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Properties of (0, 1)-Matrices with No Triangles*

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We study (0, 1)-matrices which contain no triangles (submatrices of order 3 with row and column sums 2) previously studied by Ryser. Let the row intersection of row i and row j of some matrix, when regarded as a vector, have a 1 in a given column if both row i and row j do and a zero otherwise. For matrices with no triangles, columns sums $\geqslant 2$, we find that the number of linearly independent row intersections is equal to the number of distinct columns. We then study the extremal (0, 1)-matrices with no triangles, column sums $\geqslant 2$, distinct columns, i.e., those of size $m \times \binom{m}{2}$. The number of columns of column sum l is $m-l+1$ and they form a $(l-1)$-tree. The $\binom{m}{2}$ columns have a unique SDR of pairs of rows with 1's. Also, these matrices have a fascinating inductive buildup. We finish with an algorithm for constructing these matrices.

1. INTRODUCTION

We wish to study (0, 1)-matrices with no triangles. We define a *configuration* to be an equivalence class of matrices, where two matrices *represent* the same configuration if one of the matrices is a row and column permutation of the other matrix. A *triangle* is defined as the configuration represented by

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{1.1}$$

A matrix *contains* a configuration if a submatrix represents the configuration.

Properties of (0, 1)-matrices without triangles have been studied by Ryser. For example we restate Theorem 2.1 [7]. Throughout the discussion we let A^T denote the transpose of the matrix A.

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THEOREM 1.1 (Ryser). *Let A be a $(0, 1)$-matrix of size $m \times n$ with $AA^T > 0$ and let A have no triangles. Then A has a column of m 1's.*

If S is a square symmetric matrix with positive integral entries and zeroes on the diagonal then we define the class $C(S)$ as follows:

$$C(S) = \{A \ (0, 1)\text{-matrix} \mid AA^T = S + D;\ D \text{ arbitrary diagonal matrix}\}. \quad (1.2)$$

To avoid trivialities, we require that matrices in $C(S)$ have column sums greater than 1. One consequence of Theorem 1.1 is Theorem 3.1 in [8], which we restate below.

THEOREM 1.2 (Ryser). *Every matrix in the class $C(S)$ contains a triangle or else the class contains exactly one matrix, apart from column permutations, without triangles.*

Throughout the paper, the *row intersection* of row i and row j $(i \neq j)$ of some matrix, when regarded as a vector, will have a 1 in a given column if both row i and row j do and a zero otherwise. Following the methods of Ryser in [9] which utilize Theorem 1.2, we obtain our first main result, which will be proved in Section 2.

THEOREM 1.3. *Let A be a $(0, 1)$-matrix of size $m \times n$ with column sums greater than 1 and no triangles. Then the number of linearly independent row intersections is equal to the number of distinct columns.*

Let A be a matrix satisfying the hypotheses of the theorem with, in addition, distinct columns. It follows from Theorem 1.3 that $n \leqslant \binom{m}{2}$. We are interested in the extremal matrices, i.e., those with $n = \binom{m}{2}$, which we will call solutions.

These solutions have a fascinating structure. In Section 3 we establish that, in a solution, the number of columns of column sum l $(2 \leqslant l \leqslant m)$ is $m - l + 1$ and they form a $(l - 1)$-tree. In Section 4 we establish our main structure theorem (Theorem 4.1) and show among other things that given any column of column sum l, there are $\binom{l}{2}$ columns of the solution with all their 1's contained in the same rows as the 1's of the given column. Thus they form a solution of size l. A nice inductive structure of the solutions is found in this way.

We may associate with each column of a solution the set consisting of all pairs of distinct rows containing 1's. Remarkably, we will find that these $\binom{m}{2}$ sets have a unique SDR. We also derive a few additional properties. For example, given any column of column sum l $(l < m)$ there is another column in the solution with 1's in the same rows as the given column and additional 1's. We finish the paper by presenting an algorithm to construct all solutions.

Theorems 1.1, 1.2, and 1.3 and the existence of SDR's have been generalized to (0, 1)-matrices with certain other lists of forbidden configurations. These results will appear in the author's thesis.

2. The Row Intersection Theorem

We restate Theorem 1.3.

