Dehn surgery and (1, 1)-knots in lens spaces

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Abstract

We give a necessary condition for Dehn surgery on (1, 1)-knots in lens spaces to yield the 3-sphere. As an application, we will give a partial answer for a conjecture given by Bleiler and Litherland.

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1. Introduction

Which knots in the 3-sphere $S^3$ admit Dehn surgery yielding lens spaces? This is one of the unsolved problems on study of Dehn surgery. Here, a lens space $L(p, q)$ is a 3-manifold obtained by $p/q$-surgery on a trivial knot in $S^3$ and is homeomorphic neither to $S^3$ nor to $S^2 \times S^1$. Note that $L(p, q) \cong L(-p, q) \cong L(p, -q)$. Hence we may assume $p > 1$ and $0 < q < p$. As usual, Dehn surgery is the following operation. Let $N$ be a compact 3-manifold and $K$ a knot in $N$. Let $m$ be a meridian of $K$ in $\partial \eta(K; N)$, where $\eta(B; A)$ is a regular neighborhood of $B$ in $A$. We fix a longitude $\ell$ in $\partial \eta(K; N)$ such that $m$ intersects $\ell$ transversely in a single point. When $N \cong S^3$, $\ell$ should be the preferred longitude of $K$ in $\partial \eta(K; N)$. In the following, we fix an orientation of $m$ and $\ell$ as illustrated in Fig. 1. Dehn surgery on $K$ is to attach a solid torus $\bar{V}$ to $E(K; N)$ by a boundary-homeomorphism $\varphi : \partial \bar{V} \to \partial E(K; N)$, where $E(B; A)$ means the exterior of $B$ in $A$, i.e., $E(B; A) = \text{cl}(A \setminus \eta(B; A))$. If $\varphi(\bar{m})$ is isotopic to a representative of $p[m] + q[\ell]$ for a meridian $\bar{m}$ of $\bar{V}$, then the surgery is called $p/q$-surgery. By integral surgery, we mean Dehn surgery such that $p/q$ is an integer. Set $N_\varphi = E(K; N) \cup_\varphi \bar{V}$ and let $K^* \subset N_\varphi$ be a core loop of $\bar{V}$. We call $K^*$ the dual knot of $K$ in $N_\varphi$. We remark that $E(K; N)$ is homeomorphic to $E(K^*; N_\varphi)$ and that if Dehn surgery on $K$ in $N$ yields a 3-manifold $N_\varphi$, then $K^*$ admits Dehn surgery yielding $N$.

It is proved by Gordon and Luecke that non-trivial surgery on a non-trivial knot cannot yield $S^3$ [9], and it is proved by Gabai that $S^2 \times S^1$ never comes from Dehn surgery on a non-trivial knot [7]. We remark that Dehn surgery on the torus knots in $S^3$ is characterized by Moser [11]. Also, Bleiler and Litherland [2], Wang [14] and Wu [15] independently characterized Dehn surgery on satellite knots in $S^3$ yielding lens spaces. The Cyclic Surgery Theorem, obtained by Culler et al. [4], implies that if a non-trivial, non-torus knot in $S^3$ admits Dehn surgery yielding a lens...
space, then its surgery must be integral. In [1], Berge introduced the concept of doubly primitive knots and gave integral surgery to obtain a lens space from any doubly primitive knot.

**Definition 1.1 (Berge).** Let $(V_1, V_2; S)$ be a genus two Heegaard splitting of $S^3$ and $K$ a simple loop on $S$. Then $K$ is called a **doubly primitive knot** if $K$ represents a free generator both of $\pi_1(V_1)$ and of $\pi_1(V_2)$.

In this paper, we call such a surgery **Berge’s surgery** (for details, see the proof of Proposition A.2 in Appendix A). We remark that it remains possible that a doubly primitive knot admits another surgery, other than Berge’s surgery, yielding a lens space. A list of doubly primitive knots in $S^3$ is also given by Berge. It is conjectured by Gordon that Berge’s list would be complete.

In this paper, we consider a conjecture given by Bleiler and Litherland [2].

**Conjecture 1.2.** It would be impossible to obtain a lens space $L(p, q)$ with $|p| < 18$ by Dehn surgery on a non-torus knot.

As a partial answer, we have the following.

**Theorem 1.3.** Let $K$ be a non-torus, doubly primitive knot in $S^3$ and $L(p, q)$ a lens space obtained by Berge’s surgery on $K$. Then $|p| \geq 18$.

We remark that Fintushel and Stern [6] showed that 18-surgery on the $(-2, 3, 7)$-pretzel knot yields $L(18, 5)$ and 19-surgery on the $(-2, 3, 7)$-pretzel knot yields $L(19, 8)$. We also note that Goda and Teragaito told us that they confirmed that Conjecture 1.2 is true for the knots in Berge’s list, but did not give a proof (cf. [8, p. 502]).

2. Dual knots of doubly primitive knots

If a lens space $M$ comes from Dehn surgery on a knot $K$ in $S^3$, then there is the dual knot $K^*$ in $M$ such that Dehn surgery on $K^*$ yields $S^3$. It is proved in [1] that when Berge’s surgery on a doubly primitive knot yields a lens space, its dual knot is isotopic to a knot defined as $K^* = K(L(p, q); u)$. (Unfortunately, Berge’s paper [1] is unpublished. To make this paper self-contained, we will give proofs of some results in [1] in the appendix.)

**Definition 2.1.** Let $V_1$ be a standard solid torus in $S^3$, $m$ a meridian of $V_1$ and $\ell$ a longitude of $V_1$ such that $\ell$ bounds a disk in $\text{cl}(S^3 \setminus V_1)$. We fix an orientation of $m$ and $\ell$ as illustrated in Fig. 1. By attaching a solid torus $V_2$ to $V_1$ so that $\bar{m}$ is isotopic to a representative of $p[\ell] + q[m]$, we obtain a lens space $L(p, q)$, where $p$ and $q$ are coprime integers and $\bar{m}$ is a meridian of $V_2$. The intersection points of $m$ and $\bar{m}$ are labelled $P_0, \ldots, P_{p-1}$ successively along the positive direction of $m$. Let $t_i^u$ ($i = 1, 2$) be simple arcs in $D_i$ joining $P_0$ to $P_u$ ($u = 1, 2, \ldots, p - 1$). Then the notation $K(L(p, q); u)$ denotes the knot $t_1^u \cup t_2^u$ in $L(p, q)$ (cf. Fig. 2).

We remark that it is not necessary for any knot represented by $K(L(p, q); u)$ for some integers $p, q$ and $u$ to admit integral surgery yielding $S^3$. 

![Fig. 1.](image-url)
Fig. 2. Here, $t'_2$ is a projection of $t_2$ on $\partial V_1$.

