Monotonicity and expansion of global secure sets

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1. Introduction

We consider only finite graphs without loops or multiple edges. For a graph $G = (V, E)$, by $V$ we denote the set of vertices and by $E$ the set of edges. An open neighbourhood of a vertex $v$ is the set $N(v) = \{ x \in V : vx \in E \}$, whereas the closed neighbourhood of the vertex $v$ is the set $N[v] = N(v) \cup \{ v \}$. Similarly, we define an open and closed neighbourhood of a set $X \subseteq V$, i.e., $N(X) = \bigcup_{v \in X} N(v)$ and $N[X] = N(X) \cup X$. By $\mathcal{U}(G)$ we denote the set of universal vertices of a graph $G$, i.e., vertices of degree $n - 1$, where $n$ denotes the order of $G$. We say that the graph $G = G_1 \cup G_2$ is a join of the graphs $G_1$ and $G_2$ if $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{ uv : u \in V(G_1) \text{ and } v \in V(G_2) \}$. The disjoint union of the graphs $G_1$ and $G_2$ is the graph $G = G_1 \cup G_2$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

The notion of a secure set was introduced by Brigham et al. in [2].

**Definition 1 ([2]).** Let $G = (V, E)$ be a graph. For any $S = \{ s_1, s_2, \ldots, s_k \} \subseteq V$, an attack on $S$ is any $k$ mutually disjoint sets $A = \{ A_1, A_2, \ldots, A_k \}$ for which $A_i \subseteq N[s_i] - S$, $1 \leq i \leq k$. A defence of $S$ is any $k$ mutually disjoint sets $D = \{ D_1, D_2, \ldots, D_k \}$ for which $D_i \subseteq N[s_i] \cap S$, $1 \leq i \leq k$. Attack $A$ is defendable if there exists a defence $D$ such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. Set $S$ is secure if and only if every attack on $S$ is defendable.

By $S_k(G)$ we denote a secure set of cardinality $k$ and by $\mathcal{S}_m^n$ the class of all graphs of order $n$ that have secure sets of cardinality $k$, for any $k \in \{ m, m + 1, \ldots, n \}$.

The next theorem gives the crucial property of the secure sets.

**Theorem 1 ([2]).** Set $S \subseteq V$ is secure if and only if $\forall X \subseteq S, |N[X] \cap S| \geq |N[X] - S|$.

For more results, we refer the reader to [2,6,7] and [11].

A set $D \subseteq V$ is a dominating set if $N[D] = V$. If a secure set is also a dominating set, then we say that it is a global secure set. The global secure set of the graph $G$ is denoted by $SD(G)$ and by $SD_k(G)$ we denote the global secure set of $G$ of cardinality $k$. If the graph $G$ is clear from the context then we write $SD$ and $SD_k$. The global security number is the minimum cardinality...
of a global secure set in a graph $G$ and is denoted by $\gamma_s(G)$. From Theorem 1 and the fact that $N[SD(G)] = V(G)$ it follows that for any graph $G$ of order $n$, $\gamma_s(G) \geq \lceil \frac{n}{2} \rceil$. Some of the properties of global secure sets were given in [10]. We recall two lemmata that are essential for proofs in the later sections.

**Lemma 1** ([10]). Let $G$ be a graph with a global secure set $SD$ and $v$ be a vertex that belongs to $SD$. If there exists a vertex $v'$ such that $v' \notin SD$ and $v' \in U(G)$, then $SD' = (SD - \{v\}) \cup \{v'\}$ is a global secure set of $G$.

**Lemma 2** ([10]). Let $k$ and $l$ be positive integers, and let $G_1$ and $G_2$ be disjoint graphs of order $n_1 \geq 0$ and $n_2 \geq 0$, respectively. If $k \geq \frac{n_1}{2}$ and $l \geq \frac{n_2}{2}$, then $S(G_1)_k \cup S(G_2)_l$ is a global secure set of cardinality $k + l$ in the graph $G = G_1 + G_2$.