THEOREM 2.1. *Let A be a* (0, 1)*-matrix of size $m \times n$ with column sums greater than* 1 *and no triangles. Then the number of linearly independent row intersections is equal to the number of distinct columns.*

Proof. The proof is similar to the discussion in [9]. Let $A = (a_{ij})$ be a matrix as described. A column which is identical to another column may be deleted without affecting the linear independence of the row intersections so we will assume that n is the number of distinct columns in A. Let n' equal the number of linearly independent row intersections. We have immediately that $n \geqslant n'$ since n is the dimension of the space containing them. Thus we will assume that $n > n'$ and will arrive at a contradiction.

Consider A as describing m subsets $S_1, S_2, \ldots, S_m$ of an n-set $\{x_1, x_2, \ldots, x_n\}$, where $x_j \in S_i$ if $a_{ij} = 1$ and $x_j \notin S_i$ if $a_{ij} = 0$. Let $X = \operatorname{diag}[x_1, x_2, \ldots, x_n]$, where this is the notation for a diagonal matrix with the diagonal elements as specified. Regard $x_1, x_2, \ldots, x_n$ as independent indeterminates. Then we have the fundamental matrix equation for sets [9].

$$AXA^T = Y. \tag{2.1}$$

Let $Y = (y_{ij})$ where Y is of order m. Then y_{ij} is the sum of the indeterminates in $S_i \cap S_j$. Thus the row intersections, as vectors, correspond to the entries y_{ij} $(i \neq j)$ when considered as vectors in n-dimensional rational space with basis $\{x_1, x_2, \ldots, x_n\}$.

Regarding $x_1, x_2, \ldots, x_n$ as variables, we wish to set $y_{ij} = 0$ $(i \neq j)$. By our assumption, the number of variables n exceeds the number of linearly independent equations $y_{ij} = 0$ $(i \neq j)$. Thus we can find rational and hence integral values $e_1, e_2, \ldots, e_n$ not all zero such that for $E = \operatorname{diag}[e_1, e_2, \ldots, e_n]$

$$AEA^T = D, \tag{2.2}$$

where D is some diagonal matrix. Every variable x_i occurs in some equation (column sums greater than 1) thus some e_i's are positive and some negative. Define matrices A_1 and A_2 as follows. For all i with $e_i > 0$, A_1 contains column i of A repeated e_i times. For all j with $e_j < 0$, A_2 contains column j of A repeated $-e_j$ times. Then

$$A_1A_1^T - A_2A_2^T = D. \tag{2.3}$$

This yields that A_1 and A_2 belong to the same class $C(S)$, for an appropriate choice of S, as defined in the Introduction. By Theorem 1.2, A_1 and A_2 are the same apart from a column permutation so A has a repeated column which is a contradiction that proves the theorem.

COROLLARY 2.2. *Let A be a* (0, 1)*-matrix of size $m \times n$, column sums greater than* 1, *distinct columns, no triangles. Then $n \leqslant \binom{m}{2}$.*

Proof. This follows directly from Theorem 2.1 by noting that the number of row intersections is $\binom{m}{2}$. This result is Theorem 2.2 in [9].

3. COLUMNS OF A GIVEN COLUMN SUM

In view of Corollary 2.2, we are interested in the extremal matrices with $n = \binom{m}{2}$ and we will call such matrices *solutions* (*of size m*). Ryser gives us one infinite family of solutions, but we will give another construction and also describe the structure of the solutions.

Let A be a solution of size m. From Theorem 2.1 we know that all $\binom{m}{2}$ row intersections are linearly independent. Let B be a submatrix of A of size $r \times \binom{m}{2}$ consisting of r rows of A. It has $\binom{r}{2}$ linearly independent row intersections. Deleting columns with one or no 1's or deleting repeated columns does not affect this. The process yields a submatrix B' of size $r \times \binom{r}{2}$ by Theorem 2.1 which is a solution of size r. Note that if a matrix does not contain a configuration, then no submatrix does either. The case where the r selected rows are precisely the rows containing 1's for some column is especially interesting and is discussed in Corollary 4.2.

Let C_k be the configuration represented by the (0, 1)-matrix $C = (c_{ij})$ of order k. Let $c_{ij} = 1$ if $i = j$ or $i = j + 1$ or $i = 1$, $j = k$ and let $c_{ij} = 0$ otherwise. Thus C is the incidence matrix of the cycle of length k.

