Nonzero solutions for a class of set-valued variational inequalities in reflexive Banach spaces

Fan Jianghua*, Wei Wenhong

Department of Mathematics, Guangxi Normal University, Guilin Guangxi 541004, PR China

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Abstract

In this paper, we study the existence of nonzero solutions for a class of set-valued variational inequalities involving set-contractive mappings by using the fixed point index approach in reflexive Banach spaces. Some new existence theorems of nonzero solutions for this class of set-valued variational inequalities are established.

Keywords: Set-valued variational inequality; Nonzero solutions; Fixed point index; Set-contractive mapping; Upper hemicontinuous

1. Introduction

Throughout this paper, let $X$ be a real Banach space with dual $X^*$, let $(\cdot, \cdot)$ be the duality pairing of $X^*$ and $X$, and let $K$ be a nonempty, closed and convex subset of $X$. We consider the following set-valued variational inequality, which consists in finding $u \in K$ and $u^* \in A(u)$ such that

$$
\langle u^*, v - u \rangle + j(v) - j(u) \geq \langle g(u), v - u \rangle + \langle f, v - u \rangle, \quad \forall v \in K,
$$

where $A : K \to 2^{X^*}$ is a set-valued mapping with nonempty values, $g : K \to X^*$ is a nonlinear mapping, $j : K \to R \cup \{+\infty\}$ is a functional and $f \in X^*$.

Variational inequality theory is an important part of nonlinear analysis, and has been applied intensively to mechanics, cybernetics, differential equation, quantitative economics, optimization theory and nonlinear programming (see, for example [1,2] and the references therein).

The existence of nonzero solutions for variational inequalities is an important topic of variational inequality theory. [3–7] discussed variational inequality (1.1) when $A$ is a single-valued monotone mapping and $g$ is strongly continuous; [8] considered variational inequality (1.1) when $K$ is a closed convex cone, $A$ is single-valued, $g$ is set-contractive and $f = f = 0$; [9] considered variational inequality (1.1) when $A$ is linear, $g$ is set-contractive and upper semicontinuous, and $f = f = 0$.

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* Corresponding author.

E-mail address: jhfan@gxnu.edu.cn (J. Fan).
It is of theoretical and practical significance to study the existence of nonzero solutions for set-valued variational inequalities. In this paper, under suitable assumptions, we discuss the existence of nonzero solutions for set-valued variational inequalities by using the fixed point index approach in reflexive Banach spaces. The results presented in this paper generalize the corresponding results in [8].

2. Preliminaries

In this section, we introduce some basic notations and preliminary results.

Let $X$, $X^*$ and $K$ be as in Section 1. For each $r > 0$, we denote by $K' = \{ x \in K, \| x \| < r \}$. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) := \{ f \in X^* : \langle x, f \rangle = \| x \|^2, \| f \| = \| x \| \}, \quad \forall x \in X.$$  

**Definition 2.1.** Let $A : K \to 2^{X^*}$ be a set-valued mapping with nonempty values. $A$ is said to be

(i) upper semicontinuous if, for all $x \in K$ and for each open subset $V$ in $X^*$ with $A(x) \subset V$, there exists an open subset $W \subset X$ with $x \in W$ such that $A(W \cap K) \subset V$.

(ii) upper hemicontinuous if, its restriction on line segments of $K$ is upper semicontinuous, where $X^*$ is equipped with the $w^*$-topology.

(iii) $\gamma$-strongly monotone if, for each pair of points $u, v \in K$ and for all $u^* \in A(u), v^* \in A(v)$, there exists a scalar $\gamma > 0$ such that

$$\langle u^* - v^*, u - v \rangle \geq \gamma \| u - v \|^2.$$  

The recession cone of $K$ is defined by

$$rc(K) := \{ y : x + \lambda y \in K, \forall \lambda > 0, \forall x \in K \}.$$  

It is evident that $rc(K)$ is a closed convex cone, and for any $u \in K$, $u_0 \in rc(K)$, it holds that $u + u_0 \in K$.

Let $j : K \to R \cup \{ +\infty \}$ be a proper lower semicontinuous and convex functional. The recession function $j_\infty$ of $j$ is defined by

$$j_\infty(y) := \lim_{\lambda \to +\infty} \frac{j(x + \lambda y) - j(x)}{\lambda},$$  

which follows that

$$j_\infty(y) = \lim_{t \to +\infty} \frac{j(ty)}{t}.$$  

In view of [10], if $j(0) = 0$ and $j(K) \subset R^+ \cup \{ +\infty \}$, we have that $j_\infty$ is a lower semicontinuous convex functional with $j_\infty(0) = 0$ and with the property that $j(u + v) \leq j(u) + j_\infty(v)$ for all $u, v \in K$.