In [10] the concept of $\gamma_s$-monotone graphs was presented. Let $G = (V,E)$ be a graph of order $n$. The graph $G$ is $\gamma_s$-monotone if there exist sets $SD_{\gamma(G)}, SD_{\gamma(G)+1}, \ldots, SD_n$. In this paper we introduce a new property of global secure sets. Namely, we say that a set $SD_k$ is expandable if $k < n$ and there exists a vertex $v \in V - SD$ such that the set $SD' = SD \cup \{v\}$ is a global secure set of $G$. Furthermore, we say that the graph $G$ is $\gamma_s$-expandable if every one of its global secure sets of cardinality less than $n$ is expandable. Whereas the graph $G$ is weakly $\gamma_s$-expandable if there exist sets $SD_{\gamma(G)}, SD_{\gamma(G)+1}, \ldots, SD_n$ such that for $\gamma_s(G) < i \leq n$, $SD_i = SD_{i-1} \cup \{v\}$ where $v \in V - SD_{i-1}$. Clearly, if a graph is (weakly) $\gamma_s$-expandable, then it is also $\gamma_s$-monotone, but the converse implication is not true.

In the forthcoming section we use the following lemma.

**Lemma 3** ([10]). Let both $G_1$ and $G_2$ be connected and $\gamma_s$-monotone graphs of order $n_1 > 0$ and $n_2 > 0$, respectively. If $\gamma_s(G_1) = \lceil \frac{n_1}{2} \rceil$ and $\gamma_s(G_2) = \lceil \frac{n_2}{2} \rceil$, then the graph $G = G_1 \cup G_2$ belongs to $\gamma_s^{n_1+n_2}$, where $k = \lceil \frac{n_1+n_2}{2} \rceil$.

2. Weakly quasi-threshold graphs

Cographs are well known graphs which have been intensively studied due to their practical applications. They can be defined in terms of forbidden subgraphs, i.e., cographs are the graphs without induced $P_4$ [5]. Moreover, there is also a recursive definition:

1. a graph $K_1$ is a cograph,
2. union of disjoint cographs $G_1$ and $G_2$ is a cograph,
3. join of disjoint cographs $G_1$ and $G_2$ is a cograph.

Every cograph has also a tree representation, called a cotree, which can be constructed in linear time [5]. A cotree $T$ is a rooted tree of the modular decomposition of a cograph. The leaves of $T$ represent the vertices of the cograph. Furthermore, every interior node of $T$ is either a 0-node or 1-node. The 0-node and 1-node symbolize the union and join operation, respectively. A cotree is uniquely determined and no 0-node is a child of a 0-node and no 1-node is a child of 1-node. Moreover, every interior node of $T$ has at least two children. For more information about cographs, we refer the reader to [4,8,9] and [12].

In [10] it was shown that for any cograph $\gamma_s(G) \leq \lceil \frac{n+1}{2} \rceil$, where $n$ denotes the order of $G$. Moreover, the author presented infinitely many cographs that are not $\gamma_s$-monotone. This result implies that there exist cographs that are not (weakly) $\gamma_s$-expandable. In this paper we present subclasses of cographs that are weakly $\gamma_s$-expandable or $\gamma_s$-monotone.

Weakly quasi-threshold graphs (wqt graphs) were introduced in [1] by Bapat et al.. However, we present their alternative definition in [12].

**Definition 2** ([12]). The class of wqt graphs can be defined recursively as follows:

1. an edgeless graph is a wqt graph;
2. if $G_1$ and $G_2$ are wqt graphs, then $G_1 \cup G_2$ is a wqt graph;
3. if $G$ is a wqt graph and $H$ is an edgeless graph, then $G + H$ is a wqt graph.