Remark 3.1. A solution A contains no C_k's for $3 \leqslant k \leqslant m$.

Proof. Certainly A has no C_3's since C_3 is a triangle. Let l be the smallest value of k for which A has a C_k. The associated l rows contain a solution B of size $l \times \binom{l}{2}$. Any other column of column sum t $(2 \leqslant t < l)$ in B creates a smaller C_k in B and hence in A. The column of l 1's is possible but then B has at most $1 + l$ columns and $1 + l < \binom{l}{2}$ for $l > 3$ which is a contradiction. This proves the remark.

Remark 3.1 sets up the following lemma, which plays an essential role in our study of the structure of solutions in Section 4.

LEMMA 3.2. *Let L be a* (0, 1)*-matrix of size $m \times n$, column sums l $(l \geqslant 2)$, distinct columns, and no C_k's for $3 \leqslant k \leqslant m$. Then $n \leqslant m - l + 1$ and* *in addition, for $n \geqslant 2$, there exist, two rows of L each of row sum* 1 *and with the two* 1*'s in different columns.*

We think of L as those columns of a solution that have column sum l. We will use the hypergraph terminology of Berge [2]. The rows will correspond to m vertices. Each column will be an edge, a set of vertices. Let H be the hypergraph associated with L. In the case $l=2$, H is also a graph and the condition no C_k's translates as no cycles. Thus, for $l=2$, the lemma is a well-known result about forests and trees.

The forbidden configurations translate awkwardly to hypergraphs in general. We define a *special chain* of length n to be a chain $x_1E_1x_2E_2 \cdots x_nE_nx_{n+1}$ of vertices x_i and edges E_j with $x_1, x_2,..., x_n$ distinct, $E_1, E_2,..., E_n$ distinct, and $E_i \cap \{x_1, x_2,..., x_{n+1}\} = \{x_i, x_{i+1}\}$. The definition of chains in hypergraphs only requires $E_i \cap \{x_1, x_2,..., x_n\} \supseteq \{x_i, x_{i+1}\}$. A *special cycle* of length n is a special chain as above with $x_1 = x_{n+1}$. Thus L has no C_k's ($k \geqslant 3$) if and only if H has no special cycles of length greater than 2. This makes it clear that, in some sense, we have a stronger result than the lemma in [5]. Our hypergraphs may contain the usual cycles of length greater than 2.

We define a once covered vertex as a vertex belonging to exactly one edge of a hypergraph. Also, let $v(H')$ be the union of the edges in some hypergraph H'. Before proving the lemma, we establish the following connection between chains and special chains.

Remark 3.3. If a pair of vertices are joined by a chain, then they are joined by a special chain. Thus every pair of vertices in a component of a hypergraph are joined by a special chain.

Proof. Let the shortest chain joining x and y be $xE_1x_1E_2x_2 \cdots x_{n-1}E_ny$. We will show that this is a special chain. If $x \in E_i$ for $i > 1$, then the chain $xE_ix_i \cdots x_{n-1}E_ny$ is a shorter chain joining x and y. This is a contradiction and thus $x \notin E_i$ for $i > 1$. Similarly $y \notin E_i$ for $i < n$. Say three vertices x_k, $x_{i-1}, x_i \in E_i$. If $k < i-1$, the chain $xE_1x_1 \cdots x_kE_ix_i \cdots x_{n-1}E_ny$ is a shorter chain joining x and y. Similarly for $k > i$. We conclude that the shortest chain joining x and y is a special chain.

Proof of the lemma. We will prove the lemma by induction on $m+n$ for a given l. If $m=l$, since the edges are distinct, $n \leqslant 1 = m-l+1$. The second conclusion holds vacuously. This establishes the base of the induction for the given l.

Assume the lemma is true for less than m vertices and less than n edges. H, the hypergraph given by L, has m vertices. If H has a once covered vertex, then delete it and the edge which covered it. By induction we have the inequality $n-1 \leqslant (m-1)-l+1$ and thus $n \leqslant m-l+1$. Thus to prove the lemma, it suffices to show that H has two once covered vertices, covered by different edges, for $n \geqslant 2$. The case $n = 1$ is easy to check.

