Asymptotic Behavior and Symmetry of Internal Waves in Two-Layer Fluids of Great Depth

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This paper studies certain asymptotic and geometric properties of internal waves at the interface of a two-layer fluid flow of infinite depth bounded below by a rigid bottom under influence of gravity. It is shown that if the governing equations of the flow have a nontrivial solution which approaches to a supercritical equilibrium state at infinity, then the solution decays to the equilibrium exactly with an order $O(1/x^2)$ for large $x$ where $x$ is the horizontal variable. Furthermore, the solution is symmetric. The interface is always above the equilibrium state and monotonically decreasing for positive $x$ and increasing for negative $x$. The exact decay estimates are obtained using the properties of Green’s function for an integro-differential equation and some tools from harmonic analysis. The proof of symmetry is similar to the one given by Craig and Sternberg for a two-fluid flow of finite depth using the Alexandrov method of moving planes.

1. Introduction

It has attracted a great deal of attention in recent years to investigate and determine characteristics of solitary internal waves in a stratified fluid of great depth. These waves occur rather frequently both in the atmosphere and oceans, whose height or depth is usually quite large. In fact, there has been a number of tentative observations reported in the literature. For instance, Christie et al. [11] have published microbarograph data indicating the occurrence of solitary internal waves over central Australia in a variety of circumstances. Similarly, it is likely that solitary internal waves were present in the satellite observations of oceanic wave-pockets reported by Apel et al. [6]. Therefore, the internal wave motions in fluids with great depth are supported by the oceanic and atmospheric studies.

The problem of solitary internal waves in fluids of great depth was first studied by Benjamin [7] and Davis and Acrivos [14]. They considered stratified fluids of infinite depth, whose density variations extend only over
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a limited depth but the total depth of the fluid is infinite. They used long-wave approximations of solutions of the exact governing equations to derive a model equation, called Benjamin–Ono equation, for the deviations of the solutions from an equilibrium state. In addition, Benjamin obtained an explicit form of a solitary wave solution for the Benjamin–Ono equation while Davis and Acrivos gave numerical solutions using a numerical method. Davis and Acrivos [14] and Maxworthy [19] also carried out experiments to show that such solitary waves are stable and very easy to generate in laboratory. Unlike the classical solitary wave solutions of Korteweg–de Vries (KdV) equation, which decay exponentially at infinity, the solitary wave solutions of the Benjamin–Ono equation decay only algebraically at infinity, which makes the theoretical study of solutions of the Benjamin–Ono equation more difficult.

Recently, there has been much progress on mathematical investigation of the solitary internal waves. Amick and Toland [4] reduced the Benjamin–Ono equation to an elliptic equation with nonlinear boundary conditions and used the maximum principle to prove the uniqueness of solitary wave solutions of the Benjamin–Ono equation. They also established analytic relations between the solutions of the Benjamin–Ono equation and the solutions of a complex-valued ordinary differential equation [5]. Then their ideas were extended to obtain the positivity property and uniqueness of solitary wave solutions of an intermediate long wave equation [1, 2], which forms a model-theoretical bridge between the KdV and Benjamin–Ono equations. As mentioned in a survey paper by Benjamin et al. [8], the existence of solitary internal wave solution of the exact governing equations for fluids of infinite depth was still unsettled until recently. For a two-fluid flow of infinite depth bounded below by a horizontal rigid bottom, it was shown by Amick [3] and Sun [20] using different methods that the exact governing equations of a two-fluid flow have solitary wave solutions and the first order approximations of such solutions are the solutions of the Benjamin–Ono equation. The solution is a supercritical solution and decays algebraically to the equilibrium at infinity. In their proof, the waves are assumed to be symmetric and have very long wave-length with small amplitude. This paper considers general properties of any supercritical solution if it exists. It shows that the interface is always above the equilibrium and gives an exact decay rate of the solution if the solution approaches to the equilibrium at infinity. Then it is obtained that such solution must be symmetric about a vertical axis and half of the interface is monotonically decreasing to the equilibrium while another half is monotonically increasing.

The counterpart of the problem for single-layer fluids bounded below by a rigid bottom and above by a free surface has been studied rather extensively in the past. Garabedian [15] studied periodic waves at the free
surface using variation method and showed that the waves, if exist, are symmetric and unique if the amplitude is given. Keady and Prichard [17] showed that if there is an even solitary wave and the wave is monotone in the positive horizontal direction, then it is a wave of elevation alone and the flow is supercritical at infinity. Craig and Sternberg [12] first proved without any apriori conditions that all supercritical solutions of the exact equations are symmetric and monotone on either side of the crest using asymptotic decay properties of the solutions and similar ideas as [9]. Berestycki and Nirenberg [10] obtained symmetry and monotonicity of solutions for a very general class of semilinear elliptic equations in cylindrical domains. Then Craig and Sternberg [13] extended their results to two-fluid flows with different densities bounded by two rigid planes. We note that the exponential decay estimates of solutions at infinity are crucial to obtain these results.

In this paper, we consider a two-fluid flow bounded by a rigid horizontal bottom only. The upper fluid is infinite in both vertical and horizontal directions. The fluids of constant densities are immiscible, inviscid and incompressible, and the flow is irrotational. The density of the lower fluid is greater than the density of the upper fluid and the depth of the lower fluid is \( h \). Assume that there is a wave moving with a constant speed \( U \) at the interface and a coordinate system \((x^*, y^*)\) is chosen moving with the wave.

Let \( \psi^*(x^*, y^*) \) be the stream function with \( \psi^* = 0 \) at the interface and the interface be determined by \( y^* = \eta^*(x^*) \) where \( \eta(x^*) \to 0 \) as \( |x^*| \to \infty \). Then \( \psi^*(x^*, y^*) \equiv y^* \) is the equilibrium state. Define the critical wave speed \( U_0 = \sqrt{(1-\rho)gh} \), where \( 0 < \rho < 1 \) is the density ratio of the upper fluid to the lower fluid and \( g \) is the constant of gravity. Our results can be stated as follows. If there exists a solution of the exact governing equations for \( U > U_0 \) with \( \eta^*(x^*) \) continuously differentiable and \( \psi^*(x^*, y^*) \) as its stream function, which decays to the equilibrium at infinity, then \( \psi^*(x^*, y^*) \) has following asymptotic expansion at infinity,

\[
\psi^*(x^*, y^*) - y^* = C_0^*(1 + (x^*)^2)^{-1} + O((1 + |x^*|)^{-3}),
\]

for \(-h < y^* < \eta^*(x^*)\),

\[
\psi^*(x^*, y^*) - y^* = \tilde{C}_0^* y^* (1 + (y^*)^2 + (x^*)^2)^{-1} + o(y^*(1 + (y^*)^2 + (x^*)^2)^{-1}),
\]

for \( \eta^*(x^*) < y^* < +\infty \),

where \( C_0^* \) and \( \tilde{C}_0^* \) are bounded. Also there exists a constant \( \lambda_0^* \) such that the solution is symmetric about \( x^* = \lambda_0^* \). Moreover, \( \eta^*(x^*) \) is always
positive and monotonically decreasing to zero for \( x^* \geq \lambda_g^* \) with an asymptotic expansion,

\[
\eta^*(x^*) = D_g^* (1 + (x^*)^2)^{-1} + O((1 + |x^*|)^{-3}),
\]

where \( D_g^* \) is a positive constant. In addition, the interface \( y^* = \eta^*(x^*) \) is analytic using a general theory for regularity of solutions in elliptic free boundary problems by Kinderlehrer et al. [18]. Therefore, the results obtained here give a mathematical justification of the symmetry assumption made for the solitary wave solutions of the exact equations in [3, 20] and show that the solitary wave solution in [3, 20] possesses the properties stated above, some of which cannot be obtained there.

The method and ideas to obtain the symmetry of solution of the exact equations are essentially from the work by Craig and Sternberg [13]. However, in order to prove the symmetry, one needs an accurate asymptotic expansion of the solution at infinity. If a two-fluid system by two rigid boundaries, the asymptotic behavior of a solution can be obtained using eigenfunction expansion of the solution and the solution decays exponentially at infinity [10, 12, 13]. This method will not work for a two-fluid system with only one horizontal boundary since the solution decays algebraically at infinity. In order to determine the asymptotic behavior of the solution for the two-fluid problem, we first find the solution explicitly in the upper fluid in terms of its values at boundary. Then the exact equations in the fluids are reduced to a differential equation in the lower fluid only and two boundary conditions, one of which has a term with a pseudo-differential operator. We study the transformed equation and nonlocal boundary conditions in the lower fluid and use Fourier transforms and the properties of the corresponding Green’s function to derive the asymptotic expansion of the solution. Finally the exact estimates of the solution at infinity are obtained using positivity of the solution at the interface.

We note that a very similar proof can be applied to a two-fluid problem, which has a fixed upper boundary, but no lower boundary. A wave of depression will appear in this case. The results for the monotonicity, symmetry, and asymptotic behavior of the wave, and their proofs will be very similar.

The paper is organized as follows. In Section 2, the formulation of the problem is given and the positivity of the solution at the interface is proved. Some Banach spaces are defined. Section 3 studies the asymptotic behavior of solution of the governing equations. Some estimates of solutions of a linear differential equation with nonlocal boundary conditions in the lower fluid are first found in Section 3.1 and then the asymptotic expansions of solutions of the nonlinear problem are derived in Section 3.2.
Section 4, the proof of symmetry of the solutions is sketched using the same method in [13]. Several estimates of the solution in the upper fluid and some nonlinear terms are given in Appendices.

2. FORMULATION AND TRANSFORMATIONS

We consider an irrotational flow of two immiscible, inviscid, and incompressible fluids of infinite depth with different but constant densities bounded below by a rigid horizontal bottom under influence of gravity. Assume that there exists an interface between two fluids and there is a wave moving with a constant speed $U$ at the interface. In reference to a coordinate system $(x^*, y^*)$ moving with the same speed $U$, the flow is steady and is uniform at infinity with a constant velocity $U$. Let $y^* = 0$ be the interface when $|x^*| \to +\infty$, $y^* = -h$ be the rigid bottom with $h > 0$, and $y^* = \eta^*(x^*)$ be the equation of interface. Then the upper fluid of density $\rho^+$ is in $y^* < y^* < \eta^*(x^*)$ and the lower fluid of density $\rho^-$ is in $-h < y^* < \eta^*(x^*)$, respectively, with $\rho^+ < \rho^-$. In the following, we shall use $f^+$ and $f^-$ to denote the quantity $f$ in the upper and lower fluid.

