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Ideals Generated by R-Sequences

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INTRODUCTION

In this note we search for reasonable conditions that ensure that an ideal is generated by an R-sequence. The main result (Theorem 1.1) gives a simple condition for that to happen and, as a bonus, gives some insight on a problem of Lech ([5]). There he proves a generalization of a theorem of Auslander-Buchsbaum-Serre, stating that a localization of a regular local ring is also regular to, so to speak, the nonregular case. We also answer in the affirmative to a question of Lech as to whether his basic lemma (Theorem 2 of [5]) can be extended to the unequal characteristic case. Our treatment of these questions is inspired by [4] which possesses, perhaps, the proper balance of commutative ring theory and homological algebra that we shall need here.

1. MAIN RESULTS

The rings considered in this paper will be assumed to be commutative Noetherian and to contain an identity element. For the notations and basic facts used here [1] will be the standing reference.

Let R be a ring, I an ideal, and M a finitely generated R-module. By an M-sequence in I we mean an ordered sequence of elements $x_1, \ldots, x_r$ of I such that x_i is not a zero divisor with respect to $M/(x_1, \ldots, x_{i-1}) M$. One can ask the question: What ideals in R can be generated by M-sequences for appropriate M's? It is easy to see that, if I is generated by an M-sequence, then IM/I^2M is a direct sum of copies of M/IM; also, M-sequences are useful in building modules of finite projective dimension (p.d.) from other ones. These natural conditions should then appear in the hypotheses if one tries to answer the question. In the case $M = R$ and R is a local ring, we can prove a slightly more general version of that problem.

To state the main result we have in mind, let R be a local ring and $\mathfrak{m}$ its maximal ideal.

THEOREM 1.1. *Let I be an ideal of finite projective dimension. Assume that J is an ideal satisfying $I \supset J \supset I^2$ and such that $I/J \cong (R/I)^r$. Then $I = (x_1, ..., x_r) + J$, where the x_i's form an R-sequence. If, in particular, $J \subset \mathbf{m}I$ then I is generated by an R-sequence.*

Before proceeding to the proof we assemble some basic results.

PROPOSITION 1.2. *Let R be a local ring and M a finitely generated R-module. Let x be a nonunit, nonzero divisor with respect to both R and M. Then* p.d. $_{R/(x)}M/xM =$ p.d. $_RM$.

Proof. Let $X = \{X_i\}$ be a minimal projective resolution of M:

$$\cdots X_1 \to X_0 \to M \to 0.$$

Tensoring with $R/(x)$ we get a complex $X^* = \{X_i^*\}$ which, we claim, is a minimal projective resolution of M/xM over $R/(x)$. The homology groups of X^* are, as we know, given by $H_i = \operatorname{Tor}_i(M, R/(x))$. Now $H_i = 0$ for $i > 1$ since p.d. $_RR/(x) = 1$; $H_1 = 0$ because x is also a nonzero divisor relative to M and $H_0 = M/xM$. We then get a projective resolution of M/xM over $R/(x)$ which is clearly minimal. This shows that X and X^* are either both infinite or both finite of the same length.

PROPOSITION 1.3. ([1], Proposition 6.2). *Let R be a ring and M a finitely generated R-module and let*

$$0 \to X_r \to \cdots \to X_0 \to M \to 0$$

be an exact sequence where the X_i's are finitely generated free R-modules. Then the following are equivalent:

(i) *annihilator of $M \neq 0$;*

(ii) *annihilator of M contains some nonzero divisor in R.*

As a consequence we see that an ideal of a local ring of finite projective dimension contains some nonzero divisor. We shall use this remark a number of times.

We proceed now to the proof of the theorem itself. It will come naturally after a few easy lemmas. Assume from now on that we have ideals $I \supset J \supset I^2$ with $I/J = (R/I)^r$. This means that there exists a set of elements $x_1, ..., x_r$ in I which, together with J, generate I and such that any relation of the type $\sum r_i \cdot x_i \in J$ implies $r_i \in I$ for $i \leqslant r$. We shall refer to this property of the x_i's as their "independence". It is clear that any element in $I - (\mathbf{m}I + J)$ (i.e., the elements eligible for a minimum generating set of the module I/J) can be taken as one of the x_i's.