Fig. 3. Throughout this section, we use the notations in Definition 2.1. Recall that we assume $p > 1$ and $0 < q < p$. Let $D_1$ ($D_2$ respectively) be a meridian disk in $V_1$ ($V_2$ respectively) with $\partial D_1 = m$ and $|\partial D_1 \cap \partial D_2| = p$. Let $t'_1$ ($t'_2$ respectively) be the arc in $\partial D_1$ ($\partial D_2$ respectively) whose initial point is $P_0$ and whose endpoint is $P_u$ passing in the positive direction of $m$ ($\ell$ respectively). Then $t'_1$ ($t'_2$ respectively) is a projection of $t_1$ ($t_2$ respectively). Set $\eta(t_2; V_2)$. Then $V'_1$ and $V'_2$ are genus two handlebodies. Let $V'_1$ be a genus two handlebody obtained from $\text{cl}(V'_1 \setminus \eta(K; V'_1))$ by attaching a solid torus $\bar{V}$ so that a meridian of $\bar{V}$ is identified with a loop represented by $r[m^*] + s[\ell^*]$. Set $M' = V''_1 \cup S'$. Then we say that $M'$ is obtained by $(r/s)^*$-surgery on $K$. If $r/s$ is an integer, $(r/s)^*$-surgery is called longitudinal surgery.

Definition 2.2. Let $p$ and $q$ be a coprime pair of positive integers. Let $\{u_j\}_{1 \leq j \leq p}$ be the finite sequence such that $0 \leq u_j < p$ and $u_j \equiv q \cdot j \pmod{p}$. For an integer $u$ with $0 < u < p$, $\Psi_{p,q}(u)$ denotes the integer satisfying $u \Psi_{p,q}(u) = u$ and $\Phi_{p,q}(u)$ denotes the number of elements of the following set (possibly empty set):

$$\{u_j \mid 1 \leq j < \Psi_{p,q}(u), \; u_j < u\}.$$

Example 2.3. Set $p = 18$ and $q = 5$. Then we have the finite sequence

$$\{u_j\}_{1 \leq j \leq 18} : 5, 10, 15, 2, 7, 12, 17, 4, 9, 14, 1, 6, 11, 16, 3, 8, 13, 0.$$

Hence $\Psi_{18,5}(7) = 5$ and $\Phi_{18,5}(7) = 2$. 
Remark 2.4. When one follows $\partial D_2$ from $P_0$ in the positive direction of $\ell$, $\partial D_2$ intersects $\partial D_1$ in the following order:

$$P_0 \rightarrow P_{u_1} \rightarrow P_{u_2} \rightarrow \cdots \rightarrow P_{u_{p-1}} \rightarrow P_0.$$ 

Then $\Psi_{p,q}(u)$ represents the number of intersection points between $t_2^u$ and a parallel copy of $\partial D_1$ in $\partial V_1$, and $\Phi_{p,q}(u)$ represents the number of intersection points between $t_1^u$ and the interior of $t_2^u$.

The following is a key theorem in order to prove Theorem 1.3.

**Theorem 2.5.** If longitudinal surgery on $K(L(p,q);u)$ yields $S^3$, then one of the following holds.

1. $p \cdot \Phi_{p,q}(u) - u \cdot \Psi_{p,q}(u) = \pm 1$. In this case, the surgery is $0^+$-surgery.
2. $p \cdot \Phi_{p,q}(u) - u \cdot \Psi_{p,q}(u) = \pm 1 - p$. In this case, the surgery is $1^+$-surgery.

We will prove Theorem 2.5 in Section 4. As a weaker condition, we have Corollary 2.6.

**Corollary 2.6.** If longitudinal surgery on $K(L(p,q);u)$ yields $S^3$, then $u^2 \equiv \pm q \pmod{p}$.

**Proof of Corollary 2.6 via Theorem 2.5.** By Theorem 2.5, we see

$$u \cdot \Psi_{p,q}(u) \equiv \pm 1 \pmod{p}$$

and hence

$$u \cdot \Psi_{p,q}(u) \cdot q \equiv \pm q \pmod{p}.$$ 

Note that since $u \Psi_{p,q}(u) = u$, we see that $\Psi_{p,q}(u) \cdot q \equiv u \pmod{p}$. Therefore, we obtain the desired result. □

**Proof of Theorem 1.3 via Theorem 2.5.** Let $K$ be a non-torus doubly primitive knot in $S^3$. It follows from the cyclic surgery theorem [4] that if Dehn surgery on $K$ yields a lens space, then the surgery must be integral surgery. It also follows from Proposition A.2 (Berge) that $K$ admits Berge’s surgery. By Theorem A.5 (Berge), if $L(p,q)$ is obtained by Berge’s surgery on $K$, then the dual knot of $K^*$ is isotopic to $K(L(p,q);u)$ $(p > 0, 0 < q < p)$ for some integer $u$ with $1 \leq u \leq p - 1$. We may assume that $1 \leq u \leq p/2$. Moreover, we may assume that $u \neq 1$ otherwise $K^*$ is a torus knot. Since two lens spaces $L(p,q)$ and $L(p',q')$ are (possibly orientation reversing) homeomorphic if and only if $p = p'$ and $q \equiv \pm q' \pmod{p}$, we may also assume $0 < q < p/2$.

Note that two points $P_0$ and $P_u$ makes a partition of $\partial D_1$ into two arcs. Hence $t_2'$ must pass on the two arcs, otherwise $K^*$ is isotopic into $\partial V_1 = \partial V_2$ and hence $K^*$ is a torus knot. Suppose that $p < 18$. Then by Theorem 2.5, we only have to check the cases in Table 1.

Suppose first that $K^* = K(L(7,2);3)$. Then we have the sequence

$$\{u_j\}_{1 \leq j \leq 7} = 2,4,6,1,3,5,0.$$ 

In this case, we see that $u_5 < u_6$. This implies that $t_2'$ misses exactly one of the components of $\partial D_1 \setminus (P_0 \cup P_1)$ and hence $K^*$ is isotoped into $\partial V_1 = \partial V_2$. Therefore $K^*$ is a torus knot. Since $E(K^*;L(p,q)) \cong E(K;S^3)$, we see that $E(K;S^3)$ admits Seifert fibration and hence $K$ is a torus knot, a contradiction.

Suppose next that $K^* = K(L(7,3);2)$. Then we have the sequence

$$\{u_j\}_{1 \leq j \leq 7} = 3,6,2,5,1,4,0.$$ 

In this case, we see that $u_3 < u_j$ $(1 \leq j \leq 2)$. This implies that $K^*$ is a torus knot. Hence we also have a contradiction.

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By using a similar argument, we see that $K^*$ is a torus knot in each of the other cases and hence $K$ is also a torus knot, a contradiction.  

3. The wave theorem on Heegaard splittings

For simple closed curves or arcs $x$ and $y$ properly embedded in a surface (i.e., connected compact 2-manifold), the notation $\iota(x, y)$ denotes the minimal possible number of intersections between $x'$ and $y'$, where $x'$ ($y'$ respectively) is any curve or arc which is properly embedded in the surface and is isotopic to $x$ ($y$ respectively).