Let $X$ be a Banach space, $E \subset X$. The Kuratowski measure of noncompactness of $E$ is defined by

$$\alpha(E) := \inf \left\{ \epsilon > 0 : E \subset \bigcup_{i=1}^{n} E_i \text{ and diam}(E_i) \leq \epsilon \text{ for } i = 1, 2, \ldots, n \right\},$$  

where $\text{diam}(E) = \sup \{ \| x - y \| : x, y \in E \}$. It is well-known that $\alpha(E) = 0$ if and only if $E$ is relatively compact.

Let $X, Y$ be two real Banach spaces and $E \subset X$. A continuous mapping $T : E \to Y$ is said to be $k$-set-contractive on $E$, if there exists a constant $k \geq 0$ such that $\alpha(T(S)) \leq k \alpha(S)$ for any bounded subset $S$ of $E$, where $\alpha$ is the Kuratowski measure of noncompactness. If $k < 1$, $T$ is called strictly set-contractive. A continuous mapping $T : E \to Y$ is said to be condensing, if for any subset $E$ of $X$ with $\alpha(E) \neq 0$, it holds that $\alpha(T(E)) < \alpha(E)$.

A mapping $T : E \to Y$ is said to be Lipschitz continuous with constant $\beta$, if for any $x, y \in E$, there exists a constant $\beta > 0$ such that

$$\| T(x) - T(y) \| \leq \beta \| x - y \|.$$
Let $U$ be an open and bounded subset of $X$ with $U_K = U \cap K \neq \emptyset$. The closure and boundary of $U$ relative to $K$ are denoted by $\overline{U_K}$ and $\partial(U_K)$, respectively. Suppose that $T : \overline{U_K} \to K$ is strictly set-contractive and $x \neq T(x)$ for $x \in \partial(U_K)$, in view of [11], the fixed point index $i_K(T, U)$ is well-defined.

**Lemma 2.1 ([11]).** Let $K$ be a nonempty closed and convex subset of a real Banach space $X$, $U$ be an open and bounded subset of $X$. Suppose that $T : \overline{U_K} \to K$ is strictly set-contractive and $x \neq T(x)$ for $x \in \partial(U_K)$, then the fixed point index $i_K(T, U)$ has the following properties:

(i) For any mapping $\hat{x}_0$ with constant value $x_0$, if $x_0 \in U_K$, then $i_K(\hat{x}_0, U) = 1$;

(ii) $i_K(T, U_1 \cup U_2) = i_K(T, U_1) + i_K(T, U_2)$, whenever $U_1$ and $U_2$ are disjoint open subsets of $X$ such that $x \neq T(x)$, for $x \in \partial(U_1) \cup \partial(U_2)$;

(iii) Let $H : [0, 1] \times \overline{U_K} \to K$ be continuous and bounded, $H(t, \cdot)$ be strictly set-contractive, for each $t \in [0, 1]$. Suppose that $H(t, x)$ is uniformly continuous at $t$ for all $x \in \overline{U_K}$ and for all $(t, x) \in [0, 1] \times \partial(U_K)$, $x \neq H(t, x)$, then $i_K(H(1, \cdot), U) = i_K(H(0, \cdot), U)$;

(iv) If $i_K(T, U) \neq 0$, then $T$ has a fixed point in $U_K$.

3. Main results

The following lemma is a special case of Theorem 8 in [12].

**Lemma 3.1.** Let $X$ be a real reflexive Banach space, $K$ be a nonempty closed convex subset of $X$ and $j : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex functional. Let $B : K \to 2^{X^*}$ be monotone and upper hemicontinuous with nonempty compact convex values. Suppose that there exists $v_0 \in K$ satisfying

$$\lim_{\|v\| \to +\infty} \inf_{v \in BV} \langle v^*, v - v_0 \rangle + j(v) - j(v_0) > 0,$$

then there exist $u \in K$ and $u^* \in B(u)$ such that

$$\langle u^*, v - u \rangle + j(v) - j(u) \geq 0, \quad \forall v \in K.$$

Let $X$ be a real reflexive Banach space, $K$ be a nonempty closed convex subset of $X$ and $j : K \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex functional with $j(K) \subset [0, +\infty]$. Let $A : K \to 2^{X^*}$ be $\gamma$-strongly monotone and upper hemicontinuous with nonempty compact convex values. For any $q \in X^*$, we consider the following variational inequality, which is finding $u \in K$ and $u^* \in A(u)$ such that

$$\langle u^*, v - u \rangle + j(v) - j(u) \geq \langle q, v - u \rangle, \quad \forall v \in K.$$ (3.1)

