is 0. Let $y \in R^n$ be defined by taking $y_L = x$ and $y_{\bar{L}} = 0$. Then note that A reverses the sign of the vector y. Note also that

$$(Ay)_L = A_{LL} y_L + A_{L\bar{L}} y_{\bar{L}} = 0. \tag{19}$$

Suppose $y_J \leqslant 0$ and $y_{\bar{J}} \geqslant 0$. From the sign of A and the fact that at least one coordinate of either y_J or $y_{\bar{J}}$ is nonzero it follows that

$$(Ay)_{L\cap J} = A_{L\cap J L\cap J}\, y_{L\cap J} + A_{L\cap J L\cap \bar{J}}\, y_{L\cap \bar{J}} > 0,$$

contradicting (19). Similarly, the case $y_J \geqslant 0$, $y_{\bar{J}} \leqslant 0$ does not arise. Thus A reverses the sign of a vector y, contrary to our hypothesis. This contradiction shows that no principal subdeterminant of A is zero.

In particular, it follows that $(0, A)$ has a unique solution. Also the number of solutions to any (q, A) is finite.

Now consider any $q \ngeqslant 0$. Suppose (q, A) has a solution. We then claim that the solution is unique. Suppose not. Then let (w^1, z^1) and (w^2, z^2) be two distinct solutions to (q, A). Note that A reverses the sign of the vector $z^1 - z^2$. Suppose now $(z^1 - z^2)_J \leqslant 0$ and $(z^1 - z^2)_{\bar{J}} \geqslant 0$. From the sign pattern of A and the fact that $(w^1 - w^2)_J \geqslant 0$ and $(w^1 - w^2)_{\bar{J}} \leqslant 0$, it follows that $z^1_J = 0$ and $z^2_{\bar{J}} = 0$. Now it is easy to check that $q_{\bar{J}} = w^2_{\bar{J}} - A_{\bar{J}J} z^1_J - A_{\bar{J}\bar{J}} z^1_{\bar{J}} \geqslant 0$. Similarly, $q_J \geqslant 0$. This however contradicts our assumption about q. The claim is proved.

Let $\bar{q} = -Ae$. By our sign nonreversal hypothesis about A, it is easy to see that $\bar{q} \ngeqslant 0$. Moreover (q, A) has a solution $w = 0$, $z = e$. Hence by our previous argument the solution is unique. Now the facts that $(0, A)$ and $(\bar{q}, A)$ have unique solutions imply that A is a Q-matrix. It follows that (q, A) has a unique solution whenever $q \ngeq 0$. Thus we see that A satisfies all the hypotheses of Theorem 4.2 and hence is an N-matrix of the first category.

This concludes the proof. ■

We conclude this paper with a theorem characterizing an N-matrix based on the signs of diagonal entries in each of its principal pivot transforms. This is similar to a theorem characterizing P-matrices. See [8].

THEOREM 4.4. *Let A be a square matrix of order n. Then A is an N-matrix if and only if the following hold:*

(i) *All the diagonal entries of A are negative.*

(ii) *Let $\varnothing \neq J \subseteq \{1, 2, \dots, n\}$. Let $B(J)$ be as defined in* (6), *and let $\bar{A}(J)$ be the principal pivot transform of A with respect to $B(J)$. Then whenever $|J| \geqslant 2$, all the diagonal entries of $\bar{A}(J)$ are positive.*

Proof. "Only if": When A is an N-matrix, all the principal minors are negative—in particular, the diagonal entries. Hence we can take a principal pivot transform with respect to $B(J)$ for any $J \subseteq \{1, 2, \dots, n\}$. Condition (ii) now follows easily from (7).

"If": By hypothesis, all the diagonal entries are negative. Consider any 2 by 2 principal submatrix A_{LL} of A. Let $L = \{i, j\}$. Consider $J = L \setminus \{i\}$. Since the diagonal element a_{jj} is negative, we can take a principal pivot transform with respect to $B(J)$. Now let $K = \{i\}$; then using (7)

$$\det \bar{A}_{KK} = \det A_{K \triangle J, K \triangle J} / \det A_{JJ} = \det A_{LL} / \det A_{JJ}.$$

By hypothesis this is positive. Since $\det A_{JJ} < 0$, it follows that $\det A_{LL} < 0$. We can now complete the proof by induction on the order of the principal minors. ■

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