By the assumption that the flow is irrotational and the fluid is incompressible, we can introduce a velocity potential $\varphi^*$ and stream function $\psi^*$ such that $\psi^* = 0$ at the interface and $\psi^* = -Uh$ at the rigid bottom. If we let $\chi^* = \varphi^* + i\psi^*$ with $z^* = x^* + iy^*$ in $-h < y^* < \eta^*(x^*)$ and $\eta^*(x^*) < y^* < +\infty$, then $w_1^* = u^* - iv^*$ is the complex velocity, where $u^*$, $v^*$ are the horizontal and vertical velocity components. By using nondimensional quantities

\[ z = z^*/h, \quad \eta = \eta^*/h, \quad w_1 = w_1^*/U, \]
\[ \rho = \rho^+ / \rho^- < 1, \quad \chi = \varphi + i\psi = \chi^*/(hU), \]

then $w_1 = \partial z / \partial \zeta$ and the interface and the rigid bottom become $\psi = 0$ and $\psi = -1$. Let

\[ \Omega^+ = \{ (x, y) : x \in \mathbb{R}, \eta(x) < y < +\infty \}, \]
\[ \Omega^- = \{ (x, y) : x \in \mathbb{R}, -1 < y < \eta(x) \}. \]

It is known that $\varphi^\pm(x, y) + i\psi^\pm(x, y)$ is an analytic function in $\Omega^\pm$. Since the flow is uniform at infinity, which corresponds to $\Psi(y) = y$ for $-1 < y < +\infty$, $\psi^\pm(x, y) \to \Psi(y)$ as $x$ or $y$ goes to infinity in $\Omega = \Omega^+ \cup \Omega^-$. The governing equations of the flow are given as follows:

\[ A\psi^\pm = 0 \quad \text{in} \quad \Omega^\pm, \]  \hspace{1cm} (1)

\[ \psi^+ = \psi^- = 0 \quad \text{at} \quad y = \eta(x), \]  \hspace{1cm} (2)
where the second equation at $y = \eta(x)$ is the Bernoulli’s equation at the interface using the continuity of the pressure across the interface, and

$$\gamma = (1 - \rho) \frac{gh}{U^2} = 1/c^2.$$  

where $c$ is a nondimensional wave speed and called Froude number. Obviously, there is a trivial solution $\psi = \Psi(y)$ and $\eta(x) = 0$ of Eqs. (1)-(5) for any $\gamma > 0$. In this paper, we are interested in nontrivial solutions, which approach to the uniform flow $\Psi(y)$ at infinity.

In order to study the asymptotic behavior of the solutions of Eqs. (1)-(5) at infinity, we have to transform the equations into another form. Assume that the interface $y = \eta(x)$ is continuously differentiable and does not touch the bottom, i.e., $y = \eta(x) > -1$. Then the results in [18] imply that $\eta(x)$ is analytic in $x$. Since $\psi(x,-1) = -1$ and $\psi = 0$ at $y = \eta(x)$, the harmonic function $\psi^-(x,y)$ takes its minimum at $y = -1$ and maximum at $y = \eta(x)$. Therefore, by Hopf Lemma [13], $\psi^- > 0$ at $y = -1$ and $y = \eta(x)$ if $\psi^- (x, y) \neq 0$. Also $\psi^- \to 1$ as $|y| \to +\infty$, which implies $\psi^- > 0$ in $\Omega^-$. Furthermore, $\psi^+(x,y)$ takes minimum at $y = \eta(x)$. Thus $\psi^+ > 0$ at $y = \eta(x)$ if $\psi^+ (x, y) \neq 0$. Since $\psi^+ \to 1$ as $|x| + y \to +\infty$, $\psi^+ (x, y) > 0$ by the maximum principle. We conclude that $\phi^+ = \psi^+ > 0$ in $\Omega$. This allows us to invert the analytic mapping $P: (x,y) \to (\phi(x,y), \psi(x,y))$ in $\Omega^\pm$ and $x(\phi, \psi) + iy(\phi, \psi)$ is an analytic function of $\phi + i\psi$ in $S^\pm = \{(\phi^+, \psi) \mid -\infty < \phi^+ < +\infty, 0 < \psi < +\infty\}$,

$$S^- = \{(\phi^-, \psi) \mid -\infty < \phi^- < +\infty, -1 < \psi < 0\}.$$  

Since we are interested in solutions approaching to the uniform flow $(x = \phi^+, y = \psi)$ at infinity, we write

$$x^\pm = \phi^+ + X^\pm (\phi^\pm, \psi), \quad y^\pm = \psi + Y^\pm (\phi^\pm, \psi),$$  

in $S^\pm$. Therefore, Eqs. (1)-(5) become

$$AY^\pm = 0 \quad \text{in} \quad S^\pm, \quad (6)$$

$$Y^- (\phi^-, -1) = 0 \quad \text{for} \quad -\infty < \phi^- < +\infty, \quad (7)$$

$$Y^\pm (\phi^+, \psi), Y^\pm_{\psi}, Y^\pm_{\phi} \to 0 \quad \text{as} \quad |\phi^+| + \psi \to +\infty. \quad (8)$$

In this paper, we are interested in nontrivial solutions, which approach to the uniform flow $\Psi(y)$ at infinity.
At the interface $\psi = 0$,

$$
Y^+(\varphi^+, 0) = Y^-(\varphi^-, 0),
$$

(9)

$$
(1/2)((Y^+_0)^2 + (1 + Y^+_0)^2)^{-1} - \rho((Y^+_0)^2 + (1 + Y^+_0)^2)^{-1} + \gamma Y^+(\varphi^+, 0) = (1/2)(1 - \rho).
$$

(10)

The relation of $\varphi^\pm$ on the interface is given by

$$
x^+ = \varphi^+ + X^+(\varphi^+, 0) = \varphi^- + X^-(\varphi^-, 0) = x^-,
$$

(11)

and $X^+(\varphi^+, \psi), Y^+(\varphi^+, \psi)$ satisfy the Cauchy–Riemann equations. Now we can obtain the positivity of $\eta(x)$, which is determined by $Y^+(\varphi^+, 0)$.

**Lemma 1.** Suppose that $Y^\pm(\varphi^\pm, \psi) \in C^1(S^\pm)$ is a solution of Eqs. (6)–(11). Let $m = \inf_{\varphi^+ \in R} Y^+(\varphi^+, 0) > -1$. If $\varphi \leq 1$ and $Y^\pm \not\equiv 0$, then $m = 0$ and $Y^\pm(\varphi^\pm, 0) > 0$. Thus the interface $y = \eta(x)$ is always positive.

The lemma is just a special case of the Lemma 1.5 in [13] if the depth of upper fluid goes to infinity. For the sake of completeness, we give a brief proof here.

**Proof.** We prove it by contradiction. Let $Y^\pm(\varphi^\pm, 0)$ attain a non-positive minimum at a point $\varphi^\pm_0$. Since the equations are translation invariant in $\varphi^\pm$, we can assume $\varphi^\pm_0 = 0$. Thus $m = \inf_{\varphi^+ \in R} Y^+(\varphi^+, 0) = Y^\pm(0, 0) \leq 0$ and $Y^\pm_0(0, 0) = 0$. By using the maximum principle, we have that if $Y^-(\varphi^-, \psi)$ is not identically zero,

$$
Y^-(\varphi^-, \psi) > m(\psi + 1) \quad \text{in} \quad S^-.
$$

Since $Y^- \to 0$ as $|\varphi^-| \to +\infty$, by Hopf Lemma,

$$
Y^-_\varphi(\varphi^-, \psi) < m \quad \text{at} \quad (\varphi^-, \psi) = (0, 0).
$$

Also $Y^+(\varphi^+, 0) \geq m$ and $Y^+(\varphi^+, \psi) \to 0$ as $|\varphi^+ + \psi \to \infty$. Thus the maximum principle implies $Y^+(\varphi^+, \psi) > m$ in $\Omega^+$. By Hopf Lemma again,

$$
Y^+_\varphi(\varphi^+, \psi) > 0 \quad \text{at} \quad (\varphi^+, \psi) = (0, 0).
$$

From (10), we obtain that at $(\varphi^-, \psi) = (\varphi^+, \psi) = (0, 0),

$$
(1/2)(1 - \rho) = (1/2)((1 + Y^-_0)^{-2} - \rho((1 + Y^+_0)^{-2} + \gamma m
\geq (1/2)((1 + m)^{-2} - \rho) + \gamma m.
$$
Thus we have an inequality,
\[ m(\gamma - (1 + (m^2/2))(1 + m)^{-2}) < 0, \]
which implies that \( m < 0 \) and
\[ \gamma > (1 + (m^2/2))(1 + m)^{-2} \geq 1. \]

It contradicts the assumption \( \gamma \leq 1 \). Therefore, \( Y^\pm(\varphi^\pm, 0) \) cannot attain the nonpositive minimum in \(( -\infty, +\infty) \). Since \( Y^\pm(\varphi^\pm, 0) \to 0 \) as \(|\varphi^\pm| \to +\infty\), \( m = 0 \). Thus \( Y^\pm(\varphi^\pm, 0) > 0 \) and \( \eta(x) > 0 \). This completes the proof of the lemma.

We have shown that if there exists a nonzero solution of Eqs. (6)-(11) for \( \gamma \leq 1 \), then the interface will always be positive. This conclusion will be used later to obtain the asymptotic form of the solution at infinity. We remark here that by using the proof of Lemma 1.5 in [13] and some asymptotic estimates of solutions at infinity, it seems possible to show that there are no solitary waves in the case that both fluids have infinite depth, which is subject to further study.

Before we derive the asymptotic behavior and symmetry of the solution, let us define several weighted supreme norms for later use. Let \( 0 < \alpha < 1 \) and \( n \geq 0 \) be any integer. Define the Hölder norms of a function \( f(\varphi, \psi) \) in \( S^\pm \) as follows. For \( (\varphi, \psi) \in S^\pm \), denote
\[
[f(\varphi, \psi)]_{n, S^\pm} = \sup \left\{ \frac{|f(\varphi + \delta, \psi + \iota) - f(\varphi, \psi)|}{(\delta^2 + e^2)^{n/2}} : (\delta, \iota) \in \mathbb{R}^2, |\delta| + |\iota| \leq 1 \right\}.
\]

Then let
\[
\|f\|_{C^{\alpha, \gamma}(S^\pm, w)} = \sum_{j=0}^{\infty} \left( \sum_{k + l = j} \sup_{(\varphi, \psi) \in S^\pm} \left( \frac{\partial^j f(\varphi, \psi)}{\partial \varphi^k \partial \psi^l} \right) w(\varphi) \right) + \sum_{k + l = n} \sup_{(\varphi, \psi) \in S^\pm} \left( \frac{\partial^n f(\varphi, \psi)}{\partial \varphi^k \partial \psi^l} \right) w(\varphi),
\]
where \( w(\varphi) \geq 1 \) is an even smooth weight function and \( w(\varphi) \to +\infty \) strictly monotonically as \( \varphi \to +\infty \), and denote \( C^{\alpha, \gamma}(S^\pm, w) \) as a Banach space of function \( f \) with \( \|f\|_{C^{\alpha, \gamma}(S^\pm, w)} < \infty \). Here \( w \) also satisfies \( w(x + y) \leq C_0 w(x) \) for \( x, y \geq 0 \), where \( C_0 \) is a fixed constant and may be chosen as \( C_0 = 10 \). We also use \( \|f\|_{C^{\alpha, \gamma}(\mathbb{R}, w)} \) to denote the corresponding norm for function \( f(\varphi) \) with \( \varphi \) only. For the sake of convenience, denote \( C^{\alpha, \gamma}(S^\pm, 0) = C^{\alpha, \gamma}(S^\pm) \) with \( 0 < \alpha < 1 \). Now we are ready to obtain the asymptotic estimates of solutions of Eqs. (6)-(11).
3. Asymptotic Behavior of the Solutions

In order to obtain the asymptotic expansions of solutions of Eqs. (6)–(11) for large $\varphi^\pm$, we need to separate linear and nonlinear terms of the equations and then transform them into more treatable forms.