Let $U(q)$ be the set of solutions in $K$ for the set-valued variational inequality (3.1). From Lemma 3.1, it holds that $U(q) \neq \emptyset$. We introduce a mapping $K_A : X^* \to 2^K$ defined by

$$K_A(q) = U(q), \quad \forall q \in X^*.$$ (3.2)

**Lemma 3.2.** Let $X$ be a real reflexive Banach space, $K$ be a nonempty closed convex subset of $X$ and $j : K \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex functional with $j(K) \subset [0, +\infty]$. Suppose that $A : K \to 2^{X^*}$ is $\gamma$-strongly monotone and upper hemicontinuous with nonempty compact convex values, then the mapping $K_A$ defined by (3.2) is single-valued, continuous and bounded. Moreover, $K_A$ is $\frac{1}{\gamma}$-set-contractive.

**Proof.** For any $q_1, q_2 \in X^*$, take any $u_1 \in K_A(q_1), u_2 \in K_A(q_2)$ and any $u^*_1 \in Au_1, u^*_2 \in Au_2$, such that

$$\langle u^*_1, v - u_1 \rangle + j(v) - j(u_1) \geq \langle q_1, v - u_1 \rangle, \quad \forall v \in K, i = 1, 2.$$ (3.3)

Letting $v = u_{3-i}$, $i = 1, 2$, it follows from (3.3) that

$$\langle u^*_1 - u^*_2, u_1 - u_2 \rangle \leq \langle q_1 - q_2, u_1 - u_2 \rangle \leq \|q_1 - q_2\|\|u_1 - u_2\|.$$ (3.4)

From the $\gamma$-strong monotonicity of $A$, it holds that

$$\|u_1 - u_2\|^2 \leq \|q_1 - q_2\|\|u_1 - u_2\||.$$ (3.5)

Letting $q_1 = q_2$ in (3.5), we obtain $u_1 = u_2$, which implies that $K_A$ is single-valued.
Moreover, it follows from (3.5) that
\[ \|K_A(q_1) - K_A(q_2)\| = \|u_1 - u_2\| \leq \frac{1}{\gamma} \|q_1 - q_2\|. \tag{3.6} \]
which yields that \(K_A\) is Lipschitz continuous, bounded and \(\frac{1}{\gamma}\)-set-contractive. This completes the proof. \(\Box\)

**Theorem 3.1.** Let \(X\) be a real reflexive Banach space and \(f \in X^*\), \(K\) be a nonempty closed convex subset of \(X\) with \(0 \in K\). Suppose that \(j : X \to R\) is a proper lower semicontinuous and convex functional with \(j(0) = 0\) and \(j(K) \subset [0, +\infty]\), \(A : K \to 2^{X^*}\) is \(\gamma\)-strongly monotone and upper hemicontinuous with nonempty compact convex values with \(0 \in A(0)\), and \(g : K \to X^*\) is a bounded and \(\beta\)-set-contractive mapping, where \(\beta < \gamma\). If the following assumptions hold

(a) for any sequence \(\{u_n\} \subset K\) with \(\|u_n\| \to +\infty\), we have
\[ \liminf_{\|u_n\| \to +\infty} \frac{\langle g(u_n), u_n \rangle}{\|u_n\|^2} < \gamma; \]
(b) there exist \(u_0 \in r\cdot K \setminus \{0\}\) and a neighborhood \(V(0)\) of zero point such that for all \(u \in K \cap V(0)\) and all \(u^* \in A(u)\), it holds that
\[ \langle u^*, u_0 \rangle + j_{\infty}(u_0) \leq \langle g(u) + f, u_0 \rangle. \]

Then the set-valued variational inequality (1.1) has a nonzero solution.

**Proof.** By Lemma 3.2, the mapping \(K_A\) defined by (3.2) is continuous, bounded and \(\frac{1}{\gamma}\)-set-contractive. Define a new mapping \(K_{Ag} : K \to K\) as follows:
\[ K_{Ag}(u) = K_A(g(u) + f), \quad \forall u \in K. \]
It is evident that \(K_{Ag}\) is bounded and \(\beta\)-set-contractive. Since \(\beta < \gamma\), \(K_{Ag}\) is strictly set-contractive.

Next we shall verify that \(i_K(K_{Ag}, K^R) = 1\) for large enough \(R\) and \(i_K(K_{Ag}, K^r) = 0\) for small enough \(r\).