The wqt graph presented on Fig. 1, let us denote it by $H$, is not weakly $\gamma_s$-expandable. Observe that $H$ has 1 universal vertex and 4 vertex-disjoint cliques $C_1, C_2, C_3, C_4$ of size 5. $\gamma_s(H) = 11$ but there does not exist a minimal global secure set that contains at least 2 vertices from each clique $C_i$, where $i \in \{1, \ldots, 4\}$. Moreover, any set that contains only 1 vertex from a clique $C_i$ is not secure. Thus, in every minimal global secure set $W$, there is at least one clique $C_k$ ($k \in \{1, \ldots, 4\}$) such that $C_k \cap W = \emptyset$. It follows that at some point of the process of the expansion it is impossible to add a vertex from $C_k$. Thus $H$ is not weakly $\gamma_s$-expandable.

Basing on the cliques of odd order we can build infinitely many wqt graphs, similar to $H$, that are not weakly $\gamma_s$-expandable. Despite this fact we prove that every wqt graph is $\gamma_s$-monotone.

**Theorem 2.** If a graph $G$ of order $n$ is a connected wqt graph, then $G$ is $\gamma_s$-monotone, $\gamma_s(G) = \lceil \frac{n}{2} \rceil$ and for every integer $k > \frac{n}{2}$ there exists $SD_k \subseteq V(G)$ that is a global secure set in $G + K_1$. 
Proof. Clearly, if a given wqt graph has only 1 vertex, then the theorem is true. Suppose that it is also true for every connected wqt graph of order at most $b$, where $b > 1$. Let $G$ be a wqt graph of order $n > b$. Since $G$ is connected, join was the last operation in its creation. Let $G = G_1 + H$, where $H$ is an edgeless graph (see Definition 2). By $n_1$ and $n_H$ we denote the order of $G_1$ and $H$, respectively.

Case 1. Assume $G_1$ is connected. First, let us suppose that both $n_1$ and $n_H$ are odd. By the inductive hypothesis there exist $SD_{\lceil \frac{n_1}{2} \rceil} (G_1)$ such that it is a global secure set in $G_1 + K_1$, let $B$ denote this set. Furthermore, let $W$ be a set obtained by an union of $B$ and $\lceil \frac{n_H}{2} \rceil$ vertices of $H$. Clearly, $W$ is a dominating set and $|W| = \frac{n}{2}$. Let $X$ be a subset of $W$. If $X \subseteq V(H)$ then $|N[X] \cap W| = |X| + |B| = |X| + \lceil \frac{n_H}{2} \rceil$ and $|N[X] - W| = n_1 - |B| = \lceil \frac{n_1}{2} \rceil$. Hence $|N[X] \cap W| > |N[X] - W|$. In the case when $X \subseteq V(G_1)$, $|N[X] \cap W| = |N[X] \cap B| + \lceil \frac{n_H}{2} \rceil$ and $|N[X] - W| = |N[X] \cap (V(G_1) - B)| + \lceil \frac{n_1}{2} \rceil$. $B$ is a secure set in $G_1 + K_1$, thus by Theorem 1, $|N[X] \cap B| \geq |N[X] \cap (V(G_1) - B)| + 1$, which implies that $|N[X] \cap W| \geq |N[X] - W|$. If $X \cap V(G_1) \neq \emptyset$ and $X \cap V(H) \neq \emptyset$ then $|N[X] \cap W| = \frac{n}{2} = |N[X] - W|$. From the above considerations it follows that for any $X$, $|N[X] \cap W| \geq |N[X] - W|$. Thus, by Theorem 1, $W$ is a secure set of $G$ of cardinality $\frac{n}{2}$, which implies that $W$ is a minimal global secure set of $G$. Now we show how to obtain the global secure sets of greater cardinality. For any $m > \frac{n_1 + n_H}{2}$, let $W$ be a set of cardinality $m$, obtained by the union of $SD_{m}(G_1)$ and $Y_t$, where $p + t = m$, $p \geq \lceil \frac{n_1}{2} \rceil$, $t \geq \lceil \frac{n_H}{2} \rceil$, $Y_t \subseteq V(H)$ and $|Y_t| = t$. By Lemma 2, $W$ is a global secure set of $G$. Now we show that $W$ is a global secure set in $G + K_1$. Let $X$ be a subset of $W \subseteq V(G + K_1)$. Suppose first that $X \subseteq V(G_1)$. Then $|N[X] \cap W| = |N[X] \cap SD_{m}(G_1)| + |Y_t|$ and $|N[X] - W| = |N[X] \cap (V(G_1) - SD_{m}(G_1))| + |V(H) - Y_t| + 1$. By the security of $SD_{m}(G_1)$, we have that $|N[X] \cap SD_{m}(G_1)| \geq |N[X] \cap (V(G_1) - SD_{m}(G_1))|$. Since $n_1$ and $t \geq \lceil \frac{n_H}{2} \rceil$, $|Y_t| \geq |V(H) - Y_t| + 1$. Thus $|N[X] \cap W| \geq |N[X] - W|$. Now let us assume that $X \subseteq V(H)$. Then $|N[X] \cap W| = |X| + |SD_{m}(G_1)|$ and $|N[X] - W| = |V(G_1) - SD_{m}(G_1)| + 1$. Since $|SD_{m}(G_1)| \geq |V(G_1) - SD_{m}(G_1)| + 1$, $|N[X] \cap W| \geq |N[X] - W|$. In the last case assume $X \cap SD_{m}(G_1) \neq \emptyset$ and $X \cap Y_t \neq \emptyset$. Then $|N[X] \cap W| = |W| \geq \frac{n}{2}$ and $|N[X] - W| = n - |W| + 1$, which implies that $|N[X] \cap W| \geq |N[X] - W|$. Hence for any $X \subseteq W$, $|N[X] \cap W| \geq |N[X] - W|$. Thus, by Theorem 1 it follows that $W$ is secure in $G + K_1$.

