ELSEVIER
Stochastic Processes and their Applications 92 (2001) 31–60
stochastic processes and their applications
www.elsevier.com/locate/spa

Strong approximation of fractional Brownian motion by moving averages of simple random walks

Tamás Szabados

Department of Mathematics, Technical University of Budapest, Egry u 20-22, H ép. V em., Budapest, 1521, Hungary

Received 19 December 1999; received in revised form 29 August 2000; accepted 4 September 2000

Dedicated to Pál Révész on the occasion of his 65th birthday

Abstract

The fractional Brownian motion is a generalization of ordinary Brownian motion, used particularly when long-range dependence is required. Its explicit introduction is due to Mandelbrot and van Ness (SIAM Rev. 10 (1968) 422) as a self-similar Gaussian process $W^{(H)}(t)$ with stationary increments. Here self-similarity means that $(a^{-H}W^{(H)}(at)\colon t\geqslant 0)\overset{\mathrm{d}}{=}(W^{(H)}(t)\colon t\geqslant 0)$, where $H\in(0,1)$ is the Hurst parameter of fractional Brownian motion. F.B. Knight gave a construction of ordinary Brownian motion as a limit of simple random walks in 1961. Later his method was simplified by Révész (Random Walk in Random and Non-Random Environments, World Scientific, Singapore, 1990) and then by Szabados (Studia Sci. Math. Hung. 31 (1996) 249–297). This approach is quite natural and elementary, and as such, can be extended to more general situations. Based on this, here we use moving averages of a suitable nested sequence of simple random walks that almost surely uniformly converge to fractional Brownian motion on compacts when $H\in(\frac{1}{4},1)$. The rate of convergence proved in this case is $\mathrm{O}(N^{-\min(H-1/4,1/4)}\log N)$, where N is the number of steps used for the approximation. If the more accurate (but also more intricate) Komlós et al. (1975, 1976) approximation is used instead to embed random walks into ordinary Brownian motion, then the same type of moving averages almost surely uniformly converge to fractional Brownian motion on compacts for any $H\in(0,1)$. Moreover, the convergence rate is conjectured to be the best possible $\mathrm{O}(N^{-H}\log N)$, though only $\mathrm{O}(N^{-\min(H,1/2)}\log N)$ is proved here. © 2001 Elsevier Science B.V. All rights reserved.

MSC: primary 60G18; 60F15; secondary 60J65

Keywords: Fractional Brownian motion; Pathwise construction; Strong approximation; Random walk; Moving average

1. Fractional Brownian motion

The fractional Brownian motion (fBM) is a generalization of ordinary Brownian motion (BM) used particularly when long-range dependence is essential. Though the

E-mail address: szabados@math.bme.hu (T. Szabados).

PII: S0304-4149(00)00078-8

history of fBM can be traced back to Kolmogorov (1940) and others, its explicit introduction is due to Mandelbrot and van Ness (1968). Their intention was to define a *self-similar*, centered Gaussian process $W^{(H)}(t)$ $(t \geqslant 0)$ with stationary but not independent increments and with continuous sample paths a.s. Here self-similarity means that for any $a > 0$,

$$(a^{-H} W^{(H)}(at)\colon t \geqslant 0) \overset{\mathrm{d}}{=} (W^{(H)}(t)\colon t \geqslant 0), \tag{1}$$

where $H \in (0,1)$ is the *Hurst parameter* of the fBM and $\overset{\mathrm{d}}{=}$ denotes equality in distribution. They showed that these properties characterize fBM. The case $H = \frac{1}{2}$ reduces to ordinary BM with independent increments, while the cases $H < \frac{1}{2}$ (resp. $H > \frac{1}{2}$) give negatively (resp. positively) correlated increments; see Mandelbrot and van Ness (1968). It seems that in the applications of fBM, the case $H > \frac{1}{2}$ is the most frequently used.