First, we define a mapping \(H_1 : [0, 1] \times K \to K\) as follows:
\[ H_1(t, u) = K_A(t(g(u) + f)). \]
Clearly, \(H_1\) is continuous and bounded in \([0, 1] \times K\) and for each \(t \in [0, 1]\), \(H_1(t, \cdot)\) is strictly set-contractive.

By (3.6), we have
\[ \|H_1(t_1, u) - H_1(t_2, u)\| = \|K_A(t_1(g(u) + f)) - K_A(t_2(g(u) + f))\| \]
\[ \leq \frac{1}{\gamma} \|t_1(g(u) + f) - t_2(g(u) + f)\| = \frac{1}{\gamma} \|g(u) + f\| |t_1 - t_2|, \tag{3.7} \]
which means that \(H_1(t, u)\) is uniformly continuous at \(t\) for all \(u \in K\).

Now we prove that there exists large enough \(R > 0\) such that \(u \neq H_1(t, u)\) for all \(t \in [0, 1], u \in \partial(K^R)\). Otherwise, there exist two sequences \(\{t_n\}\) and \(\{u_n\}\) with \(t_n \in [0, 1], u_n \in \partial(K^R)\) and \(\|u_n\| \to +\infty\) such that
\[ u_n = H_1(t_n, u_n) = K_A(t_n(g(u_n) + f)). \]
Then there exists \(u_n^* \in A(u_n)\) such that
\[ \langle u_n^*, v - u_n \rangle + j(v) - j(u_n) \geq t_n \langle g(u_n) + f, v - u_n \rangle, \quad \forall v \in K. \tag{3.8} \]
Letting \(v = 0, (3.8)\) yields that
\[ \langle u_n^*, u_n \rangle + j(u_n) \leq t_n \langle g(u_n) + f, u_n \rangle. \tag{3.9} \]
From the \(\gamma\)-strong monotonicity of \(A\) and \(0 \in A(0)\), we have
\[ \langle u_n^*, u_n \rangle \geq \gamma \|u_n\|^2. \tag{3.10} \]
Let $X$ be a real reflexive Banach space and $f$ and $g$ jointly yield
\begin{equation}
(3.10)
\end{equation}
We only need to show that the conditions in (1.1) are satisfied. Theorem 3.1
\begin{equation}
(1.1)
\end{equation}
Combining (3.11) and (3.12), we obtain that
\begin{equation}
\gamma \leq \liminf_{\|u_n\| \to +\infty} \frac{t_n \langle g(u_n) + f, u_n \rangle}{\|u_n\|^2} < \gamma,
\end{equation}
which is a contradiction.

On the other hand, since $j(0) = 0$ and $j(v) \geq 0$ for any $v \in K$, we have
\begin{equation}
(0, v - 0) + j(v) - j(0) \geq (0, v - 0), \quad \forall v \in K,
\end{equation}
\begin{equation}
(0, v - 0) + j(v) - j(0) \geq (0, v - 0), \quad \forall v \in K,
\end{equation}
Together with $0 \in A(0)$, which implies that $0 = K_A(0)$. Thus, we have
\begin{equation}
i_K(K_{Ag}, K^R) = i_K(H_1(1, \cdot), K^R) = i_K(H_1(0, \cdot), K^R) = i_K(\hat{0}, K^R) = 1.
\end{equation}
Let $r > 0$ be small enough such that $\overline{K^r} \subset K \cap V(0)$. From condition (b), there exists $u_0 \in rcK \setminus \{0\}$, for all $u \in \overline{K^r}$ and all $u^* \in A(u)$, it holds that
\begin{equation}
\langle u^*, u_0 \rangle + j_\infty(u_0) < \langle g(u) + f, u_0 \rangle.
\end{equation}
We now claim that $i_K(K_{Ag}, K') = 0$. If $i_K(K_{Ag}, K') \neq 0$, then, by Lemma 2.1(iv), the mapping $K_{Ag}$ has a fixed point $u \in K'$, i.e., $u = K_A(g(u) + f)$. From the definition of the mapping $K_A$, there exists $u^* \in A(u)$ such that
\begin{equation}
\langle u^*, v - u \rangle + j(v) - j(u) \geq \langle g(u) + f, v - u \rangle, \quad \forall v \in K.
\end{equation}
Since $u_0 \in rc(K)$ and $u \in K$, we have $u_0 + u \in K$. Taking $v = u_0 + u$ in (3.17), it holds that
\begin{equation}
\langle u^*, u_0 \rangle + j(u_0 + u) - j(u) \geq \langle g(u) + f, u_0 \rangle.
\end{equation}
Since $j(u_0 + u) \leq j_\infty(u_0) + j(u)$, (3.18) implies that
\begin{equation}
\langle u^*, u_0 \rangle + j_\infty(u_0) \geq \langle g(u) + f, u_0 \rangle,
\end{equation}
which contradicts (3.16). Thus we have proved $i_K(K_{Ag}, K') = 0$.