First from the boundary conditions (10) and (11) at $\psi = 0$, we have

\[ Y_\varphi^-(\varphi^-, 0) - \rho Y_\varphi^+(\varphi^+, 0) - \gamma Y^- (\varphi^-, 0) = (1/2)[(\varphi^-)^2 (2 \varphi^- - 1) + 3(\varphi^-)^2 + 2(\varphi^-)^3] \]
\[ - (\rho/2)[(\varphi^+)^2 (2 \varphi^+ - 1) + 3(\varphi^+)^2 + 2(\varphi^+)^3] \]
\[ = F_1(Y^+(\varphi^+, 0), Y^- (\varphi^-, 0)), \]
\[ (12) \]

and

\[ \varphi^+ + X^+(\varphi^+, 0) = \varphi^- + X^-(\varphi^-, 0). \]
\[ (13) \]

Since $\varphi^+_0 > 0$, we have that at $\psi = 0$,

\[ \frac{d\varphi^+}{d\varphi^-} = \frac{d\varphi^+}{dx} \left( \frac{d\varphi^-}{dx} \right)^{-1} > 0. \]

From (13), we can apply implicit function theorem to obtain a function $L(\varphi^-)$ such that

\[ \varphi^+ = L(\varphi^-), \]
\[ (14) \]

where $L(\varphi^-)$ is analytic. Since $\varphi^+_0 = \psi^+_0 \to 1$ as $|x| \to \infty$, $L'(\varphi^-) = \frac{d\varphi^+}{d\varphi^-} \to 1$ as $\varphi^- \to \pm \infty$.

Thus $L'(\varphi^-)$ is bounded and $L(\varphi^-) \to \pm \infty$ when $\varphi^- \to \pm \infty$.

Now let us consider the equations in $S^+$:

\[ AY^+ = 0 \quad \text{in} \quad S^+, \]
\[ Y^+(\varphi^+, 0) = Y^- (\varphi^-, 0), \quad Y^+(\varphi^+, \psi) \to 0 \quad \text{as} \quad |\varphi^-| + \psi \to + \infty. \]
\[ (15) \]
\[ (16) \]

The solution of (15) and (16) can be obtained for $\psi > 0$ by

\[ Y^+(\varphi^+, \psi) = \int_{-\infty}^{+\infty} \frac{Y^+(t, 0)}{(\varphi^+ - t)^2 + \psi^2} \, dt \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y^+(t, 0) \frac{\partial}{\partial \psi} (\ln((\varphi^+ - t)^2 + \psi^2)) \, dt. \]
\[ (17) \]
Then we have the following Lemma, whose proof is deferred to Appendix 1.

**Lemma 2.** Assume that \( Y^+(\varphi^+, 0) \in \mathcal{C}^n(\mathbb{R}, w) \).

(a) If \( w(\varphi) \leq C_0 (1 + \varphi^2) \) for some small \( \varepsilon > 0 \) and \( C_0 > 0 \), then \( Y^+(\varphi^+, \psi) \in \mathcal{C}^n(S^+, w) \) and

\[
\| Y^+(\varphi^+, \psi) \|_{\mathcal{C}^{n+1}(B^+, w)} \leq C(b) \| Y^+(\varphi^+, 0) \|_{\mathcal{C}^{n+1}(\mathbb{R}, w)},
\]

where \( B^+ = \mathbb{R} \times [0, b] \) for \( 0 < b < +\infty \) and \( C(b) \) may depend upon \( b \).

(b) If there exist \( C_0, C_1 > 0 \) and \( 0 < \varepsilon < 1 \) such that \( C_0 (1 + \varphi^2) \leq w(\varphi) \leq C_1 (1 + \varphi^2) \) for large \( \varphi \in \mathbb{R} \), then \( Y^+(\varphi^+, \psi) \in \mathcal{C}^n(S^+, w) \) and

\[
\| (1 + \psi^2 + w(\varphi^+)) (1 + \psi)^{-1} Y^+(\varphi^+, \psi) \|_{\mathcal{C}^{n+1}(S^+)} \leq C \| Y^+(\varphi^+, 0) \|_{\mathcal{C}^{n+1}(\mathbb{R}, w)},
\]

where \( C \) is a generic constant.

From the expression of \( Y^+ \) in (17), for \( \psi > 0 \), we have that

\[
Y^+_{\psi}(\varphi^+, \psi) = (1/2\pi) \int_{-\infty}^{+\infty} Y^+(t, 0) \frac{\partial^2}{\partial \psi^2} \left( \ln((\varphi^+ - t)^2 + \psi^2) \right) dt
\]

\[
= (-1/2\pi) \int_{-\infty}^{+\infty} Y^+(t, 0) \frac{\partial^2}{\partial t^2} \left( \ln((\varphi^+ - t)^2 + \psi^2) \right) dt
\]

\[
= (1/2\pi) \int_{-\infty}^{+\infty} Y^+_t(t, 0) - 2(\varphi^+ - t) \left( \frac{\varphi^+ - t}{(\varphi^+ - t)^2 + \psi^2} \right) dt.
\]

Thus as \( \psi \to 0 \),

\[
Y^+_{\psi}(\varphi^+, 0) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{Y^+_t(t, 0)}{\varphi^+ - t} dt,
\]

where the singular integral is interpreted as the Cauchy principle value. Note that the integral is well defined since for \( t \) large we can use integration by parts to obtain the convergence of the integral. By (9), we rewrite (19) as
Thus the boundary condition (12) becomes

\[
Y_\phi^-(\phi^-, 0) + \frac{\rho}{\pi} \int_{-\infty}^{+\infty} \frac{Y^-_\phi(v, 0)}{\phi^- - v} \, dv = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{Y^-_\phi(v, 0)}{L(\phi^-) - L(v)} \, dv.
\]

(20)

Here we note that

\[
d \phi^+ = \frac{d \phi^-}{L(\phi^-) - L(v)} = \frac{1 + Y^-_\phi(\phi^-, 0)}{1 + Y^-_\phi(L(\phi^-), 0)} > 0.
\]

(22)

In Appendix 2, we shall prove the following Lemma.

Lemma 3. If \( Y^\pm(\phi^\pm, 0) \in C^{1+\gamma}(\mathbb{R}, w) \) and \( w(\phi) \) satisfies that either \( w^2(\phi) \leq C(1 + \phi^2)^{1+\gamma/2} \) for some \( C > 0 \) and small \( \gamma > 0 \) or \( C_0(1 + \phi^2)^{1+\gamma/2} \) for some \( C_0, \ C_1 > 0 \) and \( 0 < \gamma \leq 1 \), then

\[
\| F(\phi) \|_{C^{1+\gamma}(\mathbb{R}, w)} \leq C \| Y^+(\phi^+, 0) \|_{C^{1+\gamma}(\mathbb{R}, w)} + \| Y^-(\phi^-, 0) \|_{C^{1+\gamma}(\mathbb{R}, w)}^2.
\]

3.1. The Estimates of Solutions of the Linear Equations in \( S^- \)

Here we consider the following linear equations,

\[
\Delta Y(\phi, \psi) = 0 \quad \text{in} \quad -1 < \psi < 0, \quad -\infty < \phi < +\infty,
\]

(23)

\[
Y_\phi + \frac{\rho}{\pi} \int_{-\infty}^{+\infty} Y_\phi(t, 0) \frac{dt}{\phi - t} - \gamma Y = f(\phi) \quad \text{at} \quad \psi = 0,
\]

(24)

\[\]

\[
Y = 0 \quad \text{at} \quad \psi = -1,
\]

(25)

where \( Y \) and \( \phi \) are meant to be \( Y^- \) and \( \phi^- \) in this subsection. Suppose \( \gamma < 1 \). For smooth data \( f(\phi) \) decay fast enough as \( \phi \to \pm \infty \), we can apply
the Fourier transform with respect to \( \varphi \) on both sides of Eqs. (23)–(25) to obtain

\[
Y(\varphi, \psi) = K \ast f(\varphi) \tag{26}
\]

\[
= \int_{-\infty}^{+\infty} e^{-ik\varphi} \hat{f}(k) \frac{\sinh(|k| (\psi + 1))}{|k| \cosh |k| - \gamma \sinh |k| + \rho |k| \sinh |k|} \, dk,
\]

where

\[
\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik\varphi} f(\varphi) \, d\varphi \tag{27}
\]

is the Fourier transform of \( f(\varphi) \). Note that the convolution is well defined by the following Lemma 4 even if \( f(\varphi) \) is bounded. Therefore for bounded \( f(\varphi) \) we also use (26) to define the solution \( Y(\varphi, \psi) \). Here we have used the fact that

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{Y_\alpha(r, 0)}{\varphi - v} \, dv \right) e^{ik\varphi} \, d\varphi = |k| \hat{Y}(k, 0). \tag{28}
\]

The Fourier transform of \( K(\varphi, \psi) \) with respect to \( \varphi \) is

\[
\hat{K}(k, \psi) = \frac{\sinh(|k| (\psi + 1))}{|k| \cosh |k| - \gamma \sinh |k| + \rho |k| \sinh |k|}. \tag{29}
\]

By using the inverse Fourier transform of \( \hat{K}(k, \psi) \), we can obtain following lemma.