A triplet $(V_1, V_2; S)$ is a genus $g$ Heegaard splitting of a closed orientable 3-manifold $N$ if $V_i$ ($i = 1$ and 2) is a genus $g$ handlebody with $N = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = S$. The surface $S$ is called a Heegaard surface. A properly embedded disk $D$ in a genus $g$ handlebody $V$ is called a meridian disk of $V$ if a 3-manifold obtained by cutting $V$ along $D$ is a genus $g - 1$ handlebody. The boundary of a meridian disk of $V$ is called a meridian of $V$. A collection of mutually disjoint $g$ meridians $\{x_1, \ldots, x_g\}$ of $V$ is called a complete meridian system of $V$ if $\{x_1, \ldots, x_g\}$ bounds mutually disjoint meridian disks of $V$ which cuts $V$ into a 3-ball.

Let $(V_1, V_2; S)$ be a genus two Heegaard splitting of $S^3$. Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be complete meridian systems of $V_1$ and $V_2$ respectively. We call $(S; \{x_1, x_2\}, \{y_1, y_2\})$ a Heegaard diagram. If $x_1, x_2, y_1$ and $y_2$ are isotoped on $S$ so that the number of their intersection points are minimal, then we call $(S; \{x_1, x_2\}, \{y_1, y_2\})$ a normalized Heegaard diagram. If $\iota(x_1, y_1) = 1, \iota(x_2, y_2) = 1, x_2 \cap y_1 = \emptyset$ and $x_1 \cap y_2 = \emptyset$, then the Heegaard diagram is said to be standard. Let $\Sigma_x$ ($\Sigma_y$ respectively) be the four holed 2-sphere obtained by cutting $S$ along $x_1$ and $x_2$ ($y_1$ and $y_2$ respectively), and let $x_i^+$ and $x_i^-$ ($y_i^+$ and $y_i^-$ respectively) ($i = 1, 2$) be the copies of $x_i$ ($y_i$ respectively) in $\Sigma_x$ ($\Sigma_y$ respectively).

**Lemma 3.1.** [12, Lemma 1 and Theorem 1] If $(S; \{x_1, x_2\}, \{y_1, y_2\})$ is a normalized genus two Heegaard diagram of $S^3$, then $x_i$ ($i = 1, 2$) must intersect $y_1$ and $y_2$ transversely in some (non-empty) points. Moreover, we can assume that $(y_1 \cup y_2) \cap \Sigma_x$ is one of the figures illustrated in Fig. 4, where each of $a, b, c$ and $d$ represents the number of parallel arc-components, possibly zero, of $(y_1 \cup y_2) \cap \Sigma_x$. Here, two arcs are said to be parallel if they cobound a disk, considering $x_i^\pm$ as points.

A wave $w$ associated with $x_i$ ($i = 1$ or 2) is a properly embedded arc in $\Sigma_x$ such that $w$ is disjoint from $(y_1 \cup y_2) \cap \Sigma_x$, $w$ joins $x_i^+$ or $x_i^-$ to itself and $w$ does not cut off a disk from $\Sigma_x$. Similarly, a wave $w$ associated with $y_i$ ($i = 1$ or 2) is a properly embedded arc in $\Sigma_y$ such that $w$ is disjoint from $(x_1 \cup x_2) \cap \Sigma_y$, $w$ joins $y_i^+$ or $y_i^-$ to itself and $w$ does not cut off a disk from $\Sigma_y$. A Heegaard diagram $(S; \{x_1, x_2\}, \{y_1, y_2\})$ contains a wave if there is a wave associated with $x_i$ ($i = 1$ or 2) or $y_i$ ($i = 1$ or 2). The following, so-called the wave theorem, is proved by Homma et al. [10].

**Theorem 3.2.** [10, Main Theorem] Any normalized genus two Heegaard diagram of $S^3$ is standard, or contains a wave.

![Fig. 4](image-url)
4. Proof of Theorem 2.5

Throughout this section, we use the same notation as in Section 2.

Theorem 4.1. If longitudinal surgery on $K(L(p,q); u)$ yields $S^3$, then the surgery is $0^*$-surgery or $1^*$-surgery.

Proof. Suppose that a meridian of a filling torus $\widetilde{V}$ is identified with a loop $\alpha$ in $\partial \eta(K; V'_1)$ represented by $r[m^*]+[\ell^*]$ for some integer $r$. Let $E_1$ ($E_2$ respectively) be the parallel copy of $D_1$ ($D_2$ respectively) in $V_1$ ($V_2$ respectively) and $x_1$ ($y_1$ respectively) an essential loop in $S'$ corresponding to $\partial E_1$ ($\partial E_2$ respectively). Let $x_2$ be an essential loop in $S'$ such that $x_2$ contains $x_2' := t_1'' \cap V''$ and that $x_2 \cup \alpha$ bounds an annulus in $\operatorname{cl}(V'' \setminus \widetilde{V})$. Set $y_2 := \partial D'_2$. Then $(S'; \{x_1, x_2\}, \{y_1, y_2\})$ is a Heegaard diagram of a manifold $M'$ obtained by $r^*$-surgery on $K(L(p,q); u)$.

We suppose $M' = S^3$. Let $P_x$ be the two-holed torus obtained by cutting $S'$ along $x_1$, and let $x_1^+$ and $x_1^-$ be the boundary components of $P_x$. Note that $y_1 = \partial E_2$ is a parallel copy of $\partial D_2$ and $\partial D_2$ contains $t_2''$. Hence whatever the value of $u$ is, there is an arc-component $\gamma$ of $P_x \cap y_1$ such that $\gamma$ intersects $x_2'$ transversely in a single point. There is also an arc-component $\gamma'$ of $P_x \cap y_1$ such that $\gamma'$ is disjoint from $x_2'$ in $P_x$. Let $\gamma''$ be the arc-component of $P_x \cap y_2$ such that $\gamma''$ intersects a core loop of the annulus $\partial \eta(t_2'; V'_2) \cap S'$ in a single point. Let $\Sigma_x$ be the four-holed 2-sphere obtained by cutting $P_x$ along $x_2$. Let $x_2^+$ and $x_2^-$ be the copies of $x_2$ in $\partial \Sigma_x$ such that one of the components of $\gamma \cap \Sigma_x$ joins $x_1^+$ to $x_2^+$ and the other joins $x_1^-$ to $x_2^-$.

Case 1. $r \leq -1$.

Then there is an arc-component, say $\gamma''$, of $\gamma'' \cap \Sigma_x$ which joins $x_1^+$ to $x_2^-$ and there is also an arc-component, say $\gamma_2''$, of $\gamma'' \cap \Sigma_x$ which joins $x_1^-$ to $x_2^+$. Recall that one of the components, say $y_1$, of $\gamma \cap \Sigma_x$ joins $x_1^+$ to $x_2^+$ and the other, say $y_2$, joins $x_1^-$ to $x_2^-$ (cf. Fig. 5). These imply that there are waves associated neither with $x_1$ nor with $x_2$. By a similar argument, we see that there are waves associated neither with $y_1$ nor with $y_2$. Hence $(S'; \{x_1, x_2\}, \{y_1, y_2\})$ contains no waves. Since we assume $M' = S^3$, this is a contradiction by Theorem 3.2.