From Lemma 2.1(ii), we obtain $i_K(K_{Ag}, K^R \setminus \overline{K^r}) = 1$. Thus the mapping $K_{Ag}$ has a fixed point in $K^R \setminus \overline{K^r}$, which is a nonzero solution of the set-valued variational inequality (1.1). This completes the proof.

Remark 3.1. If $A$ is single-valued and bounded, $j = f = 0$ and $K$ is a closed convex cone, Theorem 3.1 reduces to Theorem 2.2 in [8].

Corollary 3.1. Let $X$ be a real reflexive Banach space and $f \in X^*$, $K$ be a nonempty closed convex subset of $X$ with $0 \in K$. Suppose that $j : X \to R$ is a proper lower semicontinuous and convex functional with $j(0) = 0$ and $j(K) \subset [0, +\infty)$. Suppose that $A : K \to 2^{X^*}$ is $\gamma$-strongly monotone and upper semicontinuous with nonempty compact convex values with $0 \in A(0)$, $g : K \to X^*$ is Lipschitz continuous with constant $\beta$, where $\beta < \gamma$. If there exists $u_0 \in rcK \setminus \{0\}$ such that
\begin{equation}
\sup_{v_0^* \in A(0)} \langle v_0^*, u_0 \rangle < \langle g(0) + f, u_0 \rangle - j_\infty(u_0).
\end{equation}
Then the set-valued variational inequality (1.1) has a nonzero solution.

Proof. We only need to show that the conditions in Theorem 3.1 are satisfied.

First, since $g$ is Lipschitz continuous with constant $\beta$, $g$ is $\beta$-set-contractive. Moreover, we have
\begin{equation}
\|g(u) - g(0)\| \leq \beta\|u\|.
\end{equation}
which follows that 
\[ \langle g(u), u \rangle \leq \|g(u) - g(0)\|\|u\| + \|g(0)\|\|u\| \leq \beta \|u\|^2 + \|g(0)\|\|u\|. \]

This implies that 
\[ \liminf_{\|u\| \to +\infty} \frac{\langle g(u), u \rangle}{\|u\|^2} \leq \beta < \gamma. \]

Secondly, from (3.19), we have 
\[ \sup_{v^*_0 \in A(0)} \langle v^*_0 - g(0), u_0 \rangle = \langle f, u_0 \rangle - j_\infty(u_0), \]

which yields that 
\[ A(0) - g(0) \subset \{v^* \in X^* : \langle v^*, u_0 \rangle < \langle f, u_0 \rangle - j_\infty(u_0)\}. \] (3.20)

Since A is upper semicontinuous and g is continuous, A - g is upper semicontinuous. From (3.20), there exists a neighborhood V(0) of zero point, for all u \in K \cap V(0), it holds that 
\[ A(u) - g(u) \subset \{v^* \in X^* : \langle v^*, u_0 \rangle < \langle f, u_0 \rangle - j_\infty(u_0)\}, \]

which means that, for all u \in K \cap V(0) and all u^* \in Au, we have 
\[ \langle u^*, u_0 \rangle + j_\infty(u_0) < \langle g(u) + f, u_0 \rangle. \]

This completes the proof. \(\square\)

**Remark 3.2.** If A(0) = \{0\}, (3.19) can be rewritten as \(\langle g(0) + f, u_0 \rangle - j_\infty(u_0) > 0\). If, in addition, \(j_\infty(u_0) \leq 0\), (3.19) becomes \(\langle g(0) + f, u_0 \rangle > 0\).

The following example shows us that functions satisfying conditions in Corollary 3.1 and Theorem 3.1 exist.

**Example 3.1.** Let \(X = R\) be the set of real numbers with usual norm, \(K = [0, +\infty), u_0 = 1\) and \(A : K \to 2^R\) be a set-valued mapping defined by 
\[ A(u) = \begin{cases} \left\{ \frac{2}{3}u \right\}, & u \in [0, 1), \\
\left\{ \frac{2}{3}, 1 \right\}, & u = 1,
\{u\}, & u \in (1, +\infty). \end{cases} \]

It is obvious that A is \(\frac{2}{3}\)-strongly monotone and upper semicontinuous with nonempty compact convex values with 
\(A(0) = \{0\}\). Define \(j : K \to R\) as \(j(u) := \|u\| + \frac{1}{\sqrt{|u| + 2}} - \frac{1}{\sqrt{2}}\), then \(j_\infty(u) = \|u\|\). Define \(g : K \to R\) as \(g(u) := \frac{1}{3}u - \frac{1}{2}\) and choose \(f = 3\). It is easy to see that the functions g, A, j and f satisfy all conditions in Corollary 3.1 and Theorem 3.1.