Assume H does not have two once covered vertices from different edges and $n \geqslant 2$. Then there is an edge E in H all of whose vertices are covered by other edges in H. Let H' be the hypergraph consisting of the edges of H which intersect E in at most $l-1$ vertices. We define it this way so that the same arguments can be used in Theorem 4.1, where E is an edge of size greater than l. Consider a mapping of the edges of H' $\theta: H' \to H''$ where we define for an edge $E' \in H'$, $\theta(E') = E' \backslash E$. H'' is simply the hypergraph whose edges are the images of the edges in H'. H'' corresponds to the hypergraph obtained from H by deleting the vertices of E.

If two different edges E_1, E_2 of H' have $\theta(E_1) = \theta(E_2)$, then we will show that this leads to a contradiction. Take $z \in E_1 \backslash E = E_2 \backslash E$. Take $x \in E_1 \cap E \backslash E_2$ and $y \in E_2 \cap E \backslash E_1$; this being possible since $|E_1| = |E_2| = l$. Then H contains the special cycle $xEyE_2zE_1x$, a contradiction. Thus θ is 1–1 on the edges of H'. Using the usual definition, we partition the edges of H'' into components. This partitions the edges of H' into what we will call *E-components*. As a consequence of Remark 3.3, a pair of vertices x, y in the same E-component, but not in E, are joined by a special chain, none of whose vertices are in E.

We will have proven the lemma if we can show that either there are at least two E-components, each with a once covered vertex outside of E or that there is one E-component with two once covered vertices, from different edges, outside of E.

To prove this take any pair of vertices x, y, where $x \in E' \cap E \backslash E''$ and $y \in E'' \cap E \backslash E'$ and E' and E'' are in the same E-component. We will show that neither x nor y are once covered vertices. Recall that $(E' \cap E'') \backslash E = \emptyset$. Otherwise, by the same argument that θ is 1–1, H would contain a special cycle of length 3. Take $x' \in E' \backslash E$ and $y \in E'' \backslash E$. Since these vertices are in the same E-component, they are joined by a special chain $x'E_1x_1E_2x_2 \cdots x_{n-1}E_ny'$ with $x_i \notin E$ for $1 \leqslant i \leqslant n-1$. We may assume E', $E'' \notin \{E_1, E_2, \dots, E_n\}$. Otherwise we would have either $E' = E_1$ and we could replace x' by x_1 or $E'' = E_n$ and we could replace y' by x_{n-1}. The new shorter special chain would be of the desired form.

By appending to the above special chain $y'E''yExE'x'$ we would have a contradiction if a special cycle were formed. This will certainly be the case if both x and y are once covered vertices. If $x \in E_j$ and $y \notin E_j$, then replace E' by E_j, x' by x_j, and use the shorter special chain $x_jE_{j+1}x_{j+1} \cdots x_{n-1}E_ny'$. Similarly if $y \in E_j$ and $x \notin E_j$, then replace E'' by E_j, y' by x_{j-1}, and use the shorter special chain $x'E_1x_1E_2 \cdots x_{j-2}E_{j-1}x_{j-1}$. Assuming there is no edge E_j with $x, y \in E_j$, then the above changes, denoted by *'s, would eventually yield the special cycle $x'^*E_1^*x_1^*E_2^* \cdots x_{k-1}^*E_k^*y'^*E''^*yExE'^*x'^*$, a contradiction. Note that $x \in E'^*$, $y \notin E'^*$ and $x \notin E''^*$, $y \in E''^*$. Thus the special cycle that is created is of length at least 4. Thus for some j, $x, y \in E$ and so neither x nor y are once covered.

Thus an E-component cannot have two once covered vertices, covered by different edges, in E and thus by induction the E-component has a once covered vertex outside of E. The case that the E-component has only one edge is verified easily. We will now show that if the E-component has a once covered vertex in E, then there are two E-components and the lemma will be proven.

Consider an E-component with one once covered vertex, say $x \in E_1 \cap E$. If there is another edge E_2 in the E-component with $y \in (E_2 \cap E) \backslash E_1$, then by our previous argument x is not once covered. Thus the intersection of the E-component with E is $E_1 \cap E$, which consists of at most $l-1$ vertices. Since in H, the edge E contains no once covered vertices, there must be additional E-components to cover the remaining vertices of E. This completes the proof.