Case 2. $r \geq 2$.

Then we see that there is an arc-component, say $\gamma''$, of $\gamma'' \cap \Sigma_x$ which joins $x_1^-$ to $x_2^-$. Note that one of the components, say $\gamma''_1$, of $\gamma'' \cap \Sigma_x$ joins $x_1^-$ to $x_2^+$ and the other, say $\gamma''_2$, joins $x_1^-$ to $x_2^-$, and that $\gamma''_1$ joins $x_1^-$ to $x_2^-$ (cf. Fig. 6). These imply that there are waves associated neither with $x_1$ nor with $x_2$. By a similar argument, we see that there are waves associated neither with $y_1$ nor with $y_2$. This is a contradiction by Theorem 3.2.

Lemma 4.2. Let $n$ be an integer. If $n^*$-surgery on $K(L(p,q); u)$ yields an integral homology 3-sphere, then

$$\Psi_{p,q}(u) = \pm 1 - np.$$

Proof. We use notations in the proof of Theorem 4.1. In particular, recall that $(S'; \{x_1, x_2\}, \{y_1, y_2\})$ is a Heegaard diagram of a manifold $M'$ obtained by longitudinal surgery on $K(L(p,q); u)$. Suppose that $M'$ is obtained by $n^*$-surgery on $K(L(p,q); u)$. We now consider $H_1(M'; \mathbb{Z})$. Note that $\iota(y_1, x_1) = p$ and $\iota(y_1, x_2) = u$. Similarly, we see that $\iota(x_1, y_2) = \Psi_{p,q}(u)$ and $\iota(x_2, y_2) = |\Phi_{p,q}(u) + n|$. Hence we have:

$$H_1(M'; \mathbb{Z}) = \langle a, \ b \mid pa + ub = 0, \ \Psi_{p,q}(u) \cdot a + (\Phi_{p,q}(u) - 1) \cdot b = 0 \rangle.$$
Therefore we see that if $H_1(M';\mathbb{Z})$ is trivial, then the following is satisfied:

$$p \cdot (\Phi_{p,q}(u) + n) - u \cdot \Psi_{p,q}(u) = \pm 1$$

and hence we have:

$$p \cdot \Phi_{p,q}(u) - u \cdot \Psi_{p,q}(u) = \pm 1 - np. \quad \Box$$

We now easily have Theorem 2.5 by Theorem 4.1 and Lemma 4.2.

5. Examples

Let $K$ be a doubly primitive knot in $S^3$ and $M$ a lens space obtained from Berge’s surgery on $K$. Let $K^*$ be the dual knot of $K$ in $M$. We use the same notation $(S; \{x_1, x_2\}, \{y_1, y_2\})$ as in Section 4 (cf. the proof of Theorem 2.5), i.e., $(S; \{x_1, x_2\}, \{y_1, y_2\})$ is a Heegaard diagram of a manifold $M'$ obtained by $0^*$ or $1^*$-surgery on $K^*$.

**Example 5.1 (The right-hand trefoil knot).** Suppose that $K$ is the right-hand trefoil knot in $S^3$. Then it is well known that 5-surgery on $K$ yields $L(5, 4)$. Hence there is the dual knot $K^*$ in $L(5, 4)$ which has longitudinal surgery on $K^*$ yielding $S^3$. By Theorem 2.5, we see that $K^* = K(L(5, 4); 2)$. In fact, we can check that $K^*$ admits longitudinal surgery yielding $S^3$ as follows.

Let $M'$ be a 3-manifold obtained by $1^*$-surgery on $K^*$. By the following calculation, we see that $\pi_1(M')$ is trivial.

$$\pi_1(M') \cong \langle x_1^*, x_2^* | y_1 = 1, y_2 = 1 \rangle$$

$$\cong \langle x_1^*, x_2^* | (x_1^*)^3 x_2^* x_1^* x_2^* x_1^* = 1, (x_1^*)^3 x_2^* = 1 \rangle$$

$$\cong \langle x_1^*, x_2^* | x_1^* = 1, x_2^* = 1 \rangle.$$  

Since Poincaré conjecture is true for the genus two 3-manifolds (cf. [3,5]), we see that $M'$ is homeomorphic to $S^3$.

**Example 5.2 (The $(-2, 3, 7)$-pretzel knot).** Suppose that $K$ is the $(-2, 3, 7)$-pretzel knot in $S^3$. It is well known [6] that 18-surgery on $K$ yields $L(18, 5)$. Then there is the dual knot $K^*$ in $L(18, 5)$ which has longitudinal surgery on $K^*$ yielding $S^3$. By Theorem 2.5, we see that the dual knot $K^*$ of $K$ is $K(L(18, 5); 7)$.

We now consider 0*-surgery on $K^* = K(L(18, 5); 7)$. Let $M'$ be a 3-manifold obtained by 0*-surgery on $K^*$. Then we see

$$\pi_1(M') \cong \langle x_1^*, x_2^* | y_1 = 1, y_2 = 1 \rangle$$

$$\cong \langle x_1^*, x_2^* | x_1^* x_2^* (x_1^*)^3 x_2^* x_1^* x_2^* (x_1^*)^3 x_2^* (x_1^*)^3 x_2^* (x_1^*)^3 x_2^* x_1^* = 1, x_1^* x_2^* (x_1^*)^3 x_2^* x_1^* = 1 \rangle$$

$$\cong \langle x_1^*, x_2^* | x_1^* = 1, x_1^* x_2^* = 1 \rangle.$$  

This implies that $M' \cong S^3$.
Remark 5.3. We also show that $M' \cong S^3$ by using the wave argument as follows (cf. Theorem 3.2). We use the same notation $(S'; \{x_1, x_2\}, \{y_1, y_2\})$. Let $\Sigma_x$ be the four-holed 2-sphere obtained by cutting $S'$ along $x_1 \cup x_2$. Let $\tilde{y}$ be a component of $(y_1 \cup y_2) \cap \Sigma_x$ which joins a copy of $x_1$ in $\partial \Sigma_x$ to a copy of $x_2$ in $\partial \Sigma_x$. Then there is a component of $\partial \eta(x_1 \cup \tilde{y} \cup x_2; S')$, say $x_1^{(0)}$, which is isotopic neither to $x_1$ nor to $x_2$ in $S'$. Set $x_2^{(0)} = x_2$, $y_1^{(0)} = y_1$ and $y_2^{(0)} = y_2$. 

![Diagram](image-url-7)
Then we see that \((S'; \{x_{1}^{0}, x_{2}^{0}\}, \{y_{1}^{0}, y_{2}^{0}\})\) is a Heegaard diagram of \(M'\). We isotope \(x_{1}^{0}\) so that the diagram is normalized. (We remark that we also obtain this diagram by finding a wave associated with \(x_{1}^{0}\)).