**Theorem 3.2.** Let \(X\) be a real reflexive Banach space and \(f \in X^*, K\) be a nonempty closed convex subset of \(X\) with \(0 \in K\). Suppose that \(j : X \to R\) is a proper lower semicontinuous and convex functional with \(j(0) = 0\), \(j(K) \subset [0, +\infty)\), \(A : K \to 2^{X^*}\) is \(\gamma\)-strongly monotone and upper hemicontinuous with nonempty compact convex values, A is bounded with \(0 \in A(0)\), \(g : K \to X^*\) is bounded and \(\beta\)-set-contractive, where \(\beta < \gamma\). If the following assumptions hold 
(a) for any sequence \(\{u_n\} \subset K\) with \(\|u_n\| \to 0\), we have 
\[ \liminf_{\|u_n\| \to 0} \frac{\langle g(u_n) + f, u_n \rangle}{\|u_n\|^2} < \gamma; \]

(b) there exist \(u_0 \in rK \setminus \{0\}\) and a constant \(\rho > 0\) such that for all \(u \in K\) with \(\|u\| > \rho\) and for all \(u^* \in A(u)\), we have 
\[ \langle u^*, u_0 \rangle + j_\infty(u_0) < \langle g(u) + f, u_0 \rangle. \]
Then the set-valued variational inequality (1.1) has a nonzero solution.

**Proof.** By Lemma 3.2, the mapping \( K_A \) defined by (3.2) is continuous, bounded and \( \frac{1}{p} \)-set-contractive. Define \( K_{Ag} : K \rightarrow K \) as follows:

\[
K_{Ag}(u) = K_A(g(u) + f), \quad \forall u \in K.
\]

It is easy to see that \( K_{Ag} \) is \( \frac{1}{p} \)-set-contractive. Since \( \beta < \gamma \), \( K_{Ag} \) is strictly set-contractive.

Now we verify that \( i_K(K_{Ag}, K^R) = 0 \) for large enough \( R \).

Since \( A \) and \( g \) are bounded, there exist constants \( M > 0 \) and \( L > 0 \) such that

\[
\sup_{u \in K^R, \|u\| \leq L} \|u^*\| \leq M \quad \text{and} \quad \sup_{u \in K^R} \|g(u)\| \leq L.
\]

which follows that

\[
\sup_{u \in K^R, v \in A(u)} \langle u^*, u \rangle \leq M\|u_0\| \quad \text{and} \quad \sup_{u \in K^R} \langle g(u), u \rangle \leq L\|u_0\|.
\] (3.21)

Since \( u_0 \neq 0 \), there exists some \( h \in X^* \) such that \( \langle h, u_0 \rangle > 0 \). Letting \( N \) be large enough, we have

\[
M\|u_0\| + L\|u_0\| + j_{\infty}(u_0) < \langle f, u_0 \rangle + N \langle h, u_0 \rangle.
\] (3.22)

Define \( H_2 : [0, 1] \times K^R \rightarrow K \) as follows:

\[
H_2(t, u) = K_A(g(u) + f + tN h), \quad \forall (t, u) \in [0, 1] \times K^R.
\]

Then \( H_2(t, u) \) is continuous and bounded in \([0, 1] \times K^R\), and \( H(t, \cdot) \) is strictly set-contractive for each \( t \in [0, 1] \). It is easy to verify that \( H_2(t, u) \) is uniformly continuous at \( t \) for all \( u \in K^R \).

We claim that there exists large enough \( R \) such that \( u \neq H_2(t, u) \), for all \( t \in [0, 1] \) and all \( u \in \partial \delta(K^R) \). Otherwise, there exist sequences \( \{t_n\} \) with \( t_n \in [0, 1] \) and \( \{u_n\} \) with \( \|u_n\| \rightarrow +\infty \) such that \( u_n = H_2(t_n, u_n) = K_A(g(u_n) + f + t_n N h) \). Hence, it holds that

\[
\langle u_n^*, v - u_n \rangle + j(v) - j(u_n) \geq \langle g(u_n) + f + t_n N h, v - u_n \rangle, \quad \forall v \in K.
\] (3.23)

Since \( u_0 \in r c(K) \) and \( u_n \in K \), we have \( u_0 + u_n \in K \). Letting \( v = u_0 + u_n \), (3.23) yields that

\[
\langle u_n^*, u_0 \rangle + j(u_0 + u_n) - j(u_n) \geq \langle g(u_n), u_0 \rangle + \langle f + t_n N h, u_n \rangle.
\] (3.24)

Since \( j(u_0 + u_n) \leq j_{\infty}(u_0) + j(u_n) \), it follows from (3.24) that

\[
\langle u_n^*, u_0 \rangle + j_{\infty}(u_0) \geq \langle g(u_n), u_0 \rangle + \langle f + t_n N h, u_0 \rangle \geq \langle g(u_n) + f, u_0 \rangle,
\] (3.25)

which contradicts condition (b).