LEMMA 1. *If I, J and x_1 are as above then passing to $R^* = R/(x_1)$ we have $I^*/J^* \simeq (R^*/I^*)^{r-1}$ (* denotes images in R^*).*

Proof. It is clear that I^*/J^* is an R^*/I^*-module. Let $x_2^*,\ldots, x_r^*$ denote the images of $x_2,\ldots, x_r$ in R^*. Together with J^* they clearly generate I^*. A relation $\sum_{i>1} r_i^* x_i^* = \in J^*$ means that there exists $r_i \in R$ such that r_i maps into r_i^* and $\sum_{i>1} r_i x_i \in J + (x_1)$, i.e., $\sum_{i>1} r_i x_i = a + r_1 x_1$. Now by hypothesis the r's are in I. The conclusion follows.

LEMMA 2. *In the situation of Lemma 1,*

$$I/x_1 I = (x_1)/x_1 I \oplus I/(x_1).$$

Proof. Let $S = x_1 I + (x_2,\ldots, x_r) + J$. Clearly $S + (x_1) = I$. Also, by the "independence" of the x_i's, $S \cap (x_1) = x_1 I$. This says that the inclusion $(x_1)/x_1 I \to I/x_1 I$ splits, whence the result.

LEMMA 3. *If I is as in the previous lemmas and does not consist entirely of zero divisors we can choose x_1 as a nonzero divisor.*

Proof. We have to show that not all nonzero divisors in I are in $\mathbf{m}I + J$. If this were so, then $I - (\mathbf{m}I + J) \subset P_1 \cup \cdots \cup P_s$ where the P_j's are the prime ideals associated to zero. Choose in this case $y \in I - (\mathbf{m}I + J)$ and let x be an arbitrary element of $\mathbf{m}I + J$. Then $y - x^i \in I - (\mathbf{m}I + J)$ for any positive integer i. If we take enough powers, two of them will land in the same prime, say $y - x^i$ and $y - x^j \in P_1$ with $i > j$. But then

$$(y - x^i - y + x^j) = x^j(1 - x^{i-j}) \in P_1$$

and so $x \in P_1$. This means that all of I is in the union of the P's, contradicting our assumption.

Proof of Theorem 1.1. We use induction on r; the case $r = 0$ is no problem. Assume $r > 0$. By Proposition 1.3 I cannot consist entirely of zero divisors so that we can pick x_1 as in Lemma 3. By Proposition 1.2, p.d. ${}_{R/(x_1)} I/x_1 I$ is finite and from Lemma 2 $I/x_1 I$ has $I/(x_1)$ as a direct summand. It follows that $I^* = I/(x_1)$ has finite projective dimension relative to $R^* = R/(x_1)$. Now use Lemma 1 and start a new cycle.

Remark. Observe that above the module $I/(x_1,\ldots, x_r)$ has finite projective dimension over $R/(x_1,\ldots, x_r)$.

COROLLARY 1. *An ideal I in a local ring R is generated by an R-sequence if and only if* p.d. ${}_R I$ *is finite and I/I^2 is R/I-free.*

Remark. In the case $I = \mathfrak{m}$ we have $\mathfrak{m}/\mathfrak{m}^2$ $R/\mathfrak{m}$-free and then the equivalence: $\mathfrak{m}$ is generated by an R-sequence if and only if $\mathfrak{m}$ has finite projective dimension. As we know this happens only for regular local rings.

2. Applications

We concern ourselves now with a problem treated by Lech [5]. We present a simplification and complete some of his results to the case of an arbitrary local ring.

For a local ring R and a finitely generated R-module M we denote by $\nu(M)$ the minimum number of generators of M, i.e., the dimension of $M/\mathfrak{m}M$ over $R/\mathfrak{m}$.

Theorem 2.1. *Let I be an $\mathfrak{m}$-primary ideal. If I/I^2 is R/I-free then* $\nu(I) \leqslant \nu(\mathfrak{m})$.