Mandelbrot and van Ness (1968) gave the following explicit representation of fBM as a moving average of ordinary, but two-sided BM $W(s)$, $s \in \mathbb{R}$:

$$W^{(H)}(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{-\infty}^{t} [(t-s)^{H-1/2} - (-s)_{+}^{H-1/2}] \,\mathrm{d}W(s), \tag{2}$$

where $t \geqslant 0$ and $(x)_{+} = \max(x, 0)$. The idea of (2) is related to *deterministic fractional calculus*, which has an even longer history than fBM, going back to Liouville, Riemann, and others; see in Samko et al. (1993). Its simplest case is when a continuous function f and a positive integer α are given. Then an induction with integration by parts can show that

$$f_{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s) \,\mathrm{d}s$$

is the order α iterated antiderivative (or order α integral) of f. On the other hand, this integral is well-defined for non-integer positive values of α as well, in which case it can be called a fractional integral of f.

So, heuristically, the main part of (2),

$$W_{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} W'(s) \,\mathrm{d}s = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \,\mathrm{d}W(s)$$

is the order α integral of the (in ordinary sense non-existing) white noise process $W'(t)$. Thus the fBM $W^{(H)}(t)$ can be considered as a stationary-increment modification of the fractional integral $W_{\alpha}(t)$ of the white noise process, where $\alpha = H + \frac{1}{2} \in (\frac{1}{2}, \frac{3}{2})$.

2. Random walk construction of ordinary Brownian motion

It is interesting that a very natural and elementary construction of ordinary BM as a limit of random walks (RWs) appeared relatively late. The mathematical theory of BM began around 1900 with the works of Bachelier, Einstein, Smoluchowski, and others. The first existence construction was given by Wiener (1921, 1923) that was followed by several others later. Knight (1961) introduced the first construction by random walks that was later simplified by Révész (1990). The present author was fortunate enough to hear this version of the construction directly from Pál Révész in a seminar at the Technical University of Budapest a couple of years before the publication of Révész's book in 1990 and got immediately fascinated by it. The result of an effort to further simplify it appeared in Szabados (1996). From now on, the expression *RW construction* will always refer to the version discussed in the latter. It is asymptotically equivalent to applying Skorohod (1965) embedding to find a nested dyadic sequence of RWs in BM, see Theorem 4 in Szabados (1996). As such, it has some advantages and disadvantages compared to the celebrated best possible approximation by BM of partial sums of random variables with moment generator function finite around the origin. The latter was obtained by Komlós et al. (1975, 1976), and will be abbreviated *KMT approximation* in the sequel. The main advantages of the RW construction are that it is elementary, explicit, uses only past values to construct new ones, easy to implement in practice, and very suitable for approximating stochastic integrals, see Theorem 6 in Szabados (1996) and also Szabados (1990). Recall that the KMT approximation constructs partial sums (e.g. a simple symmetric RW) from BM itself (or from an i.i.d. sequence of standard normal random variables) by an intricate sequence of conditional quantile transformations. To construct any new value it uses to whole sequence (past and future values as well). On the other hand, the major weakness of the RW construction is that it gives a rate of convergence $\mathrm{O}(N^{-1/4}\log N)$, while the rate of the KMT approximation is the best possible $\mathrm{O}(N^{-1/2}\log N)$, where N is the number of steps (terms) considered in the RW.

In the sequel first the main properties of the above-mentioned RW construction are summarized. Then this RW construction is used to define an approximation similar to (2) of fBM by moving averages of the RW. The convergence and the error of this approximation are discussed next. As a consequence of the relatively weaker approximation properties of the RW construction, the convergence to fBM will be established only for $H\in(\frac{1}{4},1)$, and the rate of convergence will not be the best possible either. To compensate for this, at the end of the paper we discuss the convergence and error properties of a similar construction of fBM that uses the KMT approximation instead, which converges for all $H\in(0,1)$ and whose convergence rate can be conjectured to be the best possible when approximating fBM by moving averages of RWs.