Let \(\Sigma_{x}^{0}\) be the four-holed 2-sphere obtained by cutting \(S'\) along \(x_{1}^{0} \cup x_{2}^{0}\). Then \((y_{1}^{0} \cup y_{2}^{0}) \cap \Sigma_{x}^{0}\) is as illustrated at the upper left in Fig. 7. Then we see that there is a wave \(w^{(0)}\) associated with \(x_{1}^{0}\). Let \(x_{1}^{(1)}\) be a component of \(\partial \eta(w^{(0)} \cup x_{1}^{0})\) which is not isotopic to \(x_{2}^{0}\). Set \(x_{2}^{(1)} = x_{2}^{0}\), \(y_{1}^{(1)} = y_{1}^{0}\) and \(y_{2}^{(1)} = y_{2}^{0}\). Then we obtain a new diagram \((S'; \{x_{1}^{(1)}, x_{2}^{(1)}\}, \{y_{1}^{(1)}, y_{2}^{(1)}\})\).

For the new diagram, we can find a wave \(w^{(1)}\) associated with \(y_{1}^{(1)}\). Let \(y_{2}^{(1)}\) be a meridian of \(V_{2}\) obtained from \(y_{1}^{(1)}\) and \(w^{(1)}\), and set \(x_{1}^{(2)} = x_{1}^{(1)}, x_{2}^{(2)} = x_{2}^{(1)}\) and \(y_{2}^{(2)} = y_{2}^{(1)}\). Then \((S'; \{x_{1}^{(2)}, x_{2}^{(2)}\}, \{y_{1}^{(2)}, y_{2}^{(2)}\})\) is also a Heegaard diagram of \(M'\).

By repeating an argument similar to the above, we obtain a Heegaard diagram of \(M'\) \((S'; \{x_{1}^{(5)}, x_{2}^{(5)}\}, \{y_{1}^{(5)}, y_{2}^{(5)}\})\) as illustrated at the lower right in Fig. 7. Since there is a wave \(w^{(5)}\) associated with \(x_{2}^{(5)}\), we finally obtain the standard Heegaard diagram of \(S^{3}\).

We next consider 19-surgery on the \((-2, 3, 7)\)-pretzel knot \(K\). It is also well known [6] that 19-surgery on \(K\) yields \(L(19, 8)\). Then there is the dual knot \(K^{*}\) in \(L(19, 8)\) which has longitudinal surgery on \(K^{*}\) yielding \(S^{3}\). By Theorem 2.5, we see that the dual knot \(K^{*}\) of \(K\) is \(K(L(19, 8); 7)\).

We now consider \(1^{*}\)-surgery on \(K^{*} = K(L(19, 8); 7)\). Let \(M'\) be a 3-manifold obtained by \(1^{*}\)-surgery on \(K^{*}\). Then we see

\[
\pi_{1}(M') \cong \langle x_{1}^{*}, x_{2}^{*} | y_{1} = 1, y_{2} = 1 \rangle \\
\cong \langle x_{1}^{*}, x_{2}^{*} | (x_{1}^{*})^{3}x_{2}^{*}(x_{1}^{*})^{5}x_{2}^{*}(x_{1}^{*})^{2}x_{2}^{*}(x_{1}^{*})^{2}x_{2}^{*}x_{1}^{*} = 1, (x_{1}^{*})^{3}x_{2}^{*}(x_{1}^{*})^{2}x_{2}^{*}(x_{1}^{*})^{3}x_{2}^{*} = 1 \rangle \\
\cong \langle x_{1}^{*}, x_{2}^{*} | x_{1}^{*} = 1, x_{1}^{*}x_{2}^{*} = 1 \rangle.
\]

This implies that \(M' \cong S^{3}\).

**Comments 5.4.** The converse statement of Theorem 2.5 does not hold. One of the examples is \(K(L(22, 3); 5)\). It is easy to see that \(K(L(22, 3); 5)\) satisfies the condition (1) of Theorem 2.5. Moreover, if \(K(L(22, 3); 5)\) has longitudinal surgery yielding \(S^{3}\), then the surgery should be \(0^{*}\)-surgery. Let \(M'\) be the 3-manifold obtained by \(0^{*}\)-surgery on \(K(L(22, 3); 5)\). Then we see that \(\pi_{1}(M')\) admits the following presentation:

\[
\pi_{1}(M') \cong \langle x_{1}^{*}, x_{2}^{*} | y_{1} = 1, y_{2} = 1 \rangle \\
\cong \langle x_{1}^{*}, x_{2}^{*} | (x_{1}^{*})^{2}x_{2}^{*}(x_{1}^{*})^{7}x_{2}^{*} = 1, (x_{1}^{*})^{6}x_{2}^{*}x_{1}^{*}x_{2}^{*}(x_{1}^{*})^{6}x_{2}^{*} = 1 \rangle \\
\cong \langle x_{1}^{*}, \tilde{x}_{2} | x_{1}^{*}\tilde{x}_{2}(x_{1}^{*})^{-4}\tilde{x}_{2} = 1, (x_{1}^{*})^{-5}\tilde{x}_{2}^{3} = 1 \rangle \\
\cong \langle x_{1}^{*}, \tilde{x}_{2} | (x_{1}^{*})^{5}(x_{1}^{*})^{2}(\tilde{x}_{2})^{-5} = 1, (x_{1}^{*})^{-5}\tilde{x}_{2}^{3} = 1 \rangle \\
\cong \langle x_{1}^{*}, \tilde{x}_{2} | (x_{1}^{*})^{5} = (x_{1}^{*})^{2} = \tilde{x}_{2}^{3} \rangle.
\]

This implies that \(\pi_{1}(M')\) is isomorphic to the binary icosahedral group and hence \(\pi_{1}(M')\) is non-trivial. Therefore \(M' \not\cong S^{3}\).

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**Appendix A**

We give proofs of some results in [1] to make this paper self-contained. We remark that outlines of the proofs are the same as that in [1].
Proposition A.1. [1, Lemma 2] Let \((S; \{x_1, x_2\}, \{y_1, y_2\})\) be a normalized genus two Heegaard diagram of \(S^3\). Suppose that there is a simple closed curve \(z\) in \(S\) with \(i(z, x_1) = i(z, y_1) = 1\) and \(z \cap (x_2 \cup y_2) = \emptyset\). If \(x_2\) and \(y_2\) are oriented, then the signed intersection points of \(x_2\) and \(y_2\) have the same sign.

**Proof.** We may assume that \(|z \cap x_1| = |z \cap y_1| = 1\). Set \(z_x = z \cap \Sigma_x\) and \(z_y = z \cap \Sigma_y\). Then \(z_x\) is a simple arc in \(\Sigma_x\) joining \(x^+_1\) to \(x^-_1\) and \(z_y\) is a simple arc in \(\Sigma_y\) joining \(y^+_1\) to \(y^-_1\).