Proof. We first remark that in any local ring the lengths of R-sequences are bounded by the dimension of the ring. Thus, in Theorem 1.1 $r \leqslant \dim R$. In our case we cannot apply this because I is not supposed to have finite projective dimension. We have to, somehow, bring that condition into play. We proceed as follows: We can make $I^2 = 0$ and still keep all the hypotheses; we assume this done. In this case R is a complete local ring and we can map a regular local ring R' onto R by a loose version of the wellknown theorem of Cohen. The following situation results:

$$R' \xrightarrow{f} R;$$
$$I' = f^{-1}(I) \to I;$$
$$J' = f^{-1}(0) \to 0.$$

The statement "I is R/I-free" is equivalent to "I'/J' is $(R'/J')/(I'/J') = R'/I'$-free." Since R' is regular, p.d. ${}_{R'}I'$ is finite and we are in a position to apply our theorem. I' has then the form $I' = (y_1, ..., y_r) + J'$ where the y's form an R'-sequence and $r \leqslant \dim R'$. We claim that we can pick R' with $\dim R' = \nu(\mathfrak{m})$. In fact we have $R = R'/J'$. If $J' \not\subset \mathfrak{m}'^2$ we can divide R' by an element in $J' \cap (\mathfrak{m}' - \mathfrak{m}'^2)$ and still get a regular local ring mapping onto R. We can go on till $J' \subset \mathfrak{m}'^2$ when then $\nu(\mathfrak{m}) = \nu(\mathfrak{m}')$ $(= \dim R')$. Applying f to I' we get the desired conclusion.

As an application we shall give another proof of a theorem of Lech. For an abuse of notation, $\nu(P)$, with P a prime ideal, will denote the minimum local number of generators of P; i.e., $\nu(P) =$ minimum number of generators of

P_P in R_P. Since this is the only use we have for the ν's, it should not lead to confusion.

THEOREM 2.2. *Let R be a local ring and $\mathbf{m}$ its maximal ideal. For every prime ideal P*

$$\nu(P) + \dim \frac{R}{P} \leqslant \nu(\mathbf{m}).$$

Proof. Assume first that R is a quotient of a regular local ring R', say $R = R'/J'$ and $\nu(\mathbf{m}) = \nu(\mathbf{m}')$. If P is a prime in R let P' be its preimage in R'. Then $P \cdot R_P \cong P' \cdot R'_{P'}/J'_{P'}$ and so $\nu(P) \leqslant \nu(P')$. Since R' is regular so are its localizations and we have clearly $\nu(P') + \dim R'/P' = \nu(\mathbf{m}')$ and so

$$\nu(P) + \dim \frac{R}{P} \leqslant \nu(\mathbf{m}).$$

Now if R is any local ring let R^* denote its completion with respect to the $\mathbf{m}$-adic topology. If P is a prime ideal in R let P' be any prime minimal over $P \cdot R^* \cdot R^*_{P'}$ being R_P-flat and $P \cdot R_P/P^2 \cdot R_P$ being $R_P/P \cdot R_P$-free imply that $P \cdot R^*_{P'}/P^2 \cdot R^*_{P'}$ is $R^*_{P'}/P \cdot R^*_{P'}$-free. Applying Theorem 2.1 we get $\nu(P) \leqslant \nu(P')$. Since $R^*/P \cdot R^*$ is the completion of R/P we can pick P' with $\dim R/P = \dim R^*/P'$. We reduce the question then to R^* for which as we remarked earlier the theorem holds.

Putting together the preceding results we have

COROLLARY. *Let R be a local ring and I an ideal such that I/I^2 is R/I-free. Then $\nu(I) \leqslant \nu(\mathbf{m})$ with possible equality only if I is $\mathbf{m}$-primary.*

Now we point an interesting consequence of Theorem 2.2. We need first a definition [5]. For a local ring R with maximal ideal $\mathbf{m}$ we call $\delta(R) = \dim \mathbf{m}/\mathbf{m}^2 - \dim R$ the regularity defect of R. Let R and S be local rings and $f : (R, \mathbf{m}) \to (S, n)$ a local homomorphism making S a flat module over R. We claim that $\delta(R) \leqslant \delta(S)$.