We now claim that \( i_K(H_2(1, \cdot), K^R) = 0 \).

If \( i_K(H_2(1, \cdot), K^R) \neq 0 \), then from Lemma 2.1(iv), the mapping \( H_2(1, \cdot) \) has a fixed point \( u \in K^R \), i.e., \( u = H_2(1, u) = K_A(g(u) + f + N h) \). Then there exists \( u^* \in A(u) \) such that

\[
\langle u^*, v - u \rangle + j(v) - j(u) \geq \langle g(u) + f + N h, v - u \rangle, \quad \forall v \in K.
\] (3.26)

Taking \( v = u_0 + u \), (3.26) yields that

\[
\langle u^*, u_0 \rangle + j_{\infty}(u_0) \geq \langle g(u), u_0 \rangle + \langle f + N h, u_0 \rangle > \langle g(u) + f, u_0 \rangle.
\] (3.27)

We consider the following two cases.

Case 1. \( \|u\| > \rho \), (3.27) contradicts condition (b).

Case 2. \( \|u\| \leq \rho \), (3.27) implies that

\[
\langle f + N h, u_0 \rangle \leq \langle u^* - g(u), u_0 \rangle + j_{\infty}(u_0) \leq M\|u_0\| + L\|u_0\| + j_{\infty}(u_0),
\]

which contradicts (3.22).

Hence, we obtain \( i_K(H_2(1, \cdot), K^R) = 0 \). Furthermore, \( i_K(K_{Ag}, K^R) = i_K(H_2(0, \cdot), K^R) = i_K(H_2(1, \cdot), K^R) = 0 \).
As in the first part of the proof of Theorem 3.1, we can obtain \( i_K(K_{Ag}, K') = 1 \).

It follows from Lemma 2.1(ii) that \( i_K(K_{Ag}, K^R \setminus K') = -1 \). Thus the mapping \( K_{Ag} \) has a fixed point in \( K^R \setminus K' \), which is a nonzero solution of the set-valued variational inequality (1.1). This completes the proof. \( \square \)

**Remark 3.3.** If \( A \) is single-valued, \( j = f = 0 \) and \( K \) is a closed convex cone, Theorem 3.2 reduces to Theorem 2.1 in [8].

**Corollary 3.2.** Let \( X \) be a real reflexive Banach space and \( f \in X^* \), \( K \) be a nonempty closed convex subset of \( X \) with \( 0 \in K \). Let \( j : X \to R \) be a proper lower semicontinuous and convex functional with \( j(0) = 0 \), \( j(K) \subset [0, +\infty) \), \( A : K \to 2^{X^*} \) be \( \gamma \)-strongly monotone and upper hemicontinuous with nonempty compact convex values, and let \( A \) be bounded with \( 0 \in A(0) \). Suppose that \( u_0 \in r cK \setminus \{0\} \) and \( g : K \to X^* \) is continuous with the form \( g = g_1 + g_2 \), where \( g_1(u) = \frac{1}{\|u_0\|}h(u)v_0 \), \( h : K \to R \), \( v_0 \in Ju_0 \), \( J \) is the normalized duality mapping of \( X \), \( g_2 : X \to X^* \) is \( \beta \)-set-contractive. Assume that

(a) there exists a constant \( r > 0 \) such that \( \langle g(0) + f, u \rangle \leq 0 \) for all \( u \in K' \), and the restriction of \( g \) on \( K' \) is Lipschitz continuous with constant \( \beta \), where \( \beta < \gamma \).

(b) there exist constants \( \rho, \alpha, \beta_1, \beta_2, \beta_3, C, C_1, C_2 \) and \( C_3 \), such that for all \( u \in K \) with \( \|u\| > \rho \), the following results hold

\[
\frac{h(u)}{\|u\|^\alpha} \geq C, \quad \frac{\|g_2(u)\|}{\|u\|^{\beta_1}} \leq C_1, \quad \frac{j(u)}{\|u\|^{\beta_2}} \leq C_2 \quad \text{and} \quad \sup_{u^* \in A(u)} \frac{\|u^*\|}{\|u\|^{\beta_3}} \leq C_3,
\]

where \( \alpha > \max\{\beta_1, \beta_2, \beta_3\} \), \( \rho, \alpha, C > 0 \) and \( \beta_2 \leq 1 \).