**Claim 1.** There are no arc-components of \((y_1 \cup y_2) \cap \Sigma_x\) such that the endpoints are contained in a single boundary component \(x^+_2\) or \(x^-_2\), and there are no arc-components of \((x_1 \cup x_2) \cap \Sigma_y\) such that the endpoints are contained in a single boundary component \(y^+_2\) or \(y^-_2\).

**Proof.** Suppose that there is an arc-component \(y'\) of \((y_1 \cup y_2) \cap \Sigma_x\) such that \(y'\) joins \(x^+_2\) or \(x^-_2\), say \(x^+_2\), to itself. If \(y'\) separates \(\{x^+_1, x^-_1, x^+_2\}\) into \(\{x^+_1, x^-_1\}\) and \(\{x^+_2\}\), then each arc-component adjacent to \(\{x^+_2\}\) joins \(\{x^-_2\}\) to \(\{x^+_2\}\). This contradicts \(|(y_1 \cup y_2) \cap x^+_2| = |(y_1 \cup y_2) \cap x^-_2|\). Hence \(y'\) separates \(\{x^+_1, x^-_1, x^+_2\}\) into \(\{x^+_1, x^-_2\}\) and \(\{x^+_2\}\), or \(\{x^+_1, x^-_2\}\) and \(\{x^+_1\}\), say the latter holds. It follows from Lemma 3.1(3) that there is another arc-component \(y''\) of \((y_1 \cup y_2) \cap \Sigma_x\) such that \(y''\) joins \(x^+_2\) to itself and that \(y''\) separates \(\{x^+_1, x^-_1, x^+_2\}\) into \(\{x^+_1, x^-_2\}\) and \(\{x^+_1\}\). This follows from the condition \(|(y_1 \cup y_2) \cap x^+_2| = |(y_1 \cup y_2) \cap x^-_2|\). Particularly, each of \(y'\) and \(y''\) intersects \(z_x\) transversely in some (non-empty) points. This implies that \(|(y_1 \cup y_2) \cap z_x| \geq 2\). However, this contradicts the assumption \(|z \cap y_1| = 1\) and \(z \cap y_2 = \emptyset\). Hence we see that there are no arc-components of \((y_1 \cup y_2) \cap \Sigma_x\) such that the endpoints are contained in a single boundary component \(x^+_2\) or \(x^-_2\).

Similarly, we also see that there are no arc-components of \((x_1 \cup x_2) \cap \Sigma_y\) such that the endpoints are contained in a single boundary component \(y^+_2\) or \(y^-_2\). Hence we have Claim 1. □

**Claim 2.** We may assume that there are no arc-components of \((y_1 \cup y_2) \cap \Sigma_x\) such that the endpoints are contained in a single boundary component \(x^+_1\) or \(x^-_1\), and there are no arc-components of \((x_1 \cup x_2) \cap \Sigma_y\) such that the endpoints are contained in a single boundary component \(y^+_1\) or \(y^-_1\).

**Proof.** We fix \(S, z, x_2\) and \(y_2\), and replace, if necessary, \(x_1\) and \(y_1\) so that \(x_1\) \((y_1\) respectively) bounds a meridian disk of \(V_1\) \((V_2\) respectively), \(|z \cap x_1| = |z \cap y_1| = 1, x_1 \cup x_2 = y_1 \cup y_2 = \emptyset\) and \(|(x_1 \cup x_2) \cap (y_1 \cup y_2)|\) is minimal among such all normalized Heegaard diagrams.

Suppose that there is an arc-component \(\tilde{y}\) of \((y_1 \cup y_2) \cap \Sigma_x\) such that \(\tilde{y}\) joins \(x^+_1\) or \(x^-_1\), say \(x^+_1\), to itself. Then there is a wave \(w\) associated with \(x^+_1\) such that \(w\) is a component of the frontier of \(\eta(\tilde{y}; \Sigma_x)\) in \(\Sigma_x\). By an argument similar to the above, we see that \(w\) separates \(\{x^-_1, x^+_1, x^+_2\}\) into \(\{x^-_1, x^+_2\}\) and \(\{x^+_1\}\). Let \(x'_1\) and \(x''_1\) be the arcs obtained by cutting \(x^+_1\) along \(\partial w\). Note that \(x'_1 \cup w\) or \(x''_1 \cup w\), say \(x'_1 \cup w\), is isotopic to \(x^+_1\) in \(\Sigma_x\). Note also that \((S; \{x_1, x_2\}, \{y_1, y_2\})\) is a normalized Heegaard diagram with \(i(z, x_1) = i(z, y_1) = 1\) and \(z \cap (x_2 \cup y_2) = \emptyset\).

Suppose first that \(x'_1 \cap z_x = \emptyset\). This implies that \(x'_1\) contains an endpoint of \(z_x\) and that \(w\) is disjoint from \(z_x\). Set \(\tilde{x}_1 = x'_1 \cup w\). Since \(|z \cap \tilde{x}_1| = |z \cap y_1| = 1\) and \(z \cap (x_2 \cup y_2) = \emptyset\), this contradicts the minimality of \(|(x_1 \cup x_2) \cap (y_1 \cup y_2)|\).

Suppose next that \(x''_1 \cap z_x = \emptyset\). This implies that \(x'_1\) contains an endpoint of \(z_x\) and hence \(w\) intersects \(z_x\) transversely in a single point. Set \(\tilde{x}_1 = x''_1 \cup w\) again. Since \(|z \cap \tilde{x}_1| = |z \cap y_1| = 1\) and \(z \cap (x_2 \cup y_2) = \emptyset\), this also contradicts the minimality of \(|(x_1 \cup x_2) \cap (y_1 \cup y_2)|\). Hence we see that there are no arc-components of \((y_1 \cup y_2) \cap \Sigma_x\) such that the endpoints are contained in a single component \(x^+_1\) or \(x^-_1\).

By an argument similar to the above, we also see that there are no arc components of \((x_1 \cup x_2) \cap \Sigma_y\) such that the endpoints are contained in a single component \(y^+_2\) or \(y^-_2\). Hence we have Claim 2. □

If the diagram \((S; \{x_1, x_2\}, \{y_1, y_2\})\) is standard, then we are done. Otherwise, it follows from Theorem 3.2 that there is a wave in \(\Sigma_x\) or \(\Sigma_y\), say \(\Sigma_x\). Recall that \(x_i\) \((i = 1, 2)\) must intersect \(y_1 \cup y_2\) transversely in some (non-empty) points and that \((y_1 \cup y_2) \cap \Sigma_x\) corresponds to one of the figures illustrated in Fig. 4 (cf. Lemma 3.1).