COROLLARY 3.4. *In a solution of size m, the number of columns of column sum l is precisely $m-l+1$ for $2 \leqslant l \leqslant m$.*

Proof. Let a_l be the number of columns of column sum l. Then by Lemma 3.2

$$\binom{m}{2} = \sum_{l=2}^{m} a_l \leqslant \sum_{l=2}^{m} (m-l+1) = \binom{m}{2}, \tag{3.1}$$

and so equality holds for every l.

We define *k-trees* as a graph following Beineke and Pippert [1]. The definition is inductive on the number of vertices n in the graph. A set of k mutually adjacent vertices is a k-tree and is analogous to the initial root vertex in a usual tree. A k-tree on $n+1$ vertices is obtained from one on n vertices by joining the $(n+1)$st vertex to some set of k mutually adjacent vertices. We think of a k-tree as a collection of superimposed complete graphs on $k+1$ vertices.

Remark 3.5. In a solution of size m, the columns of column sum $k+1$ $(m > k)$ can be thought of as a k-tree.

Proof. Let H be the hypergraph associated with the columns of column sum $k+1$. Let G be the graph obtained as the set union of the edges of size 2 contained in the edges of H, on the m vertices of H. Using the inductive definition, we will verify that G is a k-tree.

Since H comes from a solution, it has $m-k$ edges, the maximum number. If $m = k+1$, then H has one edge and G corresponds to the second stage in the inductive definition of a k-tree. Nest consider a once covered vertex x in some edge E. By Lemma 3.2 we may repeatedly delete other once covered vertices, not in E, as well as the edges that cover them until the new hypergraph H' has only $k+2$ vertices and hence only two edges E and E'. But then we can think of x as the $(n+1)$st vertex in the definition of k-trees joined to k other vertices already mutually adjacent, in this case because of the edge E'. We can repeat this process with a new hypergraph H'' obtained from H by deleting x and E. Thus the remark is proven by induction on m.

Harary and Palmer have proven Lemma 3.2 in the case $l=3$ by looking at acyclic simply connected 2-plexes, which is their version of 2-trees [6]. Note that k-trees, as a set of columns of column sum $k+1$, may contain triangles for $k>1$.

4. The Main Structure Theorem

Using the techniques of Lemma 3.2 we are able to establish the main structure theorem. Define one column to *cover* another column if the latter column has nonzero entries only in the rows where the former column does.

THEOREM 4.1. *Let A be a solution of size m. Then a column of l 1's in A covers precisely $l-t+1$ columns of t 1's of A for $2 \leqslant t \leqslant l$.*

Proof. Since A is a solution, A has the maximum number $m-t+1$ columns of column sum t. Let H be the hypergraph associated with these $m-t+1$ columns. Let E be an edge consisting of the l vertices specified by the column of l 1's. Delete a once covered vertix in H that lies outside E and delete the edge which covers it. The remaining hypergraph has the maximum number of edges of size t on the remaining vertices. Repeat this process, deleting as many vertices as possible, to obtain a hypergraph H' having the maximum number of edges of size t on the vertices $v(H')$.

We will show that $v(H')=E$. In analogy to Lemma 3.2, we look at the E-components of H'. The E-components consist of the edges of H' that intersect E in at most $t-1$ vertices. The fact that $|E|=l \geqslant t$ causes no difficulty. By an argument in the lemma, if there is an E-component, it must have a once covered vertex outside of E even if the E-component consists of one edge. But this vertex would already have been deleted. Thus there are no E-components and $v(H')=E$. We conclude by noting that in H' and hence in H there are the maximum number $l-t+1$ edges of size t covered by the l vertices of E which proves the theorem.

From Theorem 4.1 we are able to derive a great deal about the structure of solutions.

COROLLARY 4.2. *Given any column of column sum l in a solution A,* *there are $\binom{l}{2}$ columns of the solution, covered by the given column, which form a solution of size l.*

Let the first column of A be the column of l 1's in the first l rows. After a permutation of the remaining columns of A we have

$$A = \begin{bmatrix} A' & * \\ 0 & * \end{bmatrix}. \tag{4.1}$$

In (4.1) A' is a solution of size l and 0 is a matrix of 0's.

Proof. Apply Theorem 4.1 for all possible values of t yielding $\binom{l}{2}$ distinct columns of column sum greater than 1 covered by the given column of l 1's.