Assume that $n_1$ or $n_H$ is even. By Lemma 2 we can obtain a set $SD_{m}(G)$ as an union of the set $SD_{m}(G_1)$ and $r$ vertices of $H$, where $p \geq \lceil \frac{n_1}{2} \rceil$, $r \geq \lceil \frac{n_H}{2} \rceil$ and $p + r = m \geq \lceil \frac{n}{2} \rceil$. If $p > \frac{n}{2}$ then we choose $SD_m(G_1)$ that is secure in $G_1 + K_1$. Similarly as previously, we can prove that for any $m > \frac{n}{2}$, the obtained set $SD_m(G)$ is also a global secure set of $G + K_1$.

Case 2. Suppose that $G_1$ is disconnected. Let $G_{11}, \ldots, G_{1q}$ ($q \geq 2$) be connected components of $G_1$. We join in disjoint pairs the components whose orders are of the same parity. If $q$ is odd then we leave one component without a pair. Only in the case when $q$ is even and $n_1$ is odd, we create one special pair, we call it a mixed pair, in which the components have different parity of orders. Observe that we cannot have both a mixed pair and a component without a pair. Let us denote the pairs obtained in the above procedure by $P_1, \ldots, P_{\lceil \frac{q}{2} \rceil}$.

First we show how to obtain the minimal global secure set of $G$. Let $G'$ and $G''$ be graphs that belong to the pair $P_i$. Without loss of generality we can assume that $n' \geq n''$, where $n'$ and $n''$ denote the order of $G'$ and $G''$, respectively. By the inductive hypothesis there exists a set $SD_{\lceil \frac{n'}{2} \rceil}(G')$ that is also a global secure set in $G' + K_1$, let us denote it by $C^1$. Analogously we obtain the set $C^1$ for every pair $P_i$, where $2 \leq i \leq \lceil \frac{q}{2} \rceil$. If $q$ is odd then we have one component $G_{1r}$ ($r \in \{1, \ldots, q\}$) of order $n_{1r}$ that does not belong to any pair. If $n_{1r}$ is odd, then there exists a set $SD_{\lceil \frac{n_{1r}}{2} \rceil}(G_{1r})$ that is a global secure set in $G_{1r} + K_1$, let $C^0$ be such a set; otherwise let $C^0$ be any $SD_{\lceil \frac{n_{1r}}{2} \rceil}(G_{1r})$. If $q$ is even then $C^0 = \emptyset$. The most problematic situation is when $n_1$ is odd and $n_{1r} = 1$. Suppose first that this is not the case. Then we obtain the set $SD_{\frac{n_1 + n_{1r}}{2}}(G)$ as an union of the sets $C^0, \ldots, C^1$ and $x$ vertices of $H$, where if $n_1$ is odd then $x = \lceil \frac{n_1}{2} \rceil$; otherwise $x = \lceil \frac{n_{1r}}{2} \rceil$. Our choice of the sets $C^0, \ldots, C^1$ implies that in the case when $n$ is odd, the obtained set is secure in $G + K_1$. Now let us consider the case when $n_1$ is odd and $V(H) = \{v\}$. We modify either $C^0$ or $C^1$, where $P_j$ is a mixed pair. If $q$ is odd, we replace an arbitrary vertex from $C^0$ with $v$, otherwise we do that for $C^1$. We leave the remaining sets $C^2 (z \neq j, 0 \leq z \leq \lceil \frac{q}{2} \rceil)$ without a change. Now let $W$
denote the union of the sets $C_0, \ldots, C_{1.5}$. Let $X \subseteq W$. If $v \not\in X$, then by Lemma 1 and the choice of $C_0, \ldots, C_{1.5}$, we have $|N[X] \cap W| \geq |N[X] \setminus W|$; otherwise since $v$ is a universal vertex in $G$, $|N[X] \cap W| = \frac{n}{2} = |N[X] \setminus W|$. Hence $W$ is a minimal global secure set in $G$.

To obtain a set $SD_m$, where $m \geq \gamma_s(G)$, we use the inductive hypothesis and Lemma 3 (we can apply it to the pairs of components) to find the set $S_p(G_1)$ (a secure set of cardinality $p$), where $n_1 \geq p \geq \lceil \frac{n}{2} \rceil$. By Lemma 2 the set $SD_m(G)$ (that is also a global secure set in $G + K_1$) can be obtained as the union of $S_p(G_1)$ and $t$ vertices of $H$, where $p \geq \lceil \frac{n}{2} \rceil$ and $t \geq \lceil \frac{n}{2} \rceil - 1$. \hfill \Box

In [12] it was proven that we can recognize wqt graphs in $O(n+m)$ time. The recognition algorithm is based on the cotree of the given graph. On the basis of the above proof we can obtain a linear time algorithm that can find a global secure set of any proper cardinality in a given wqt graph.

3. Co-quasi-threshold graphs

Co-quasi-threshold graphs (co-qt graphs) are the graphs without induced $P_4$ and $2K_2$. This class of graphs is equivalent to co-trivially perfect graphs. Co-qt graphs can be recognized in linear time by an algorithm presented by Chu [3]. In this section we prove that co-qt graphs are weakly $\gamma_s$-expansible.

**Theorem 3.** If a graph $G$ is a connected co-qt graph of order $n$, then $G$ is weakly $\gamma_s$-expansible, $\gamma_s(G) = \lceil \frac{n}{2} \rceil$ and there exist sets $SD_{\gamma_s(G)}, SD_{\gamma_s(G) + 1}, \ldots, SD_n$ such that for $\gamma_s(G) < i \leq n$, $SD_i = SD_{i-1} \cup \{v\}$, where $v \in V - SD_{i-1}$, and for every integer $k > \frac{n}{2}$, $SD_k$ is a global secure set in $G + K_1$.