Suppose first that \((y_1 \cup y_2) \cap \Sigma_x\) corresponds to (1) of Fig. 4. If exactly one of \(a, b, c\) and \(d\) is equal to zero, then there are no waves in \(\Sigma_x\), a contradiction. If \(a = b = c = 0\) or \(a = b = d = 0\), then this implies that \(x^+_1\) or \(x^-_1\) is disjoint from \(y_1 \cup y_2\). This contradicts Lemma 3.1. If \(b = c = d = 0\) or \(a = c = d = 0\), then we easily obtain the desired result. Hence we may assume that exactly two of \(a, b, c\) and \(d\) are equal to zero. If \(a \neq 0\) and \(b \neq 0\), then
there are no waves in $\Sigma$. Suppose that $a = 0$ and $b \neq 0$. Then there is an arc-component, say $\tilde{y}$, of $(y_1 \cup y_2) \cap \Sigma$ joining $x_1^+$ to $x_2^-$. If we fix orientation of $\tilde{y}$ from $x_1^+$ to $x_2^-$, then the other arc-components are oriented as in Fig. 8. Hence if $x_2$ and $y_2$ are oriented, then the signed intersection points of $x_2$ and $y_2$ have the same sign. Suppose that $a \neq 0$ and $b = 0$. Then there is an arc-component, say $\tilde{y}$, of $(y_1 \cup y_2) \cap \Sigma$ joining $x_1^+$ to $x_2^-$. If we fix orientation of $\tilde{y}$ from $x_1^+$ to $x_2^+$, then the other arc-components are oriented as in Fig. 8. Hence if $x_2$ and $y_2$ are oriented, then the signed intersection points of $x_2$ and $y_2$ have the same sign. Suppose that $a \neq 0$ and $b = 0$ (possibly $c = 0$ or $d = 0$). Then there is an arc-component, say $\tilde{y}$, of $(y_1 \cup y_2) \cap \Sigma$ joining $x_1^+$ to $x_2^+$. If we fix orientation of $\tilde{y}$ from $x_1^+$ to $x_2^+$, then the other arc-components are oriented as in Fig. 9 and hence we also have the desired conclusion. If $a = b = 0$, we easily have the desired result.

Suppose next that $(y_1 \cup y_2) \cap \Sigma$ corresponds to (2) of Fig. 4. If $a = b = 0$, then $x_2$ is disjoint from $y_1 \cup y_2$, a contradiction. If $a \neq 0$ and $b \neq 0$, then there are no waves in $\Sigma$. Suppose that $a = 0$ and $b \neq 0$ (possibly $c = 0$ or $d = 0$). Then there is an arc-component, say $\tilde{y}$, of $(y_1 \cup y_2) \cap \Sigma$ joining $x_1^+$ to $x_2^-$. If we fix orientation of $\tilde{y}$ from $x_1^+$ to $x_2^-$, then the other arc-components are oriented as in Fig. 9. Hence if $x_2$ and $y_2$ are oriented, then the signed intersection points of $x_2$ and $y_2$ have the same sign. Suppose that $a \neq 0$ and $b = 0$ (possibly $c = 0$ or $d = 0$). Then there is an arc-component, say $\tilde{y}$, of $(y_1 \cup y_2) \cap \Sigma$ joining $x_1^+$ to $x_2^+$. If we fix orientation of $\tilde{y}$ from $x_1^+$ to $x_2^+$, then the other arc-components are oriented as in Fig. 9 and hence we also have the desired conclusion.

Suppose finally that $(y_1 \cup y_2) \cap \Sigma$ corresponds to (3) of Fig. 3. It follows from Claims 1 and 2 that $b$ is equal to zero. Note that (3) of Fig. 4 with $b = 0$ is the same as (2) of Fig. 4 with $b = 0$ and hence we are done.

We have completed the proof of Proposition A.1. \(\square\)
Proposition A.2. [1, Theorem 1] Let $K$ be a non-trivial doubly primitive knot in $S^3$. Then there is integral surgery on $K$ yielding a lens space.

Proof. Let $(V_1, V_2; S)$ be a genus two Heegaard splitting of $S^3$ with $K \subset S$. Since $K$ represents a free generator of $V_1$, there are mutually disjoint meridian disks $D_1$ and $D'_1$ of $V_1$ with $\partial D_1 \cap K = 1$ and $\partial D'_1 \cap K = \emptyset$. Similarly, there are mutually disjoint meridian disks $D_2$ and $D'_2$ of $V_2$ with $\partial D_2 \cap K = 1$ and $\partial D'_2 \cap K = \emptyset$. Let $K'$ be a simple loop obtained by pushing $K$ into the interior of $V_2$ and $A_K$ a spanning annulus in $cl(V_2 \setminus \eta(K'; V_2))$ with $\partial A_K \subset K$, i.e., $A_K$ is an annulus properly embedded in $cl(V_2 \setminus \eta(K'; V_2))$ such that one of the boundary components of $A_K$ lies in $\partial V_2$ and the other, say $K''$, lies in $\partial \eta(K'; V_2)$. We now fill $cl(V_2 \setminus \eta(K'; V_2))$ with a solid torus $\tilde{V}$ so that a meridian of $\tilde{V}$ is identified with $K''$. Then the resulting manifold $\tilde{V}$ is also a genus two Heegaard splitting. Hence $M := V_1 \cup V_2$ is obtained by integral surgery on $K$ and $M$ admits a genus two Heegaard splitting $(V_1, V_2; S)$. Note that $D_K := \tilde{D} \cup A_K$ is a meridian disk of $V_2$, where $\tilde{D}$ is a meridian disk of $\tilde{V}$ with $\partial \tilde{D} = K''$. Hence we see that $(S; \{\partial D_1, \partial D'_1\}, \{K, \partial D'_2\})$ is a Heegaard diagram of $M$. Since $|\partial D_1 \cap K| = 1$, we see that $\tilde{V}_1 := V_1 \cup \eta(D_K; V_2')$ and $\tilde{V}_2 := cl(V_2' \setminus \eta(D_K; V_2'))$ are solid tori and $M = \tilde{V}_1 \cup \tilde{V}_2$. This implies that $M$ admits a genus one Heegaard splitting. Recall that non-trivial surgery on a non-trivial knot cannot yield $S^3$ and Dehn surgery on a non-trivial knot cannot yield $S^2 \times S^1$. This implies that $M$ is a lens space. \hfill \Box

In this paper, Berge’s surgery means the surgery in the proof of Proposition A.2.

Definition A.3. A knot $K$ in a closed orientable 3-manifold $M$ is called a (1, 1)-knot if $(M, K) = (V_1, t_1) \cup_3 (V_2, t_2)$, where $(V_1, V_2; S)$ is a genus one Heegaard splitting and $t_1$ ($t_2$ respectively) is a trivial arc in $V_1$ ($V_2$ respectively). Here, an arc $t$ properly embedded in a solid torus $V$ is said to be trivial if there is a disk $D$ in $V$ with $t \subset \partial D$ and $\partial D \cap t = \emptyset$. We call such a disk $D$ a cancelling disk of $t$. Set $W_i = (V_i, t_i)$ $(i = 1, 2)$ and $P = (S, S \cap K)$. The triplet $(W_1, W_2; P)$ is called a (1, 1)-splitting of $(M, K)$.