The RW construction of BM summarized here is taken from Szabados (1996). We start with an infinite matrix of i.i.d. random variables $X_m(k)$,

$$\boldsymbol{P}\{X_m(k)=1\}=\boldsymbol{P}\{X_m(k)=-1\}=\tfrac{1}{2}\quad(m\geqslant 0,\ k\geqslant 1),$$

defined on the same underlying probability space $(\Omega,\mathscr{A},\boldsymbol{P})$. Each row of this matrix is a basis of an approximation of BM with a certain dyadic step size $\Delta t=2^{-2m}$ in time and a corresponding step size $\Delta x=2^{-m}$ in space, illustrated by the next table.

The second step of the construction is *twisting*. From the independent random walks (i.e. from the rows of Table 1), we want to create dependent ones so that after shrinking temporal and spatial step sizes, each consecutive RW becomes a refinement of the previous one. Since the spatial unit will be halved at each consecutive row, we define stopping times by $T_m(0)=0$, and for $k\geqslant 0$,

$$T_m(k+1)=\min\{n\colon n>T_m(k),|S_m(n)-S_m(T_m(k))|=2\}\quad(m\geqslant 1).$$

Table 1
The starting setting for the RW construction of BM

Δt	Δx	i.i.d. sequence	RW
1	1	$X_0(1), X_0(2), X_0(3), \ldots$	$S_0(n) = \sum_{k=1}^{n} X_0(k)$
2^{-2}	2^{-1}	$X_1(1), X_1(2), X_1(3), \ldots$	$S_1(n) = \sum_{k=1}^{n} X_1(k)$
2^{-4}	2^{-2}	$X_2(1), X_2(2), X_2(3), \ldots$	$S_2(n) = \sum_{k=1}^{n} X_2(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

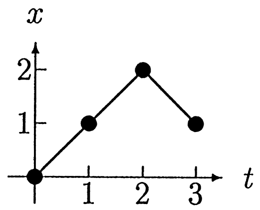

Fig. 1. $B_0(t;\omega) = S_0(t;\omega)$.

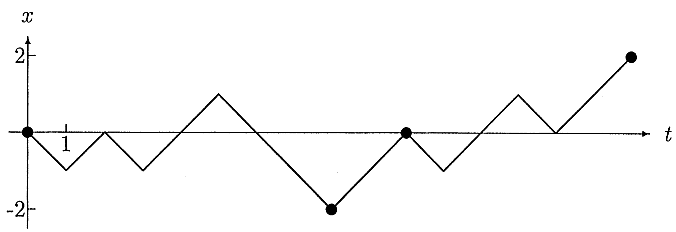

Fig. 2. $S_1(t;\omega)$.

These are the random time instants when a RW visits even integers, different from the previous one. After shrinking the spatial unit by half, a suitable modification of this RW will visit the same integers in the same order as the previous RW. (This is what we call a refinement.) We will operate here on each point $\omega \in \Omega$ of the sample space separately, i.e. we fix a sample path of each RW appearing in Table 1. Thus each *bridge* $S_m(T_m(k+1)) - S_m(T_m(k))$ has to mimic the corresponding step $X_{m-1}(k+1)$ of the previous RW. We define twisted RWs $\tilde{S}_m$ recursively for $m = 1, 2, 3, \ldots$ using $\tilde{S}_{m-1}$, starting with $\tilde{S}_0(n) = S_0(n)$ $(n \geqslant 0)$. With each fixed m we proceed for $k = 0, 1, 2, \ldots$ successively, and for every n in the corresponding bridge, $T_m(k) < n \leqslant T_m(k+1)$. Any bridge is flipped if its sign differs from the desired (Figs. 1–3):

$$\tilde{X}_m(n) = \begin{cases} X_m(n) & \text{if } S_m(T_m(k+1)) - S_m(T_m(k)) = 2\tilde{X}_{m-1}(k+1), \\ -X_m(n) & \text{otherwise,} \end{cases}$$

and then $\tilde{S}_m(n) = \tilde{S}_m(n-1) + \tilde{X}_m(n)$. Then each $\tilde{S}_m(n)$ $(n \geqslant 0)$ is still a simple, symmetric RW; see Lemma 1 in Szabados (1996). Moreover, the twisted RWs have the