Define K_m to be a (0, 1)-matrix of size $m \times \binom{m}{2}$ with column sums 2 and distinct columns. Thus K_m has all possible columns with column sum 2. The columns are ordered so that if column i has 1's in rows j and k and column p has 1's in rows q and r, then $i < p$ if $j < q$ or $j = q$ and $k < r$. Thus K_m is the incidence matrix of the complete graph K_m on m vertices.

COROLLARY 4.3. *For a solution A of size m, there is a unique permutation matrix P of order $\binom{m}{2}$ such that $A \geqslant K_m P$.*

Proof. For each column i of A let

$$S_i = \{\{j, k\} \mid \text{column } i \text{ has 1's in rows } j \text{ and } k, j \neq k\}. \tag{4.2}$$

Then this corollary states that the $\binom{m}{2}$ sets S_i have a unique SDR. We will prove this by induction on the column size l. There is no choice and no conflicts for $l = 2$. At column size $l = k - 1$ we assume that there have been no conflicts up to this point in the selection of an SDR and that it is unique. Consider a column of column sum k. It covers two columns of column sum $k - 1$ by Theorem 4.1 and so the only choice left for a pair of rows is precisely the two rows not in common to both columns of column sum $k - 1$. This pair cannot already have been chosen since it would imply the existence of a column of column sum less than or equal to k with 1's in those two rows and hence a zero somewhere in the k rows of the column we are considering. In conjunction with the two columns of column sum $k - 1$ a triangle will be formed. This is a contradiction and so the corollary follows by induction.

Corollary 4.3 tells us that any solution A of size m can be obtained from the matrix K_m by adding 1's. Now K_m has $\binom{m}{3}$ triangles and the zeroes of the triangles are disjoint. Thus at least $\binom{m}{3}$ 1's must be added to "kill" the $\binom{m}{3}$ triangles. However from Corollary 3.4 we know that this is precisely the number of 1's added. As a consequence no further triangles are created by the $\binom{m}{3}$ 1's since additional 1's would be required to kill them.

It is easy, now, to construct an infinite family of solutions. Let A be the matrix obtained from K_m by replacing column i with 1's in rows j and k $(j < k)$ by the column with a 1 in row l for each l, $j \leqslant l \leqslant k$. It is easy to verify that A has distinct columns. But clearly A has the consecutive 1's property for columns. We recall that a matrix has the consecutive 1's property for columns if a row permutation (in this case the identity) of the matrix has the 1's appearing consecutively in each column. This property was studied by Fulkerson and Gross and they remarked that such matrices have no triangles [4]. Thus A is a solution and apart from row and column permutations it is the unique solution of size m with the consecutive 1's property. Of course there are infinitely many solutions apart from these and the infinite family of Ryser [9].

COROLLARY 4.4. *In a solution A of size m there are two rows of sum $m-1$. After suitable row and column permutations we have*

$$A = \left[\begin{array}{ccc|cccc} & & & & & & 1 \\ & A' & & & 0 & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 1 & & & 1 \\ \hline 0 & 0 \;\cdots & 0 & 1 & 1 & \cdots & 1 \end{array}\right]. \tag{4.3}$$

In (4.3) *A' is a solution of size $m-1$.*

Proof. Consider the two columns of column sum $m-1$ in A. Each will yield a solution A' of size $m-1$ as above by Corollary 4.2. We note that there is one column of column sum l left over from A' in A for each l $2 \leqslant l \leqslant m$. Each of these $m-1$ columns must have a 1 in the bottom row because otherwise A could not have the SDR as given in Corollary 4.3. Thus the bottom row has row sum $m-1$. By the uniqueneness of the SDR and Corollary 1.2 of Brualdi [3] there is a row and column permutation which leaves the 1's arranged in the triangular pattern as shown in (4.3). This gives us a nice inductive buildup of solutions.

COROLLARY 4.5. *In a solution A of size m, a column of column sum l is covered by a column of column sum $l+1$ for $l < m$.*

Proof. If $l = m-1$, then the column of column sum l is covered by the column of m 1's. If $l < m-1$, use Corollary 4.2 repeatedly. The column of column sum l is covered by one of the two columns of column sum $m-1$ because otherwise a triangle is formed. Thus the column of column sum l is contained in a solution of size $m-1$. Repeat this argument on the new solution until the given column is contained in a solution of size $l+1$. The column of column sum $l+1$ in the solution of size $l+1$ is the desired column.