We shall need the

LEMMA. *In the conditions above*

$$\dim S = \dim R + \dim \frac{S}{\mathbf{m}S}.$$

Proof. We can assume that R has no nilpotent elements. If $\dim R = 0$, R is a field and the relation follows. If $\dim R > 0$, $\mathbf{m}$ contains some nonzero divisor, say x, and so $R/(x)$ has dimension one less than R. By the flatness x is not a zero divisor with respect to S either. The rest follows from the fact that S/xS is $R/(x)$-flat.

We can now proceed to prove the claim. Let P be a prime ideal in S minimal over $\mathbf{m}S$ such that $\dim S/P = \dim S/\mathbf{m}S$. By Theorem 2.2,

$$\nu(P) + \dim \frac{S}{P} \leqslant \nu(\mathbf{n}).$$

On the other hand, by Theorem 2.1 we have that $\nu(\mathbf{m}) \leqslant \nu(P)$, and so

$$\nu(\mathbf{m}) + \dim \frac{S}{\mathbf{m}S} \leqslant \nu(\mathbf{n}).$$

The conclusion follows now from the lemma.

We now extend certain results of [5] to the unequal-characteristic case.

THEOREM 2.3. *Let R be a local ring and I an $\mathbf{m}$-primary ideal with I/I^2 R/I-free. If $\nu(\mathbf{m}) - \nu(I) \leqslant 1$ then R/I is a complete intersection (i.e. $R/I \simeq R'/I'$ where R' is a regular local ring and I' is an ideal generated by an R'-sequence).*

Proof. As we have done previously we may assume $I^2 = 0$ and map a regular local ring R' onto R with $\nu(\mathbf{m}') = \nu(\mathbf{m})$. Using the notation of Theorem 2.1 we have $I'/J' \simeq (R'/I')^r$ with $r = \nu(I)$. By Theorem 1.1 we can write $I' = (x_1',..., x_r') + J'$ with the x's forming an R'-sequence. More is true: as remarked earlier from Theorem 1.1 it follows that $I'/(x_1',..., x_r')$ has finite projective dimension with respect to the ring $R'/(x_1',..., x_r')$. There are two cases to examine. If $\nu(I) = \nu(\mathbf{m}) = \nu(\mathbf{m}')$ then $I' = (x_1',..., x_r')$ for otherwise $I^* = I'/(x_1',..., x_r')$ would contain some nonzero divisor x' in $R^* = R'/(x_1',..., x_r')$ which is absurd for R^* is zero-dimensional. If $\nu(I) = \nu(\mathbf{m}) - 1$ we have $I^* \neq 0$ and having finite projective dimension over the one-dimensional ring R^*, i.e., I^* is principal. In conclusion: in both cases we have that I' is generated by an R'-sequence, and this is just what we wanted to show.

In order to go a step further in this matter we shall need a small calculation on the homology of Macaulay rings. For the definitions and facts here we refer to [3]. We recall enough of it to make sense. For an irreducible ideal I in a ring R we mean that I is not an intersection of two properly larger ideals. It is easy to see that if R is noetherian any ideal I can be written as an intersection $J_1 \cap \cdots \cap J_n$ of irreducible ideals without superfluous ones; the integer n is an invariant of such decompositions. In the case when R is a local ring and I is an $\mathbf{m}$-primary ideal $n = \dim_{R/m}(I : \mathbf{m}/I)$. A local ring R is called a Macaulay ring when there exists an R-sequence in R of length equal to $\dim R$. In this case it can be proved that ideals generated by maximal R-sequences have the same number of irreducible components, denoted by $\mu(R)$. A Macaulay ring with $\mu(R) = 1$ is called a Gorenstein ring, i.e., maximal R-sequences generate always irreducible ideals.