**Lemma 4.** The kernel \( K(\varphi, \psi) \) satisfies following properties:

1. If \( |\varphi| > 1/2 \) and \( \psi \in (-1, 0) \), then
   \[ |K(\varphi, \psi)| + |K_\varphi(\varphi, \psi)| \leq C(\psi + 1)\varphi^{-2} \quad \text{and} \quad |K_\varphi(\varphi, \psi)| \leq C\varphi^{-2}. \]

2. If \( |\varphi| \leq 1/2 \) and \( \psi \in (-1, 0) \), then
   \[
   K(\varphi, \psi) = (2/\pi)^{1/2} (1 + \psi) \times \left[ (1/\rho) \ln |\varphi| - (1/2(1 + \rho)) \ln(\varphi^2 + \psi^2) + B_1(\varphi, \psi) \right],
   \]
   \[
   K_\varphi(\varphi, \psi) = (2/\pi)^{1/2} (1 + \psi) \left[ (1/\rho) \psi - (\varphi/(1 + \rho))(\varphi^2 + \psi^2) \right] + B_2(\varphi, \psi),
   \]
   \[
   K_\psi(\varphi, \psi) = (2/\pi)^{1/2} \left[ (1/\rho) \ln |\varphi| - 1/(1 + \rho)(\varphi^2 + \psi^2) \right] + (1/2) \ln(\varphi^2 + \psi^2) + B_3(\varphi, \psi),
   \]
   where \( B_1, i = 1, 2, 3, \) are Hölder continuous with exponent \( 0 < \alpha < 1 \).
A similar proof can be found in [3] and the proof of Lemma 4 is omitted.

After having the expression of the solutions and the integral kernel \( K(\varphi, \psi) \), we can estimate \( Y(\varphi, \psi) \) and its derivatives at \( \psi = 1 \). Let

\[
\mathcal{L}[f](\varphi) = K_0 * f(\varphi),
\]

\[
\frac{\partial \mathcal{L}[f]}{\partial \varphi} = K_1 * f(\varphi), \quad \frac{\partial (K * f(\varphi))}{\partial \varphi} \bigg|_{\psi = 1} = K_2 * f(\varphi),
\]

(30)

where

\[
\hat{K}_0(k) = \frac{\sinh(|k|)}{|k| \cosh |k| - \gamma \sinh |k| + \rho |k| \sinh |k|},
\]

\[
\hat{K}_1(k) = \frac{i k \sinh(|k|)}{|k| \cosh |k| - \gamma \sinh |k| + \rho |k| \sinh |k|},
\]

\[
\hat{K}_2(k) = \frac{|k| \cosh(|k|)}{|k| \cosh |k| - \gamma \sinh |k| + \rho |k| \sinh |k|}.
\]

We note that \( \hat{K}_i(k), i = 0, 1, 2 \) are not analytic in any neighborhood of the real axis \( \mathbb{R} \). If we can obtain estimates of \( K_0 * f, K_1 * f \) and \( K_2 * f \), it is straightforward to see that the same estimates hold for \( K * f \) and its derivatives in \(-1 < \psi < 0\) by using classical elliptic operator theory with Dirichlet boundary conditions in a stripe domain \(-1 < \psi < 0\) [16]. Therefore in the following, we only derive the estimates for the operators in (30).

We introduce a class \( \mathscr{A} \) of admissible weight functions \( w(\varphi) \) satisfying conditions stated in the definition of \( C^{n+\gamma}(\mathbb{R}, w) \) and \( w(\varphi) \leq C_0(1 + \varphi^2)^{1+\gamma/2} \) for some \( C_0 > 0 \), where \( \varepsilon > 0 \) is a small fixed constant. Then we have the following lemma, whose proof can be found in [12].

**Lemma 5.** If \( m(\varphi) \) is an integrable kernel such that for \( w(\varphi) \in \mathscr{A} \),

\[
\int_{-\infty}^{+\infty} |m(\varphi) w(\varphi)| \, d\varphi \leq C_1,
\]

then

\[
\|m * f\|_{C^1(\mathbb{R}, w)} \leq C_2 \|f\|_{C^0(\mathbb{R}, w)},
\]

where \( C_1, C_2 \) are constants and \( C_2 \) may depend on \( C_1 \).
In order to use the results in [12] and obtain the estimates of the convolutions in (30), we further decompose the kernels \( K_0(\phi), K_1(\phi) \) and \( K_2(\phi) \). Write
\[
\tilde{K}_0(k) = (|k| \coth |k| - \gamma + \rho |k|)^{-1} = (|k| \coth |k| (1 + \rho \tanh |k|) - \gamma)^{-1}
\]
\[
= (|k| \coth |k| (1 + \rho + \rho (\tanh |k| - 1)) - \gamma)^{-1}
\]
\[
= ((1 + \rho) |k| \coth |k| (1 - (\rho/(2(1 + \rho \cosh |k|))) e^{-2|k|}) - \gamma)^{-1}
\]
\[
= ((1 + \rho) |k| \coth |k| - \gamma)^{-1} + \tilde{R}_0(k) = (1 + \rho)^{-1} \hat{r}(k) + \tilde{R}_0(k),
\]
where
\[
\tilde{R}_0(k) = \rho |k| (e^{-2|k|/\sinh |k|})
\times [(1 + \rho) |k| \coth |k| - \gamma] (|k| \coth |k| (1 + \rho \tanh |k|) - \gamma)^{-1}.
\]
By using a similar derivation, we have
\[
\tilde{K}_1(k) = \hat{p}(1 + \rho) |k| \coth |k| - \gamma)^{-1} + i \tilde{R}_1(k) = (1 + \rho)^{-1} \hat{p}(k) + \tilde{R}_1(k),
\]
\[
\tilde{K}_2(k) = |k| ((1 + \rho) |k| \coth |k| - \gamma)^{-1} + |k| \cosh |k| (\sinh |k|)^{-1} \tilde{R}_2(k) + \tilde{R}_2(k).
\]
Since \( \gamma < 1 \), thus \( \gamma/(1 + \rho) < 1 \). The kernels \( r(\phi), p(\phi) \) and \( q(\phi) \) have been studied in [12] and following lemma was obtained [Lemma 3.8, 12].

**Lemma 6.** The following estimates for integral operators with kernels \( r(\phi), p(\phi) \) and \( q(\phi) \) hold:
\[
\|r(\phi) * f(\phi)\| \leq C \|f\|, \quad \|p(\phi) * f(\phi)\| \leq C \|f\|, \quad \|q(\phi) * f(\phi)\| \leq C \|f\|
\]
where \( C \) is a fixed constant and \( w \in A \).

Next we prove a lemma for the estimates of operators with kernels \( R_0, R_1 \) and \( R_2 \).

**Lemma 7.** If \( w(\phi) \in A \), then
\[
\|R_0(\phi) * f(\phi)\| \leq C \|f\|, \quad \|R_1(\phi) * f(\phi)\| \leq C \|f\|, \quad \|R_2(\phi) * f(\phi)\| \leq C \|f\|
\]
Proof. Since the proofs for three estimates are similar, we only give proof of the first estimate. By Lemma 5, we only need to show that for \( w \in \mathcal{A} \),

\[
\int_{-\infty}^{+\infty} |R_0(\varphi) w(\varphi)| \, d\varphi < +\infty.
\]

Here we note that \( \tilde{R}_0(k) \) is not analytic for \( k \) near the origin. From the definition of \( R_0(\varphi) \) and the exponential decay of \( \tilde{R}_0(k) \) as \(|k| \to +\infty\), we have that for \( \varphi \in \mathbb{R} \) large,

\[
R_0(\varphi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{R}_0(k) e^{-ik\varphi} \, dk
\]

\[
= \frac{1}{2\pi} \int_{0}^{+\infty} \tilde{R}_0(k) \cos k\varphi \, dk
\]

\[
= -\frac{1}{\pi\varphi^2} \left[ \left( \frac{d\tilde{R}_0(k)}{dk} \right) \Bigg|_{k=0} + \int_{0}^{+\infty} \cos k\varphi \left( \frac{d^2\tilde{R}_0(k)}{dk^2} \right) \, dk \right],
\]

by using integration by parts twice. Since \( \tilde{R}_0(k) \) and its derivatives decay exponentially as \( k \to +\infty \), we have

\[
\int_{-\infty}^{+\infty} |R_0(\varphi) w(\varphi)| \, d\varphi \leq C \int_{-\infty}^{+\infty} (1/(1 + \varphi^2))(1 + \varphi^2)^{(1-\varepsilon)/2} \, d\varphi
\]

\[
= C \int_{-\infty}^{+\infty} (1 + \varphi^2)^{-(1+\varepsilon)/2} \, d\varphi < +\infty,
\]

where \( \varepsilon > 0 \) is small. Thus the lemma is proved.

After combining the above lemmas and applying the classical elliptic operator theory in a stripe domain [16], we have

**Lemma 8.** If \( w(\varphi) \in \mathcal{A} \), then

\[
\|K(\varphi, \psi) * f(\varphi)\|_{C^{1,1}(\mathbb{R}, \mathbb{R})} \leq C \|f\|_{C^{1}(\mathbb{R}, \mathbb{R})}.
\]

To obtain accurate estimates of the solutions of Eqs. (23)–(25) as \(|\varphi| \to +\infty\), we need following more refined estimates of the integral operator \( K * f(\varphi) \). Now we assume that

\[
\|f(\varphi)(1 + \varphi^2)^{\beta/2}\|_{C^{1}(\mathbb{R})} < +\infty \quad \text{for some} \quad \beta > 1.
\]

Then we shall show that \( K * f \) will decay with order of \(|\varphi|^{-2} + |\varphi|^{-\beta} \) as \(|\varphi| \to +\infty \). Note that a similar proof of Lemma 8 does not work here since \( K(\varphi, \psi) \) is not integrable with respect to weight \(|\varphi|^{\beta} \) for \( \beta > 1 \).
**Lemma 9.** If \( f(\varphi) \in C^\infty(\mathbb{R}) \) with \( \| f(\varphi)(1 + \varphi^2)^{\beta/2} \|_{C^\infty(\mathbb{R})} < +\infty \) and \( \beta > 1 \),

then \[
\|(1 + \varphi^2)^{-\kappa/2} (K * f(\varphi))\|_{C^{\infty, \kappa}(\mathbb{R})} \leq C \| f(\varphi)(1 + \varphi^2)^{\beta/2} \|_{C^\infty(\mathbb{R})},
\]

where \( \kappa = \min(2, \beta) \).