Lemma A.4. [1, Lemma 1] Let $K$ be a doubly primitive knot, $M$ a lens space obtained by Berge’s surgery on $K$, and $K^*$ the dual knot of $K$ in $M$. Then $K^*$ is a (1, 1)-knot.

Proof. We use the same notations as in the proof of Proposition A.2. Recall that $\tilde{V}$ is the filling solid torus. Note that all longitudinal loops in $\partial \tilde{V}$ are mutually isotopic in $\tilde{V}$. This implies that $K^*$ is isotopic to $\partial D_2$ in $V_2$. Recall that $\tilde{V}_1 = V_1 \cup \eta(D_K; V_2')$ and $\tilde{V}_2 = cl(V_2' \setminus \eta(D_K; V_2'))$. Set $t_1 = \partial D_2 \cap \eta(D_K; V_2')$ and $t_2 = cl(\partial D_2 \setminus t_1)$. Let $t_2$ be an arc in $V_2$ obtained by pushing the interior of $t_2$ into the interior of $V_2$. Then we see that $t_1$ and $t_2$ are trivial in $\tilde{V}_1$ and $\tilde{V}_2$ respectively and therefore $\partial D_2 = t_1 \cup t_2$ (hence $K^*$) is a (1, 1)-knot in $M$. \hfill \Box

Lemma A.4 implies that it is very important to study Dehn surgery on (1, 1)-knots in lens spaces yielding $S^3$ as well as to study Dehn surgery on knots in $S^3$ yielding lens spaces. Moreover, the following implies that such a (1, 1)-knot is in a good position with respect to the Heegaard surface of the lens space.

Theorem A.5. [1, Theorem 2] Let $K$, $M$ and $K^*$ be as in Lemma A.4. Let $(V_1, V_2; S)$ be a genus one Heegaard splitting of $M$ and $D_i$ $(i = 1, 2)$ meridian disks in $V_i$ such that $(S; \{\partial D_i\}, \{\partial D_2\})$ is a normalized Heegaard diagram of $M$. Then $K$ is isotopic to $t_1 \cup t_2$ in $M$, where $t_1$ ($t_2$ respectively) is a properly embedded arcs in $D_1$ ($D_2$ respectively).

Proof. Since $K$ is a (1, 1)-knot in $M$, we may assume that $K \cap V_i = \{t_i\}$ is a trivial arc in $V_i$ for each $i = 1$ and 2. Let $E_i$ $(i = 1, 2)$ be meridian disks in $V_i$ with $E_i \cap t_i = \emptyset$. Set $V'_1 = V_1 \cup \eta(t_2; V_2)$ and $V'_2 = cl(V_2 \setminus \eta(t_2; V_2))$. Then we see that $V'_2 \cup \eta(E_1; V'_1)$ is homeomorphic to $E(K; M)$, i.e., $cl(V'_1 \setminus \eta(E_1; V'_1))$ is regarded as a regular neighborhood of $K$ in $M$. We now consider longitudinal surgery on $K$ yielding $S^3$. Let $\delta$ be a simple loop in $\partial V'_1 \setminus \partial E_1$ such that $\delta$ is identified with a meridian of a filling solid torus. Set $S' = \partial V'_1$ and $E'_2 = \delta \cap V'_2$, where $\delta$ is a cancelling disk of $t_2$. Then $(S; \{\partial E_1, t\}, \{\partial E_2, \partial E'_2\})$ is a genus two Heegaard diagram of $S^3$ and we may assume that the diagram is normalized. Set $c \subset S'$ be a core loop of the annulus $S' \cap \eta(t_2; V_2)$. Note that $c \cap \partial E_1 = \emptyset$ and $c \cap \partial E_2 = \emptyset$ and $\iota(c, \partial E'_2) = 1$. Moreover, since we consider longitudinal surgery on $K$, we see that $\iota(c, t) = 1$. It follows from Proposition A.1 that if $\partial E_1$ and $\partial E_2$ are oriented, then the signed intersection points of $\partial E_1$ and $\partial E_2$ have the same sign. Therefore we obtain the desired result by the next lemma. \hfill \Box
Lemma A.6. Let $E_1$ and $E_2$ be as in the proof of Theorem A.5. Suppose that if $\partial E_1$ and $\partial E_2$ are oriented, then the signed intersection points of $\partial E_1$ and $\partial E_2$ have the same sign. Then the conclusion of Theorem A.5 holds.

Proof. We use the same notation as in the proof of Theorem A.5. Recall that $V_i$ is a solid torus and $t_i$ is a trivial arc in $V_i$. Let $D_1$ be a parallel copy of $E_1$ which contains $t_1$. We suppose that $|\partial D_1 \cap \partial E_2|$ is minimal among such all meridian disks of $V_1$. We first prove the following.

Claim. If $\partial D_1$ and $\partial E_2$ are oriented, then the signed intersection points of $\partial D_1$ and $\partial E_2$ have the same sign.

Proof. Suppose that the claim does not hold. Let $A_P$ be the annulus with two specified points $P \cap K$ which is obtained by cutting $P$ along $\partial E_1$. Let $\gamma$ be a component of $\partial E_2 \cap A_P$. By the assumption of Lemma A.6, we see that $\gamma$ joins distinct boundary components of $A_P$. Let $D_P$ be the disk with the specified points which are obtained by cutting $A_P$ along $\gamma$.

Suppose that there are no components of $\partial E_2 \cap D_P$ separating the specified points in $D_P$. Then this implies that each component of $\partial E_2 \cap D_P$ is parallel to $\gamma$ in $A_P \setminus K$. Hence we can regard $D_P$ as a square $[0, 1] \times [0, 1]$ such that each component of $\partial E_2 \cap D_P$ is vertical, i.e., each component of $\partial E_2 \cap D_P$ corresponds to $\{p\} \times [0, 1]$. We may assume that the specified points are in $[0, 1] \times \{1/2\}$. Let $\alpha$ be a loop on $P$ such that $\alpha$ corresponds to $[0, 1] \times \{1/2\}$ in the square $D_P$. Then we see that $\alpha$ bounds a meridian disk $D_\alpha$ of $V_1$ and $t_1$ is isotoped into $D_\alpha$ relative to the endpoints (cf. [13, Section 3]). Since we suppose that the claim does not hold, we see that $|\partial D_\alpha \cap \partial E_2| < |\partial D_1 \cap \partial E_2|$. This contradicts the minimality of $|\partial D_1 \cap \partial E_2|$. Hence we have the claim.

Let $D_2$ be a parallel copy of $E_2$ with $\partial D_2 \supset (P \cap K)$. Then $t_2$ is isotoped into $D_2$ relative to the endpoints. This completes the proof of Lemma A.6.

Theorem A.5 (Berge) implies that if a lens space $M = L(p, q)$ is obtained by Berge’s surgery on a doubly primitive knot $K$, then the dual knot $K^*$ is isotopic to $K(L(p, q); u)$ for some integer $u$.

References