Then the set-valued variational inequality (1.1) has a nonzero solution.

**Proof.** We only need to verify that the conditions in Theorem 3.2 are satisfied.

It is easy to see that \( g_1 \) is compact, then \( g_1 \) is \( 0 \)-set-contractive, thus \( g \) is \( \beta \)-set-contractive. It is obvious that

\[
\langle g(u) + f, u \rangle = \langle g(u) - g(0), u \rangle + \langle g(0) + f, u \rangle.
\]

From condition (a), we obtain that

\[
\langle g(u) - g(0), u \rangle \leq \beta \|u\|^2 \quad \text{and} \quad \langle g(0) + f, u \rangle \leq 0, \quad \forall u \in K'.
\]

(3.28) and (3.29) jointly yield that

\[
\langle g(u) + f, u \rangle \leq \beta \|u\|^2, \quad \forall u \in K'.
\]

For any sequence \( \{u_n\} \subset K \) with \( \|u_n\| \to 0 \), we have

\[
\liminf_{\|u_n\| \to 0} \frac{\|g(u_n) + f, u_n\|}{\|u_n\|^2} \leq \beta < \gamma.
\]

Since \( g_1(u) = \frac{1}{\|u_0\|}h(u)v_0 \) and \( v_0 \in J(u_0) \), we have

\[
\langle g_1(u), u_0 \rangle = h(u)v_0.
\]

From condition (b), for any \( u \in K \) with \( \|u\| > \rho \), we have \( j(u) \leq C_2 \|u\|^{\beta_2} \). For \( t > 0 \) big enough with \( \|tu_0\| > \rho \), it holds that \( j(tu_0) \leq C_2 \|tu_0\|^{\beta_2} \), which yields that \( j(tu_0) \leq C_2 t \|u_0\|^{\beta_2} \). It follows from \( \beta_2 \leq 1 \) that \( j_{\infty}(u_0) < \infty \).

Moreover, we obtain

\[
\liminf_{\|u\| \to +\infty} \frac{\|g_1(u), u_0\|}{\|u\|^\alpha} \geq C, \quad \limsup_{\|u\| \to +\infty} \frac{\|g_2(u) + f, u_0\|}{\|u\|^\alpha} = 0 \quad \text{and} \quad \limsup_{\|u\| \to +\infty} \sup_{u^* \in A(u)} \frac{\langle u^*, u_0 \rangle}{\|u\|^\alpha} = 0.
\]

Thus, the following results hold

\[
\liminf_{\|u\| \to +\infty} \frac{\|g(u) + f, u_0\|}{\|u\|^\alpha} \geq \liminf_{\|u\| \to +\infty} \frac{\|g_1(u), u_0\|}{\|u\|^\alpha} - \limsup_{\|u\| \to +\infty} \frac{\|g_2(u) + f, u_0\|}{\|u\|^\alpha} \geq C.
\]
and
\[
\limsup_{\|u\| \to +\infty} \sup_{u^* \in A(u)} \frac{\langle u^*, u_0 \rangle + j_\infty(u_0)}{\|u\|^\alpha} = 0.
\]

Therefore, there exists a constant \( \rho_1 \), for all \( u \in K \) with \( \|u\| > \rho_1 \) and for all \( u^* \in A(u) \), it holds that
\[
\langle u^*, u_0 \rangle + j_\infty(u_0) < \langle g(u) + f, u_0 \rangle.
\]

This completes the proof. \( \square \)

**Remark 3.4.** If \( g(0) + f = 0 \) in condition (a) of Corollary 3.2, then, for any \( r > 0 \), all \( u \in K' \), we have \( \langle g(0) + f, u_0 \rangle = 0 \).

The following example shows us that functions satisfying conditions in Corollary 3.2 and Theorem 3.2 exist.

**Example 3.2.** Let \( X \) be a Hilbert space, \( K \) be a nonempty closed convex subset of \( X \) with \( 0 \in K \) and \( u_0 \in rcK \setminus \{0\} \). Define \( g : K \to X^* \) as \( g(u) := \frac{\|u\|^2}{\|u_0\|^2} \), define \( j : K \to R \) as \( j(u) := \|u\| \) and define \( A : K \to X^* \) as \( A(u) := u \), choose \( f = 0 \). It is easy to see that the functions \( g, A, j \) and \( f \) satisfy all conditions in Corollary 3.2 and Theorem 3.2.

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**References**