**Proof.** From the definition of \( K(\varphi, \psi) \) and Lemma 4, we have that for \( \varphi \) large,

\[
|K(\varphi, \psi) * f(\varphi)| = \left| \int_{-\infty}^{+\infty} K(\varphi - t, \psi) f(t) \, dt \right|
\leq C \sup_{\varphi} (\| f(\varphi) \| (1 + \varphi^2)^{\beta/2}) \left[ \int_{|t| \geq 1/2} (\varphi - t)^{-2}(1 + t^2)^{-\beta/2} \, dt \right.
+ \left. \int_{|t| \geq 1/2} (-\ln |\varphi - t|)(1 + t^2)^{-\beta/2} \, dt \right]
\leq C \| f(\varphi)(1 + \varphi^2)^{\beta/2} \|_{C^\infty(\mathbb{R})}
\times \left( \int_{|t| \geq 1/2} t^{-2}(1 + (\varphi - t)^2)^{-\beta/2} \, dt + (1 + \varphi^2)^{-\beta/2} \right)
\leq C \| f(\varphi)(1 + \varphi^2)^{\beta/2} \|_{C^\infty(\mathbb{R})} (1 + \varphi^2)^{-\beta/2}
\times t^{-2}(1 + (\varphi - t)^2)^{-\beta/2} \, dt + (1 + \varphi^2)^{-\beta/2}
\leq C \| f(\varphi)(1 + \varphi^2)^{\beta/2} \|_{C^\infty(\mathbb{R})} (1 + \varphi^2)^{-\beta/2}
\leq C \| f(\varphi)(1 + \varphi^2)^{\beta/2} \|_{C^\infty(\mathbb{R})} (1 + \varphi^2)^{-\kappa}.
\]

By differentiating \( K * f \) with respect to \( \varphi \), we have

\[
(K * f)_\varphi = \left[ \int_{-\infty}^{+\infty} K_\varphi(\varphi - t, \psi) f(t) \, dt \right]
= \left( \int_{-\infty}^{-1/2} + \int_{-1/2}^{+1/2} + \int_{+1/2}^{+\infty} \right) K_\varphi(\varphi - t, \psi) f(t) \, dt
= I + II + III.
\]
The terms I, III and their Hölder norms can be estimated in a similar way as above. Since $K(\varphi, \psi)$ is even in $\varphi$,

$$\left| \left[ \int_{\varphi - 1/2}^{\varphi + 1/2} K_\varphi(\varphi - t, \psi) f(t) \, dt \right|^2 \right|$$

$$= \left| \left[ \int_{\varphi - 1/2}^{\varphi + 1/2} K_\varphi(\varphi - t, \psi)(f(t) - f(\varphi)) \, dt \right|^2 \right|$$

$$= C \| f(\varphi)(1 + \varphi^2)^{\beta_2} \|_{C^1(R)} \left[ \int_{\varphi - 1/2}^{\varphi + 1/2} |\varphi - t|^{\gamma - 1} (1 + t^2)^{-\beta_2} \, dt \right]$$

$$\leq C \| f(\varphi)(1 + \varphi^2)^{\beta_2} \|_{C^1(R)} (1 + \varphi^2)^{1 - \beta_2}.$$ 

Also from

$$(K * f)_\varphi = \left[ \int_{-\infty}^{+\infty} K_\varphi(\varphi - t, \psi) f(t) \, dt \right],$$

we only need to estimate

$$\left| \left[ \int_{\varphi - 1/2}^{\varphi + 1/2} K_\varphi(\varphi - t, \psi) f(t) \, dt \right|^2 \right|$$

$$\leq C \| f(\varphi)(1 + \varphi^2)^{\beta_2} \|_{C^1(R)}$$

$$\times \left( C_1 (1 + \varphi^2)^{\beta_2} + \left[ \int_{-1/2}^{1/2} (\psi/(t^2 + \psi^2))(1 + (\varphi + t)^2)^{-\beta_2} \, dt \right] \right)$$

$$\leq C \| f(\varphi)(1 + \varphi^2)^{\beta_2} \|_{C^1(R)} (1 + \varphi^2)^{1 - \beta_2}.$$ 

The Hölder norms of the derivatives can be obtained similarly using a standard procedure for deriving the Hölder norms of solutions of classical elliptic equations \[16\] and the proof is omitted here.

Therefore we have all estimates of the solutions for the linear equations (23)–(25) with $(\varphi, \psi) \in S^-$. 

3.2. Asymptotic Behavior of Solutions of Nonlinear Problem

From the expression of solutions of Eqs. (23)–(25), we can write the solution of (6), (7) and (21) as

$$Y^+(\varphi^-, \psi) = K * F(Y^+(L(\varphi^-), 0), Y^-(\varphi^-, 0)).$$
Since \( Y^-(\varphi^-, 0) \) and \( Y^+(\varphi^+, 0) \) decay to zero as \(|\varphi^\pm| \to +\infty\), it is straightforward to construct a \( w(\varphi) \in \mathcal{A} \) such that \( \|Y^\pm(\varphi^\pm, 0)\|_{C^{1+\eta, w}} < +\infty \) for \( w \in \mathcal{A} \). Therefore, Lemmas 8 and 3 imply

\[
\|Y^-(\varphi^-, \psi)\|_{C^{1+\eta, S^+, w}} = \|K \ast F(Y^+(L(\varphi^-), 0), Y^-(\varphi^-, 0))\|_{C^{1+\eta, S^+, w}} \\
\leq C\|F(Y^+(L(\varphi^-), 0), Y^-(\varphi^-, 0))\|_{C(S^+, w)} \\
\leq C\|Y^-(\varphi^-, 0)\|_{C^{1+\eta, w}} + \|Y^+(\varphi^+, 0)\|_{C^{1+\eta, w}}^2.
\]

Then by Lemma 2, 9 and (22), \( \|Y^+(\varphi^+, \psi)\|_{C^{1+\eta, S^+, w^2}} \) is also bounded.

Now let a subset \( \mathcal{A}_0 \) of \( \mathcal{A} \) be all \( w \in \mathcal{A} \) such that \( \|Y^\pm(\varphi^\pm, \psi)\|_{C^{1+\eta, S^+, w}} \) are bounded and \( w(\varphi) \leq C_0(1 + \varphi^2)^{(1-\epsilon)/2} \) for some fixed \( C_0 > 0 \) and small \( \epsilon > 0 \). Then \( \mathcal{A}_0 \) is nonempty since \( Y^\pm(\varphi^\pm, 0) \) decay to zero at infinity. Also we can make \( \mathcal{A} \) a partially order set by introducing a bilinear relation \( w_1 \prec w_2 \) if and only if \( w_1(\varphi) \leq w_2(\varphi) \) for all \( \varphi \in \mathbb{R} \). Then by the above discussion, \( w_2(\varphi) \in \mathcal{A}_0 \) if \( w(\varphi) \in \mathcal{A}_0 \). Obviously the set \( \mathcal{A}_0 \) has an upper bound. Therefore, by Zorn’s Lemma, the set \( \mathcal{A}_0 \) attains the maximum \( w(\varphi) = C_0(1 + \varphi^2)^{(1-\epsilon)/2} \). Hence we obtain following Lemma.

**LEMMA 10.** If a solution of Eqs. (6)–(11) is \( C^{1+\eta}(S^+) \), then

\[
\|Y^\pm(\varphi^\pm, \psi)(1 + (\varphi^\pm)^2)^{(1-\epsilon)/2}\|_{C^{1+\eta, S^+}} < +\infty
\]

for any small \( \epsilon > 0 \).

Thus, by Lemmas 3 and 10,

\[
\|F(Y^+(L(\varphi^-), 0), Y^-(\varphi^-, 0))(1 + (\varphi^-)^2)^{\beta/2}\|_{C(\mathbb{R})} < +\infty,
\]

for some \( \beta > 1 \). By Lemma 9, (13) and Lemma 2, we obtain that

\[
\|Y^-(\varphi^-, \psi)(1 + (\varphi^-)^2)^{\beta/2}\|_{C^{1+\eta, S^-}} < +\infty,
\]

\[
\|Y^+(\varphi^+, \psi)(1 + (\varphi^+)^2)^{\beta/2}\|_{C^{1+\eta, S^+}} < +\infty.
\]

Then we use Lemmas 2, 3 and 9 again to obtain that

\[
\|Y^\pm(\varphi^\pm, \psi)(1 + (\varphi^\pm)^2)\|_{C^{1+\eta, S^\pm}} < +\infty,
\]

\[
\|F(Y^+(L(\varphi^-), 0), Y^-(\varphi^-, 0))(1 + (\varphi^-)^2)^{\beta/2}\|_{C(S^-)} < +\infty.
\]

By the expression (26) and (29) of the solution \( Y^-(\varphi^-, \psi) \) and integration by parts twice, we have
\( \varphi^2 Y^{-}(\varphi^{-}, \psi) \)
\[
= \varphi^2 \int_{-\infty}^{+\infty} (\cos k\varphi - i \sin k\varphi) \hat{K}(k, \psi) \hat{F}(k) \, dk
\]
\[
= \varphi^2 \left( \int_{0}^{+\infty} + \int_{-\infty}^{0} \right) (\cos k\varphi - i \sin k\varphi) \hat{K}(k, \psi) \hat{F}(k) \, dk
\]
\[
= \varphi^2 \left( \int_{0}^{+\infty} \cos k\varphi \hat{K}(k, \psi)(\hat{F}(k) + \hat{F}(-k)) \, dk
\]
\[
- i \int_{0}^{+\infty} \sin k\varphi \hat{K}(k, \psi)(\hat{F}(k) - \hat{F}(-k)) \, dk \right)
\]
\[
= C_0(1 + \psi) - \int_{0}^{+\infty} \cos k\varphi \hat{K}(k, \psi)(\hat{F}(k) + \hat{F}(-k)))_{k\varphi} \, dk
\]
\[
+ i \int_{0}^{+\infty} \sin k\varphi \hat{K}(k, \psi)(\hat{F}(k) - \hat{F}(-k)))_{k\varphi} \, dk, \quad (31)
\]

where \( C_0 = (\pi(1 - \gamma))^{-1} \int_{-\infty}^{+\infty} F(\varphi^{-}) \, d\varphi^{-} \). Note that \((\hat{F}(k))_{k\varphi} \in L^2(\mathbb{R})\) for \( k \geq 0 \) since \( F(\varphi^{-})(1 + (\varphi^{-})^2) \) is bounded. Thus by integration by parts once more, (31) implies

\[ \varphi^2 Y^{-}(\varphi^{-}, \psi) \]
\[
= C_0(1 + \psi) + \varphi^{-1} \left( C_1 + \int_{-\infty}^{+\infty} e^{-ik\varphi} \hat{K}(k, \psi)(\hat{F}(k) + \hat{F}(-k)))_{k\varphi} \, dk \right),
\]

where \( \hat{R}(k) \) is the remainder and has terms with derivatives of \( \hat{F}(k) \) up to second order. Thus we can still use integration by parts to get

\[ \int_{-\infty}^{+\infty} e^{-ik\varphi} \hat{R}(k) \, dk = O(1/\varphi) \quad \text{as} \quad |\varphi| \to +\infty. \]

Since \( \hat{K}(k, \psi) \in L^2(\mathbb{R}), \hat{K}(k, \psi)(\hat{F}(k))_{k\varphi} \in L^1(\mathbb{R}) \). By Riemann–Lebesgue lemma,

\[ \int_{-\infty}^{+\infty} e^{-ik\varphi} \hat{K}(k, \psi)(\hat{F}(k))_{k\varphi} \, dk \to 0 \quad \text{as} \quad \varphi \to \pm \infty. \]

Therefore from Lemma 2, we have the following Lemma.