Of some interest on its own is the following

PROPOSITION 2.4. *Let R be a two-dimensional local Macaulay ring and I an $\mathfrak{m}$-primary irreducible ideal. If I has finite projective dimension then R is Gorenstein and I is generated by an R-sequence.*

Proof. Let

$$0 \to F_1 \xrightarrow{f} F_0 \to I \to 0$$

be a minimal projective resolution of I and this entails $f(F_1) \subset \mathfrak{m}F_0$. Taking Hom with $k = R/\mathfrak{m}$ we get the long exact sequence

$$\begin{aligned} &0 \to \operatorname{Hom}(k, F_1) \to \operatorname{Hom}(k, F_0) \to \operatorname{Hom}(k, I) \\ &\quad \to \operatorname{Ext}^1(k, F_1) \to \operatorname{Ext}^1(k, F_0) \to \operatorname{Ext}^1(k, I) \to \operatorname{Ext}^2(k, F_1) \xrightarrow{f^*} \operatorname{Ext}^2(k, F_0). \end{aligned}$$

Since R is a 2-dimensional Macaulay ring, the first five terms drop out. On the other hand, $\operatorname{Ext}^2(k, F_0)$ and $\operatorname{Ext}^2(k, F_1)$ are direct sum of copies of $\operatorname{Ext}^2(k, R)$ which is a vector space over k. Since f^* is induced by multiplications by elements in $\mathfrak{m}$ we have $f^* = 0$. We get then the isomorphism $\operatorname{Ext}^1(k, I) \simeq \operatorname{Ext}^2(k, F_1)$.

More information is obtained from the exact sequence

$$0 \to I \to R \to \frac{R}{I} \to 0,$$

which gives

$$0 \to \operatorname{Hom}(k, I) \to \operatorname{Hom}(k, R) \to \operatorname{Hom}\left(k, \frac{R}{I}\right) \to \operatorname{Ext}^1(k, I) \to \operatorname{Ext}^1(k, R).$$

Using again that R is Macaulay of dimension two, we get $\operatorname{Ext}^1(k, I) \simeq \operatorname{Hom}(k, R/I)$. But this last module is isomorphic to $I : \mathfrak{m}/I \simeq k$, for I is irreducible. Altogether we have $k \simeq \operatorname{Ext}^2(k, F_1)$. By [3] this last module is isomorphic to $k^{\operatorname{rank} F_1 \mu(R)}$ and we have rank $F_1 = 1$ and $\mu(R) = 1$. The last equality says that R is Gorenstein, and the first one implies that rank $F_0 = 2$, i.e., I is generated by two elements, and this is all that is needed.

A result analogous to Theorem 2.3 is the following

THEOREM 2.5. *Let R be a local ring and I an $\mathfrak{m}$-primary ideal with $I/I^2R/I$-free. If $\nu(\mathfrak{m}) = \nu(I) + 2$ and I is irreducible, then R/I is a complete intersection.*

A proof can easily be obtained by tracing thru the general remarks in Theorem 2.3 and reducing the question to the previous Proposition.

What prompts us to consider this is the observation that in the cases $\nu(\mathfrak{m}) - \nu(I) \leqslant 1$ I turns out to be irreducible and, in the simple counterexample of Lech, $R = K[x, y, z]/(x^2, (y, z)^2)$, $I = (x)$, I is not irreducible. In a sense this is the best possible, for there exist Gorenstein rings in three generators which are not complete intersections.

Finally we make some remarks on the global case. We state the question in the following form:

Conjecture. Let R be a Noetherian ring for which projectives are free, I an ideal of R with I/I^2 R/I-free and finite projective dimension. Then I can be generated by an R-sequence.

This seems to be the proper generalization of our local case as expressed by the Corollary to Theorem 1.1. We have been able to verify it only in the case where p.d. $_RI = 1$.

To conclude let us remark that in the global case there are bounds for the economical generation of ideals I for which I/I^2 is R/I-projective. If p.d. $_RI$ is finite and R has finite Krull dimension, say d, we can always generate I with $d + 1$ elements.

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References

1. Auslander, M. and Buchsbaum, D. Homological dimension in local rings. *Trans. Am. Math. Soc.* **85** (1957), 390-405.
2. Auslander, M. and Buchsbaum, D. Codimension and multiplicity, *Ann. Math.* **68** (1958), 625-657.
3. Bass, H. On the ubiquity of Gorenstein rings, *Math. Zeit.* **82** (1963), 8-28.
4. Kaplansky, I. Homological dimension of modules and rings (Mimeographed Notes, University of Chicago, 1957).
5. Lech, C. Inequalities related to certain couples of local rings, *Acta Math.* **12** (1964), 69-89.