COROLLARY 4.6. *In a solution, if a column of column sum l and a column of column sum k have exactly t rows where both have* 1's $(t \geqslant 2)$, *then these t rows are the rows of some column of column sum t. In addition the $l+k-t$ rows where either column has a* 1 *covers precisely $(l+k-t)-s+1$ columns of column sum s for $2 \leqslant s \leqslant t$.*

Proof. As given, the column of column sum t if added to the solution cannot create any new triangles and so it is in the solution. Now by Theorem 4.1 the column of column sum l covers $l-s+1$ columns of column sum s. A similar argument holds for l replaced by k or t. Thus the $l+k-t$ rows in question cover

$$(l-s+1)+(k-s+1)-(t-s+1)=(l+k-t)-s+1 \tag{4.4}$$

columns of column sum s.

The following 6×15 matrix is an example of a solution of size 6. Our results on the structure of solutions may be checked for this matrix. An appropriate submatrix is outlined so that Corollary 4.4 may be verified more easily.

$$\begin{bmatrix} 1&0&0&0&0&1&0&0&0&1&0&0&1&0&1\\ 0&1&0&0&0&1&1&0&0&1&1&0&1&1&1\\ 1&1&1&0&0&1&1&1&0&1&1&1&1&1&1\\ 0&0&0&1&0&0&0&1&1&0&1&1&1&1&1\\ 0&0&1&1&1&0&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&0&0&0&1&0&0&1&0&1&1 \end{bmatrix}. \tag{4.5}$$

We will finish by giving an algorithm for constructing all solutions of size m. This will involve the notion of k-trees as well as the result from Theorem 4.1 that a column of column sum l covers two columns of column sum $l-1$. The algorithm to generate an arbitrary solution A follows below. To generate all solutions simply try all possibilities at each choice in the algorithm and then test for isomorphisms.

Step 1. Choose a spanning tree on m vertices. Its incidence matrix yields $m-1$ columns of column sum 2 for A. Note that a tree is a 1-tree.

Step 2. Repeat for $k=3,4,\ldots,m$ in turn. For each k we will add $m-k+1$ columns of column sum k to A as follows. Recall that the columns of column sum $k-1$ form a $(k-2)$-tree. Select two columns of column sum $k-1$ that have exactly $k-2$ rows where both have 1's. Add a column of k 1's covering both columns. Repeat what follows until the remaining $m-k$ columns of column sum k have been added. Select a column of column sum $k-1$ not already covered by a column of column sum k such that there is a covered column of column sum $k-1$ with $k-2$ rows where both have 1's. Add a column of k 1's covering both of the above columns.

In the above algorithm a column of column sum $k-1$ only gets covered when explicitly selected. This follows since the columns of column sum $k-1$ form a $(k-2)$-tree. Thus Step 2 will be completed and a $(k-1)$-tree generated.

The columns of column sum k form an arbitrary $(k-1)$-tree as described in Remark 3.5 subject only to the condition that a column of column sum k covers two columns of column sum $k-1$, a result of Theorem 4.1. Thus A is indeed arbitrary. We need only verify that the matrix A has no triangles.

Let H be the hypergraph associated with the matrix A where the rows correspond to vertices. Assume H has some special cycle of length 3: $x_1E_1x_2E_2x_3E_3x_1$. A short inductive argument verifies that an edge of size l covers the maximum number $l-1$ edges of size 2. For $l=2$ this is obvious. For an edge of size l, recall that it covers two edges of size $l-1$. By induction the edges of size $l-1$ cover $l-2$ edges of size 2. For $l=3$ we are done. Also the 2 edges have $l-2$ vertices in common which is an edge of size $l-2$ by the algorithm and so by induction the edge of size $l-2$ covers $l-3$ edges of size 2. We compute that the edge of size l covers $l-1$ edges of size 2 as desired. Thus every edge covers a spanning tree on its vertices. But then x_1 and x_2 are joined by a path in a spanning tree that involves the vertices of E_1 and hence does not include x_3. Similarly for x_2 and x_3 and also x_3 and x_1. This yields some cycle in our original spanning tree, which is a contradiction. Thus A has no triangles.

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