**Proof.** If $G$ is a complete graph, then clearly the theorem holds. Let us suppose that the theorem is true for every co-qt graph of order at most $n'$, where $n' > 2$, and let $G$ be a connected co-qt graph of order $n > n'$ that is not a complete graph. Since $G$ is a connected graph, join was the last operation in its creation. Hence, let $G = G_1 \cup G_2$ and $n_1, n_2$ denote the order of $G_1$ and $G_2$, respectively. From the properties of the cotree and the fact that $G$ is not a complete graph it follows that, without loss of generality, we can assume that $G_1$ is disconnected. Since $G$ does not have $2K_2$ as an induced subgraph, neither $G_1$ nor $G_2$ has two connected components of order greater than 1. Let $G_1 = G_{11} \cup H_1 (G_{21} \cup H_2)$, where $H_1 (H_2)$ is an edgeless graph and $G_{11} (G_{21})$ is a connected component of $G_1 (G_2)$ of the greatest cardinality. Furthermore, let $n_{11} > 0 (n_{21} > 0)$ and $n_{12}, n_{22} > 0 (n_{12}, n_{22} > 0)$ denote the order of $G_{11} (G_{21})$ and $H_1 (H_2)$, respectively. Since being a co-qt graph is an induced hereditary property, we can apply the inductive hypothesis to $G_{11}$ and $G_{21}$. Hence, for $j \in \{1, 2\}$ there exists sets $A_j = (SD_0(G_{j1}), SD_1(G_{j1}), \ldots, SD_{\gamma_s(G_{j1})}(G_{j1}))$ such that for $\gamma_s(G_{j1}) < i \leq n_{j1}$, $SD_i(G_{j1}) = SD_{i-1}(G_{j1}) \cup \{v\}$, where $v \in V(G_{j1}) - SD_{i-1}(G_{j1})$, and for every $k > \frac{n_{j1}}{2}$, $SD_k(G_{j1})$ is a global secure set in $G_{j1} + K_1$. From now on we choose only global secure sets of $G_{11}$ and $G_{21}$ that belong to $A_1$ and $A_2$, respectively. First we show how to obtain the minimal global secure set of $G$. For $i \in \{1, 2\}$, let $X_i$ denote the subset of cardinality $k$ of the vertices of $H_i$. Let us assume that $n_1$ or $n_2$ is even. For $i \in \{1, 2\}$, we define the set $F_i$ in the following way:

$$F_i = \begin{cases} X_i \cup SD_{\gamma_s(G_{i1})} - 1 (G_{i1}) & \text{if } n_{i1} \text{ is odd}, \\ SD_{\gamma_s(G_{i1})} & \text{if } n_{i1} \text{ is even and } n_{i1} > 0, \\ SD_{\gamma_s(G_{i1})} & \text{if } n_{i1} \text{ is even and } n_{i1} = 0. \end{cases}$$

By Lemma 2, the set $F = F_1 \cup F_2$ is a global secure set of $G$ of cardinality $\lceil \frac{n}{2} \rceil$. Furthermore, if $n$ is odd, then it is a global secure set in $G + K_1$. Now we consider the case when both $n_1$ and $n_2$ are odd. We define the set $F_2$ as previously, whereas to obtain the minimum global secure set of cardinality $\frac{n}{2}$ we decrease the cardinality of $F_1$. If $n_{i1} > 1$, let $F_1 = X_1 \cup SD_{\gamma_s(G_{i1})} - 1 (G_{i1})$; otherwise $n_{i1}$ is even and $F_1 = SD_{\gamma_s(G_{i1})} (G_{i1})$. Our choice of $F_2$ implies that even though $|F_1| < \frac{n}{2}$ the set $F = F_1 \cup F_2$ is a global secure set of $G$. To finish the proof we describe one of the possible procedures of the expansion of $F$.

Let $P_1 = F$.

1. For $i := 1$ to $n - |P_1|$. If $V(G_{i1}) - P_1 \neq \emptyset$ then $v := SD_{k+1}(G_{i1}) - SD_k(G_{i1}), \text{ where } k = |V(G_{i1}) \cap P_1|$. Else $v := \text{any vertex that belongs to } V(H_1) - P_1$.