**Lemma 11.** If a nonzero solution \( Y^{\pm}(\varphi^{\pm}, \psi) \) of Eqs. (6)-(11) is in \( C^{1+}(\mathbb{S}^2) \), then \( Y^{\pm}(\varphi^{\pm}, \psi) \) must have the following asymptotic expansion as \( |\varphi| \to +\infty \).
Y^+(\varphi^+, \psi) = \hat{C}_0 (1 + \psi)(1 + \psi^2 + (\varphi^+)^2)^{-1} + o(1 + \psi)(1 + \psi^2 + (\varphi^+)^2)^{-1},
Y^-(\varphi^-, \psi) = C_0 (1 + \psi)(\varphi^-)^{-2} + C_1 (\varphi^-)^{-3} + o((\varphi^-)^{-3}),

where \( C_0 \) is a fixed constant and \( \hat{C}_0, C_1 \) may depend upon \( \psi \) and are bounded.

Next we shall show that \( C_0 \) in Lemma 11 is positive. It is obvious that \( C_0 \geq 0 \) since \( Y^-(\varphi^-, 0) > 0 \) by Lemma 1 for \( Y^\pm \neq 0 \). Therefore, we only need to show \( C_0 \neq 0 \). We prove this by contradiction. Assume \( C_0 = 0 \). By Lemma 11,

\[ Y^-(\varphi^-, \psi) = O((\varphi^-)^{-3}) \quad \text{as} \quad \varphi^- \to \pm \infty. \]

Also \( F(\varphi^+, \varphi^-) = O((\varphi^-)^{-4}) \) as \( \varphi^- \to \pm \infty \). From (28), we rewrite (21) at \( \psi = 0 \) as follows:

\[
\rho \int_{-\infty}^{+\infty} |k| e^{-ik\varphi^-} \hat{Y}^-(k, 0) \, dk = -Y^-_x + Y^- - F(\varphi^+, L(\varphi^-), 0, Y^-(\varphi^-, 0)). \quad (32)
\]

Then we use integration by parts twice to obtain that for \( \varphi = \varphi^- \neq 0 \)

\[
\int_{-\infty}^{+\infty} |k| e^{-ik\varphi^-} \hat{Y}^-(k, 0) \, dk
= \left( \int_{-\infty}^{+\infty} + \int_{-\infty}^{0} \right) |k| e^{-ik\varphi^-} \hat{Y}^-(k, 0) \, dk
= \int_{0}^{+\infty} k(\cos k\varphi)(\hat{Y}^-(k, 0) + \hat{Y}^-(0, k)) \, dk
+ i \int_{0}^{+\infty} k(\sin k\varphi)(\hat{Y}^-(0, k) - \hat{Y}^-(k, 0)) \, dk
= (2/\varphi^2) \left( (1/2\pi) \cos k\varphi(k\hat{Y}^-(k, 0)) |_{k=0}^{k=+\infty} \right)
- (1/2\pi) \int_{0}^{+\infty} \cos k\varphi \left( k \int_{-\infty}^{+\infty} \cos ksY^-(s, 0) \, ds \right)_{kk} \, dk
- (1/2\pi) \int_{0}^{+\infty} \sin k\varphi \left( k \int_{-\infty}^{+\infty} \sin ksY^-(s, 0) \, ds \right)_{kk} \, dk
\]
However, from (31) and the form of $\hat{K}(k, \psi)$ in (29), we obtain that

$$\int_{-\infty}^{\infty} |k| e^{-i\alpha k} e^{i k^2 Y^- (s, 0)} ds dk$$

is in $L^2(\mathbb{R})$ by using the Parseval equality for the Fourier transforms and the boundedness of $|k| \hat{K}(k, 0)$.

By applying the Parseval equality to the other terms, we can show that $R(\varphi)$ is in $L^2(\mathbb{R})$. Here we have used the fact that $C_0 = 0$ and $(1 + \varphi^2) \sigma^2 F$ is bounded. Then we take the derivative of (31) with respect to $\psi$ and let $\psi = 0$. Since $C_0 = 0$ and $\hat{K}_0(k, \psi)$ is bounded, $\sigma^2 Y^- (\varphi, 0) \in L^2(\mathbb{R})$ by using the Parseval equality several times. Therefore, from (32) and (33) we have

$$\int_{-\infty}^{\infty} |k| e^{-i\alpha k} e^{i k^2 Y^- (s, 0)} ds dk$$

is in $L^2(\mathbb{R})$ by using the Parseval equality for the Fourier transforms and the boundedness of $|k| \hat{K}(k, 0)$. By applying the Parseval equality to the other terms, we can show that $R(\varphi)$ is in $L^2(\mathbb{R})$. Here we have used the fact that $C_0 = 0$ and $(1 + \varphi^2) \sigma^2 F$ is bounded. Then we take the derivative of (31) with respect to $\psi$ and let $\psi = 0$. Since $C_0 = 0$ and $\hat{K}_0(k, \psi)$ is bounded, $\sigma^2 Y^- (\varphi, 0) \in L^2(\mathbb{R})$ by using the Parseval equality several times. Therefore, from (32) and (33) we have

$$\int_{-\infty}^{\infty} Y^- (s, 0) ds = -2\varphi R(\varphi) - \varphi^2 Y^- \psi + \gamma \varphi^2 Y^-$$

By the above argument and Lemma 11, the right hand side of (34) is in $L^2(\mathbb{R})$. Since Lemma 1 implies $Y^- (\varphi^-, 0) > 0$ for all $\varphi^- \in \mathbb{R}$, the left hand side of (34) is a nonzero constant, which is not in $L^2(\mathbb{R})$. Therefore, it is a contradiction. Hence the assumption $C_0 = 0$ is incorrect and $C_0 > 0$.

Finally from Lemma 11 and Lemma 2, we have following Theorem.

**Theorem 1.** Assume that for $\gamma < 1$, there exists a nonzero solution $Y^\pm (\varphi^\pm, \psi)$ of Eqs. (6)–(11) in $C^1(S^+)$.

Then the solution $Y^\pm (\varphi^\pm, \psi)$ must have the asymptotic behavior

$$Y^- (\varphi^-, \psi) = C_0 (1 + \psi) (\varphi^-)^{-2} + C_1 (\varphi^-)^{-3} + o((\varphi^-)^{-3}),$$

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for large \( \varphi^- \), where \( C_0>0 \) is a fixed constant, \( C_1 \) is bounded and \((\varphi^-)^3 \) \( o((\varphi^-)^{-3}) \to 0 \) uniformly for \( \psi \in [-1, 0] \) as \( |\varphi^-| \to +\infty \). The solution \( Y^+(\varphi^+, \psi) \) satisfies

\[
\sup_{\varphi^+ \in \mathbb{R}, \psi \in [0, +\infty)} \left( (1 + \psi^2 + (\varphi^+)^2)(1 + \psi)^{-1} |Y^+(\varphi^+, \psi)| \right) < +\infty.
\]

4. Symmetry of Solutions

The proof of symmetry of the solution of Eqs. (6)–(11) is almost same as the one given in [13] for two-layer fluids bounded by two rigid horizontal boundaries. The only difference is that the solutions have different asymptotic behavior at infinity. Therefore, we only give a very brief sketch of the proof of symmetry. The most details can be found in [13]. We note that there is a sign difference for the stream functions defined here and in [13].

In order to study the symmetry, we consider Eqs. (1)–(5), which are equivalent to Eqs. (6)–(11). By Theorem 1 and the relation (11) between \( x \) and \( \varphi \), we have that

\[
\eta(x) = D_0 x^{-2} + D_1 |x|^{-3} + o(|x|^{-3}),
\]

where \( D_0 \) is a positive constant. Let \( x_* = 2\lambda - x, \psi^+(x, y) = \psi^+(x_*, y), \) and \( w^+(x, y) = \psi^+(x, y) - \psi^-(x, y) \). Define two subdomains in \( \Omega^2 \) as follows:

\[
\Sigma^+_0 = \{(x, y) \mid x > \lambda, \eta(x) = \eta(2\lambda - x) < y < +\infty\},
\]

\[
\Sigma^-_0 = \Sigma^+_0 \setminus \{(x, y) \mid x > \lambda, -1 < y < \eta(x)\},
\]

and assume that

\[
\lambda_0 = \inf \{ \lambda : \eta(x) - \eta(x) > 0 \text{ for all } x > \lambda \}.
\]

Since \( \eta(x) \) satisfies (35), \( \lambda_0 \) is finite and \( \lambda_0 \geq 0 \). For some technical reasons, we assume that \( \lambda_0 > 0 \). Otherwise, we can always use translation in \( x \)-direction to achieve this condition since the equations (1)–(5) are translation invariant in \( x \)-direction.

Now let the functions \( w^0_{\lambda_0} \) be restricted to the sets \( \Sigma^+_0 \). Then \( w^0_{\lambda_0} \) are harmonic in \( \Sigma^+_0 \) and have nonpositive boundary values. Therefore, either at least one is identically zero, which implies both are zero functions by properties of harmonic functions, or both are strictly negative in \( \Sigma^+_0 \). We assume that \( w^0_{\lambda_0} < 0 \) in \( \Sigma^+_0 \). Then following lemmas were obtained in [13] for two-layer fluids bounded by two rigid boundaries.
Lemma 12. There is no point $x > \lambda_0$ such that $\eta(x) = \eta(x_0)$.

Although Lemma 12 was proved in [13] for two fluids bounded by two boundaries, the argument can be carried over in our case except that the upper boundary condition in [13] is replaced by $w_x^+ \to 0$ as $y \to +\infty$. Also we have another similar lemma from [13].

Lemma 13. $\eta_x(\lambda_0) = 0$ and $\eta_x(x) < 0$ for $x > \lambda_0$.

Sketch of Proof. The proof of $\eta_x(x) < 0$ for $x > \lambda_0$ is the same as the one in [13]. In order to obtain $\eta_x(\lambda_0) = 0$, we need have the asymptotic behavior of $\eta(x_0) - \eta(x)$. From (35), we obtain that for $x$ large

$$\begin{align*}
\eta(x_0) - \eta(x) &= D_0 \left( \frac{1}{(2\lambda_0 - x)^3} - \frac{1}{x^3} \right) \\
&\quad + D_1 \left( \frac{1}{(2\lambda_0 - x)^3} - \frac{1}{x^3} \right) + o(x^{-3}) \\
&= D_0 \lambda_0 x^{-3} + o(x^{-3}).
\end{align*}$$

Using this asymptotic behavior of $\eta(x_0) - \eta(x)$ together with the continuity of $\eta(x_0) - \eta(x)$ with respect to $\lambda$, $\lambda_0 > 0$ and Lemma 12, the Lemma can be proved in a same way as the one in [13]. Details can be found in [13].

Thus by a same proof in [13] using Lemmas 12 and 13, we can show that $w_+^0(x, y)$ violate Hopf’s Corner Point Lemma [13] at $(\lambda_0, \eta(\lambda_0))$. Hence $w_+^0 \equiv 0$. We note that the boundaries of $\Sigma_0^\pm$ at $(\lambda_0, \eta(\lambda_0))$ form a right angle since $\eta_x(\lambda_0) = 0$ by Lemma 13, which implies that Hopf’s Corner Point Lemma can be applied. Therefore $\psi^\pm(x, y)$ and $\eta(x)$ is symmetric about $x = \lambda_0$ and by Lemma 13 the function $\eta(x)$ is strictly decreasing for $x > \lambda_0$ and increasing for $x < \lambda_0$. Finally we summarize above results into following Theorem.

Theorem 2. Assume that Eqs. (1)-(5) have a solution $\psi^\pm(x, y)$ in $S^\pm$ and $\eta(x) > -1$, and $\psi^\pm(x, y)$, $\eta(x)$ and their derivatives have uniformly bounded Hölder norms with exponent $\alpha$ between zero and one. Then $\eta(x) > 0$ for $x \in \mathbb{R}$ and there exists a real number $\lambda_0$ such that the solution is symmetric about $x = \lambda_0$ and $\eta(x)$ is strictly decreasing to zero for $x > \lambda_0$ and increasing for $x < \lambda_0$. Furthermore, $\eta(x)$ has following asymptotic behavior for large $x$,

$$\eta(x) = D_0 x^{-2} + D_1 |x|^{-3} + o(|x|^{-3}),$$

where $D_0$ is a positive constant.
First let us prove (a). For $0 < \psi < b$, (17) implies

$$|Y^+(\phi^+, \psi)| = \left| \frac{\psi}{\pi} \int_{-\infty}^{+\infty} \frac{Y^+(t, 0)}{(\phi^+ - t)^2 + \psi^2} dt \right|$$

$$= \left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{Y^+(\phi^+ - \psi t, 0)}{t^2 + 1} dt \right|$$

$$\leq C \|Y^+(\phi^+, 0)\|_{C^1(\mathbb{R}, \omega)} \int_{-\infty}^{+\infty} \frac{1}{(t^2 + 1)(w(\phi^+ - \psi t))} dt$$

$$\leq C \|Y^+(\phi^+, 0)\|_{C^1(\mathbb{R}, \omega)} (w(\phi^+))^{-1} \int_{-\infty}^{+\infty} \frac{w(\psi t)}{(t^2 + 1)} dt$$

$$\leq C \|Y^+(\phi^+, 0)\|_{C^1(\mathbb{R}, \omega)} (w(\phi^+))^{-1} \times \int_{-\infty}^{+\infty} \frac{(1 + (\psi t)^2)^{1-\epsilon/2}}{(t^2 + 1)} dt$$

$$\leq C(b) \|Y^+(\phi^+, 0)\|_{C^1(\mathbb{R}, \omega)} (w(\phi^+))^{-1}. \quad (A.1)$$

The estimate for the Hölder norm with respect to $\phi^+$ can be obtained similarly. In order to have the estimate for the Hölder norm with respect to $\psi$, we assume $\psi > 0$ and $\psi + \delta > 0$. Without loss of generality, let $1 > \delta > 0$. Then

$$|Y^+(\phi^+, \psi + \delta) - Y^+(\phi^+, \psi)|$$

$$= \left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \left( \frac{\psi + \delta}{t^2 + (\psi + \delta)^2} - \frac{\psi}{t^2 + \psi^2} \right) Y^+(\phi^+ - t, 0) dt \right|$$

$$\leq \frac{1}{\pi} \left( \int_{-\infty}^{+\infty} \left( \frac{\psi + \delta}{t^2 + (\psi + \delta)^2} - \frac{\psi}{t^2 + \psi^2} \right) \left(Y^+(\phi^+ - t, 0) - Y^+(\phi^+ - 0)\right) dt \right)$$

$$+ \frac{1}{\pi} \left( \int_{-\infty}^{+\infty} \left( \frac{\psi + \delta}{t^2 + (\psi + \delta)^2} - \frac{\psi}{t^2 + \psi^2} \right) Y^+(\phi^+ - t, 0) dt \right)$$

$$+ \left| \int_{-\infty}^{+\infty} \left( \frac{\psi + \delta}{t^2 + (\psi + \delta)^2} - \frac{\psi}{t^2 + \psi^2} \right) \left(Y^+(\phi^+ - 0) - Y^+(\phi^+, 0)\right) dt \right|$$

$$= |I| + |II| + |III|. \quad (A.2)$$

Since there is no singularity in $I$, $|II|$ can be estimated in a similar way as (A.1). By boundness of $\psi$ and $t$ in $II$ and using the Hölder continuity of $Y^+(\phi^+, 0)$, we have

$$|I| + |II| + |III|.$$
\[ |II| \leq C \| Y^+(\varphi^+, 0)\|_{C^{n}(\mathbb{R}, u)}^{-1} \times \left| \int_{-1}^{1} \left( \frac{t + \delta}{t^2 + (\psi + \delta)^2} - \frac{\psi}{t^2 + \psi^2} \right) |t|^d dt \right| \leq C \| Y^+(\varphi^+, 0)\|_{C^{n}(\mathbb{R}, u)}^{-1} \delta^5(w(\varphi^+))^{-1} \]

The integrals in \( \mathcal{H} \) can be found explicitly and they are differentiable with respect to \( \psi \). Therefore,

\[ |(Y^+(\varphi^+, \psi + \delta) - Y^+(\varphi^+, \psi)) \delta^{-n}w(\varphi^+)| \leq C \| Y^+(\varphi^+, 0)\|_{C^{n}(\mathbb{R}, u)}^{-1} \]

and

\[ \| Y^+(\varphi^+, \psi)\|_{C^{n}(\mathbb{R}, u)} \leq C \| Y^+(\varphi^+, 0)\|_{C^{n}(\mathbb{R}, u)}^{-1} \]

The estimates of the Hölder norm for \( \varphi^+ \)-derivatives of \( Y^+(\varphi^+, \psi) \) can be obtained similarly. For the estimate of \( \psi \)-derivative of \( Y^+(\varphi^+, \psi) \) and \( \psi > 0 \), from (17) we have

\[
Y_{\psi}^+(\varphi^+, \psi) = \frac{1}{2\pi} \left( \int_{-\infty}^{1} + \int_{1}^{\infty} \right) Y^+(\varphi^+ - t, 0)(\ln(t^2 + \psi^2))_{\psi} dt \\
- \frac{1}{2\pi} \int_{-1}^{1} Y^+(\varphi^+ - t, 0)(\ln(t^2 + \psi^2))_{\psi} dt \\
= I_1 + I_2.
\]

Since \( I_1 \) has no singularity in the integral, \( I_1 \) and its Hölder norms can be estimated similarly as (A.1). \( I_2 \) can be rewritten as

\[
I_2 = - \frac{1}{2\pi} \int_{-1}^{1} Y_{\psi}^+(\varphi^+ - t, 0) \frac{2t}{t^2 + \psi^2} dt \\
- \frac{1}{\pi(1 + \psi^2)} (Y^+(\varphi^+ - 1, 0) - Y^+(\varphi^+ + 1, 0)) \\
= I_1 + I_2.
\]

\( I_1 \) can be estimated in a similar way as (A.2), while the estimates for \( I_2 \) are obtained easily. Therefore, we have proved (a) for \( n = 0, 1 \). For \( n \geq 2 \), the equation \( Y_{\varphi}^+ + Y_{\psi}^+ = 0 \) can be used to obtain the estimates. Thus (a) is proved for all \( n \geq 0 \).

For proof of (b), we only need give an estimate similar to (A.1). Other parts of the proof are same as the ones in (a). Here we assume that \( |\varphi^+| \geq 4 \). The case for \( |\varphi^+| \leq 4 \) is similar and simpler. By (17), we have
Next we estimate each term.

$$\| III_3 \| \leq (1 + (|\phi^+| - 1)^2)^{-1/2} \int_{-1}^{1} \frac{\psi}{t^2 + \psi^2} \, dt$$

$$\leq C(1 + \psi)^{-1} (1 + (\phi^+)^2)^{-1/2}$$

$$\leq C(1 + \psi)/\psi^2 + (1 + (\phi^+)^2)^{(1 + \epsilon)/2},$$

$$\| III_4 \| \leq C(1 + (\phi^+)^2)^{-1/2} \int_{1}^{\infty} \psi(t^2 + \psi^2)^{-1} \, dt$$

$$\leq C(1 + (\phi^+)^2)^{-1/2} \arctan t \left| \frac{\psi^+}{t^{1/2}} \right|$$

$$\leq C(1 + \psi)/\psi^2 + (1 + (\phi^+)^2)^{(1 + \epsilon)/2},$$

$$\| III_5 \| \leq \psi \int_{|\phi^+|/2}^{\infty} ((t^2 + \psi^2)(1 + |\phi^+ - t|^{1 + \epsilon}))^{-1} \, dt$$

$$\leq C(\psi/((\phi^+)^2 + \psi^2)) \int_{|\phi^+|/2}^{\infty} (1 + (\phi^+ - t))^{-1/2} \, dt$$

$$\leq C(\psi((\phi^+)^2 + \psi^2))^{-1}.$$
By the definition of $F_1$ and $F$ in (12) and (21), it is obvious that we only need to obtain the estimates for a term with an integral operator. In the following, we let $\varphi = \varphi^-, \ Y = Y^-$ and $G(Y^+ \varphi^-, 0) = \|Y^{-}(\varphi^-, 0)\|_{C^{2, \gamma}(\overline{\mathbb{R}_+, \mathbb{R}_-})} + \|Y^+(\varphi^+, 0)\|_{C^{2, \gamma}(\mathbb{R}_+, \mathbb{R}_-)}$. Write

\[ I_1(\varphi) = \int_{-\infty}^{+\infty} \left( \frac{1}{\varphi - t} - \frac{1}{L(\varphi) - L(t)} \right) Y(t, 0) \, dt \]

\[ = \int_{-\infty}^{\varphi - 1} + \int_{\varphi - 1}^{\varphi + 1} + \int_{\varphi + 1}^{+\infty} \left( \frac{1}{\varphi - t} - \frac{1}{L(\varphi) - L(t)} \right) Y(t, 0) \, dt \]

\[ = I_3(\varphi) + I_2(\varphi) + I_4(\varphi). \]