Recall that also in this procedure we use only the global secure sets that belong to $A_1$ and $A_2$. The existence of the sets $P_i (1 \leq i \leq n - |P_1| + 1)$ proves that $G$ is weakly $\gamma_s$-expansible and the choice of the $P_i$ guarantees that if $|P_i| > \frac{n}{2}$, then $P_i$ is a global secure set in $G + K_1$. This observation finishes the proof. \hfill \Box
As in previous section we can construct a linear time algorithm that can find a global secure set of any proper cardinality in a given co-qt graph.

4. **k**-trees

In this section we consider the possibility of expansion of a global secure set in the class of **k**-trees. A **k**-tree [13] is defined inductively as follows:
1. the complete graph on **k** vertices is a **k**-tree;
2. let **G** be a **k**-tree on **n** vertices and let **K** be a **k**-clique in **G**; then the (**n** + 1)-vertex graph **G**' formed from **G** by adding a new vertex **v** adjacent to all vertices of **K** is a **k**-tree.

A perfect elimination order is an ordering (**v**₁, **v**₂, . . . , **v**ₙ) of the vertices of the graph such that **N**(**v**ᵢ) induces a clique in a graph induced by the vertices **{v**₁+1, . . . , **v**ₙ}. From the definition of the **k**-tree it follows that every **k**-tree has a perfect elimination order.

**Theorem 4.** If a graph **G** is a **k**-tree where **k** ≤ 2, then **G** is **γ**₂-expandable.

**Proof.** We prove the above theorem for **k** = 2. Similarly, we can prove it for **k** = 1. Let **v**₁, . . . , **v**ₙ be a perfect elimination order of **G** that is accordant with the reverse order of adding vertices in the construction of **G**. Furthermore, let **A** (|**A**| < **n**) be a global secure set in **G**. Let **i** be the smallest **i** such that **v**ᵢ /∈ **A**. Thus **i** = 1 or **v**₁, . . . , **v**ᵢ₋₁ ∈ **A**. Since **A** is a dominating set and **G** is a 2-tree, **v**₁ has at least 1 neighbour in **A** and at most 2 neighbours in **{v**₁+1, . . . , **v**ₙ} that do not belong to **A**. If at most 1 neighbour of **v**ᵢ does not belong to **A**, then clearly we can add **v**ᵢ to **A**. So suppose that **i** > 1 and **v**ᵢ has 2 neighbours **v**₁, **v**ᵢ /∈ **A**, where **j**, **t** ∈ [1, . . . , **n**]. Since **G** is a 2-tree and **A** is a dominating set such that **v**₁, . . . , **v**ᵢ₋₁ ∈ **A**, there exists a vertex **v**ᵢ⁺ ∈ **E** such that **v**ᵢ **v**ᵢ⁺ ∈ **E** and **v**ᵢ **v**ᵢ⁺ ∈ **E** or **v**ᵢ **v**ᵢ⁺ ∈ **E**. Without loss of generality we can assume that **v**ᵢ **v**ᵢ⁺ ∈ **E**. Since **A** is a dominating set, it is enough to show that **A** ∪ **{v**ᵢ} is a secure set. From the security of **A** it follows that, if in an attack the vertex **v**ᵢ or **v**ᵢ attacks a vertex different from **v**ᵢ, then such an attack is defendable, since **v**ᵢ can defend itself against one attacker. The only problematic case is when both **v**ᵢ and **v**ᵢ attack **v**ᵢ. We show that in this situation, **v**ᵢ can defend **v**ᵢ. Suppose conversely that **v**ᵢ cannot defend **v**ᵢ. It follows that **v**ᵢ has a neighbour **v**ᵢ⁺ where 1 ≤ **r** ≤ **p** − 1, that has at least 1 (otherwise **v**ᵢ does not require any defence) and at most 2 neighbours in **V** − (**A** ∪ **{v**ᵢ}). Let us consider the neighbourhood of **v**ᵢ. From the choice of **v**ᵢ and the perfect elimination order it follows that all the neighbours of **v**ᵢ in **{v**₁, . . . , **v**ᵢ₋₁} belong to **A**. However, **v**ᵢ has only two neighbours in **{v**ᵢ₊₁, . . . , **v**ₙ}. One of them is **v**ᵢ⁺, let us denote the second one by **y**. From the construction of **G** it follows that **v**ᵢ **v**ᵢ⁺ ∈ **E**. However, **N**(**v**ᵢ) ∩ (**V** − (**A** ∪ **{v**ᵢ}))) = **{v**ᵢ⁺}. Hence, **y** = **v**ᵢ and **v**ᵢ attacks **v**ᵢ (otherwise **v**ᵢ would not have to participate in the defence of **v**ᵢ), which gives us the contradiction with our previous assumption that **v**ᵢ is the attacker of **v**ᵢ. This observation finishes the proof.