By (13) and integration by parts,

\[ I_3(\varphi) = \int_{-\infty}^{+\infty} \frac{L(\varphi) - \varphi + t - L(t)}{(\varphi - t)(L(\varphi) - L(t))} Y(t, 0) \, dt \]

\[ = \int_{-\infty}^{+\infty} \frac{X^-(\varphi, 0) - X^+(L(\varphi), 0) + X^+(L(t), 0) - X^-(t, 0)}{(\varphi - t)(L(\varphi) - L(t))} Y(t, 0) \, dt \]

\[ = X^-(\varphi, 0) - X^+(L(\varphi), 0) + X^+(L(t), 0) - X^-(t, 0) \frac{L(\varphi) - L(\varphi + 1)}{L(\varphi) - L(\varphi + 1)} \]

\[ \times Y(\varphi + 1, 0) \]

\[ - \int_{-\infty}^{+\infty} \frac{X^+(L(\varphi), 0) L'(t) - X^-(t, 0)}{(\varphi - t)(L(\varphi) - L(t))} Y(t, 0) \, dt \]

\[ - \int_{-\infty}^{+\infty} \frac{X^-(\varphi, 0) - X^+(L(\varphi), 0) + X^+(L(t), 0) - X^-(t, 0)}{(\varphi - t)^2(L(\varphi) - L(t))^2} \]

\[ \times (L'(t)(\varphi - t) + L(\varphi) - L(t)) Y(t, 0) \, dt. \]

Then we use the following equalities,

\[ L(\varphi) - L(t) = (\varphi - t) \int_0^t L'(s\varphi + (1 - s) t) \, ds, \]

\[ 0 < C_1 \leq L'(\varphi) = (1 + Y_\varphi(\varphi, 0))(1 + Y_\varphi^+(L(\varphi), 0))^{-1} \leq C_2 < +\infty, \]

\[ X^+_\varphi(L(t), 0) L'(t) - X^-_\varphi(t, 0) = Y_\varphi^+(L(t), 0) L'(t) - Y^-_\varphi(t, 0), \]
\[
X^-(\phi, 0) - X^-(t, 0) = (\phi - t) \left[ \int_0^1 X^-_\phi (\phi s + (1 - s) t, 0) \, ds \right]
= (\phi - t) \left[ \int_0^1 Y^-_\phi (\phi s + (1 - s) t, 0) \, ds \right],
\]
\[
X^+(L(\phi), 0) - X^-(L(t), 0) = (L(\phi) - L(t)) \times \left[ \int_0^1 X^+_\phi (L(\phi)s + (1 - s) L(t), 0) \, ds \right]
= (L(\phi) - L(t)) \times \left[ \int_0^1 Y^+_\phi (L(\phi)s + (1 - s) L(t), 0) \, ds \right]
\]
and a similar proof of the decay estimate of the integrals at infinity as the proof of Lemma 2 to obtain
\[
\| I_3(\phi) \|_{C_0^1(\mathbb{R}, w_{-1})} \lesssim C(G(Y^\pm))^2,
\]
when \( w^2(\phi) \) satisfies the conditions of the Lemma. We note that the product of two functions in \( C^{n+\eta}(\mathbb{R}, w) \) is in \( C^{n+\eta}(\mathbb{R}, w^2) \) for any weight function \( w(\phi) \) and norm of the product is bounded by the product of norms of two functions in \( C^{n+\eta}(\mathbb{R}, w) \). The estimate for \( I_1(\phi) \) is obtained similarly.

Now we study \( I_2(\phi) \). Write
\[
I_2(\phi) = \left[ \int_{\phi - 1}^{\phi + 1} \left( \frac{1}{t} \frac{1}{L(\phi) - L(\phi - t)} \right) Y\phi (\phi - t, 0) \, dt \right]
= \left[ \int_{\phi - 1}^{\phi + 1} \left( \frac{1}{t} \int_0^1 L'(\phi - st) - 1 \, ds \right) Y\phi (\phi - t, 0) \, dt \right]
= \left[ \int_{\phi - 1}^{\phi + 1} t^{-1} III(\phi, t) \, ds = \int_{\phi}^{\phi + 1} t^{-1} \left( III(\phi, t) - III(\phi, -t) \right) \, dt \right]. \tag{A.4}
\]
By (A.4) and (22),
\[
III(\phi, t) = \left( \int_0^1 L'(\phi - st) \, ds \right)^{-1} Y\phi (\phi - t, 0)
\times \left[ \int_{\phi - 1}^{\phi + 1} Y\phi (\phi - st, 0) - Y\phi (L(\phi - st), 0) \frac{1}{1 + Y\phi (L(\phi - st), 0)} \, dt \right].
\]
It is straightforward to show that
\[
|III(\phi, t) - III(\phi, -t)| \lesssim C \tau^2 (G(Y^\pm))^2 w^{-2}(\phi).
\]
Thus $I_2(\varphi) \in C^0(\mathbb{R}, \mathbb{R}^2)$. Let

$$V(\varphi, y) = -\int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{t^2}{t^2 + y^2} \frac{2}{h(\varphi, t)} \, dt$$

Then for $y > 0$,

$$|I_2(\varphi) + V(\varphi, y)| = \left| \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{t^2}{t(t^2 + y^2)} \frac{2}{h(\varphi, t)} \, dt \right| \leq \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{2}{h(\varphi, t)} \, dt \leq Cw^{-2}(\varphi)(G(Y^2))^2$$

and

$$V_\varphi(\varphi, y) = -\frac{\partial}{\partial \varphi} \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{\varphi - t}{(\varphi - t)^2 + y^2} \left( 1 - \frac{\varphi - t}{L(\varphi) - L(t)} \right) Y_\varphi(t, 0) \, dt$$

$$= -\frac{\partial}{\partial \varphi} \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{(\varphi - t)^2}{(\varphi - t)^2 + y^2} \left( \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{L'(s)}{L(s)} ds \right) Y_\varphi(t, 0) \, dt$$

$$= B(\varphi) + \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{(\varphi - t) \left( \left( \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} L'(s) ds \right) \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} L(s) ds \right)}{(\varphi - t)^2 + y^2} Y_\varphi(t, 0) \, dt$$

where $B(\varphi)$ is the boundary terms and bounded by $Cw^{-2}(\varphi)(G(Y^2))^2$.

The estimate for $IV_2(\varphi)$ is obtained as follows.

$$|IV_2(\varphi)| = \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{(\varphi - t) \left( \left( \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} L'(s) ds \right) \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} L(s) ds \right)}{(\varphi - t)^2 + y^2} Y_\varphi(\varphi - t, 0) \, dt$$

$$\leq Cw^{-2}(\varphi)(G(Y^2))^2 \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{|t|^2}{t^2 + y^2} \, dt$$

$$\leq Cw^{-2}(\varphi)(G(Y^2))^2 y^3.$$

Then we rewrite $IV_1(\varphi)$ as

$$|IV_1(\varphi)| = \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} \frac{(\varphi - t) \left( \left( \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} L'(s) ds \right) \int_{-1}^{t \frac{1}{\sqrt{t^2 + y^2}}} L(s) ds \right)}{(\varphi - t)^2 + y^2} Y_\varphi(\varphi - t, 0) \, dt.$$
If we let
\[ g(\varphi, t) = \frac{\int_{s_{\varphi}}^{t} (L'(s) - 1) \, ds}{\int_{s_{\varphi}}^{t} L'(s) \, ds} \, Y_{\varphi}(\varphi - t, 0) \]
and \( g(\varphi, 0) = (L'(\varphi) - 1) \, Y_{\varphi}(\varphi, 0) \, (L'(\varphi))^{-1} \), then
\[
|IV_{1}(\varphi)| \leq \left| \int_{-1}^{1} \frac{y^2 - t^2}{(t^2 + y^2)^2} \, (g(\varphi, t) - g(\varphi, 0)) \, dt \right|
\]
\[
+ \left| \int_{-1}^{1} g(\varphi, 0) \frac{y^2 - t^2}{(t^2 + y^2)^2} \, dt \right|
\]
\[
\leq Cw^{-2}(\varphi)(G(Y^z))^2 \times \left( \int_{-1}^{1} \left| \frac{y^2 - t^2}{(t^2 + y^2)^2} \right| |t|^n \, dt + \left| \int_{-1}^{1} \frac{(t^2 - 1)}{y(t^2 + 1)^2} \, dt \right| \right)
\]
\[
\leq Cw^{-2}(\varphi)(G(Y^z))^2 \left( y^{s-1} + (2/3) \left[ \int_{0}^{y} \frac{1}{(s^2 + 1)^2} \, ds \right] \right)
\]
\[
\leq Cw^{-2}(\varphi)(G(Y^z))^2 \left( y^{s-1} + 1 \right).
\]

Here we have used the fact that \( \int_{0}^{\infty} (t^2 - 1)(t^2 + 1)^{-2} \, dt = 0 \). Thus for \( 0 < y < 1 \),
\[
|V_{\varphi}(\varphi, y)| \leq Cw^{-2}(\varphi)(G(Y^z))^2 \left( y^{s-1} \right).
\]

Hence taking \( 1 > \delta > 0 \),
\[
|I_{2}(\varphi + \delta) - I_{2}(\varphi)| \leq |I_{2}(\varphi + \delta, V(\varphi + \delta, \delta))| + |V(\varphi + \delta, \delta) - V(\varphi, \delta)|
\]
\[
\leq C \left( w^{-2}(\varphi)(G(Y^z))^2 \, \delta^s + \left| \int_{\varphi}^{\varphi + \delta} V_{s}(s, \delta) \, ds \right| \right)
\]
\[
\leq Cw^{-2}(\varphi)(G(Y^z))^2 \left( \delta^s + \int_{\varphi}^{\varphi + \delta} \delta^s - 1 \, ds \right)
\]
\[
\leq Cw^{-2}(\varphi)(G(Y^z))^2 \delta^s.
\]

Therefore the norm of \( I_{2}(\varphi) \) in \( C^s(\mathbb{R}, w^2) \) is bounded by \( C(G(Y^z))^2 \). By the definition of \( F \) in (21), we have that
\[
\|F(\varphi + (L(\varphi -), 0), Y^-(\varphi - 0, 0))\|_{C^s(\mathbb{R}, w^2)} \leq C(G(Y^z))^2.
\]
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REFERENCES