The next theorem concerns **k**-trees, where **k** ≥ 3.

**Theorem 5.** For every integer **k** ≥ 3, there exists a **k**-tree that is not **γ**₂-expandable.

**Proof.** We divide the proof into two cases. First suppose that **k** = 3. Consider the graph presented on Fig. 2, let us denote it by **G**. The black vertices form a global secure set, let us denote it by **W**. That is not expandable. It follows from the fact that for every white vertex **x**, |**N**(**x**) ∩ (**W** ∪ **{x}**)| < |**N**(**x**) − (**W** ∪ **{x}**)|. Hence by Theorem 1, the set **W** ∪ **{x}** is not secure.

Now assume **k** ≥ 4. Let **G** be a **k**-tree with a set of vertices **V** = **{v**₁, **v**₂, . . . , **v**₄⁺}. Furthermore, the following sets of vertices induce cliques: **{v**₁, . . . , **v**₄⁺}, **{v**₂, . . . , **v**ᵢ⁺} for 1 ≤ **i** ≤ **k**, **{v**₂, . . . , **v**ᵢ⁺} for 2**k** + 1 ≤ **j** ≤ 3**k**, and |**v**ᵢ⁺| for 3**k** + 1 ≤ **p** ≤ 4**k**. Let us consider a set **W** = **{v**₂⁺, . . . , **v**₄⁺} | **{v**ᵢ⁺} for 3**k** + 1 ≤ **p** ≤ 4**k**. From the construction of **G** it follows that **v**₄⁺ is an universal vertex. Hence **W** is a dominating set. Now we show that **W** is secure. Let **X** be a subset of **W**. First suppose that **v**₄⁺ /∈ **W**. If **v**ᵢ ∈ **W**, where **t** ∈ [**k** + 2, . . . , 2**k**], then |**N**(**X**) ∩ **W**| = 2**k** and |**N**(**X**) − **W**| ≤ **k**; otherwise |**N**(**X**) ∩ **W**| ≥ |**X**| + **k** and |**N**(**X**) − **W**| = 0. Now suppose that **v**₄⁺ ∈ **W**. Then |**N**(**X**) ∩ **W**| = |**N**(**X**) − **W**|. In every considered case |**N**(**X**) ∩ **W**| ≥ |**N**(**X**) − **W**|. Thus by Theorem 1 the set **W** is secure.
In the last step of the proof we show that there does not exist \( y \in (V(G') - W) \) such that \( W \cup \{y\} \) is secure. Consider the vertices \( v_{2k+1}, \ldots, v_{3k} \). Each of these vertices has exactly 1 neighbour in \( W \), and \( k \) neighbours in \( V(G') - W \). Whereas each of the vertices \( v_1, \ldots, v_k \) has at most \( k \) neighbours in \( W \), and \( 2k - 1 \) neighbours in \( V(G') - W \). Thus, by Theorem 1, none of the considered vertices can be used to expand the set \( W \). It follows that \( W \) is a global secure set of \( G' \) that is not expandable, which implies that \( G' \) is not \( \gamma_s \)-expandable.

The following questions remain open.

**Problem 1.** Are \( k \)-trees weakly \( \gamma_s \)-expandable, for \( k \geq 3 \)?

**Problem 2.** Are \( k \)-trees \( \gamma_s \)-monotone, for \( k \geq 3 \)?

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**References**