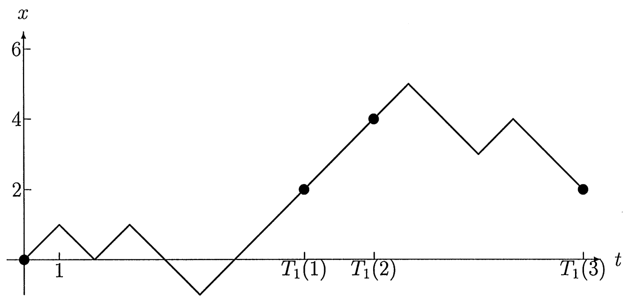

Fig. 3. $\tilde{S}_1(t;\omega)$.

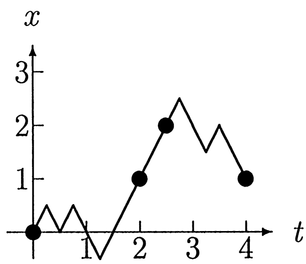

Fig. 4. $B_1(t;\omega)$.

desired refinement property:

$$\tfrac{1}{2}\tilde{S}_m(T_m(k)) = \tilde{S}_{m-1}(k) \quad (m\geqslant 1, k\geqslant 0). \tag{3}$$

The last step of the RW construction is *shrinking*. The sample paths of $\tilde{S}_m(n)$ $(n\geqslant 0)$ can be extended to continuous functions by linear interpolation. This way one gets $\tilde{S}_m(t)$ $(t\geqslant 0)$ for real t. Then we define the *mth approximation of BM* (see Fig. 4) by

$$B_m(t) = 2^{-m}\tilde{S}_m(t2^{2m}). \tag{4}$$

Compare three steps of a sample path of the first approximation $B_0(t;\omega)$ and the corresponding part of the second approximation $B_1(t;\omega)$ on Figs. 1 and 4. The second visits the same integers (different from the previous one) in the same order as the first, so mimics the first, but the corresponding time instants differ in general: $2^{-2}T_1(k)\neq k$. Similarly, (3) implies the general *refinement property*

$$B_{m+1}(T_{m+1}(k)2^{-2(m+1)}) = B_m(k2^{-2m}) \quad (m\geqslant 0, k\geqslant 0), \tag{5}$$

but there is a *time lag*

$$T_{m+1}(k)2^{-2(m+1)} - k2^{-2m} \neq 0 \tag{6}$$

in general. The basic idea of the RW construction of BM is that these time lags become uniformly small if m gets large enough. It can be proved by the following simple lemma.

Lemma 1. *Suppose that* $X_1, X_2, \ldots, X_N$ *is an i.i.d. sequence of random variables,* $\boldsymbol{E}(X_k) = 0$, $\mathbf{Var}(X_k) = 1$, *and their moment generating function* $\mathbf{E}(e^{uX_k})$ *is finite for* $|u|\leqslant u_0$,

$u_0 > 0$. Let $S_j = X_1 + \cdots + X_j$, $1 \leqslant j \leqslant N$. Then for any $C > 1$ and $N \geqslant N_0(C)$ one has

$$\boldsymbol{P}\left\{ \max_{0 \leqslant j \leqslant N} |S_j| \geqslant (2CN \log N)^{1/2} \right\} \leqslant 2N^{1-C}.$$

This basic fact follows from a large deviation inequality, see, e.g. Section XVI.6 in Feller (1966). Lemma 1 easily implies the uniform smallness of time lags in (6).

Lemma 2. *For any $K > 0, C > 1$, and for any $m \geqslant m_0(C)$, we have*

$$\boldsymbol{P}\left\{ \max_{0 \leqslant k2^{-2m} \leqslant K} |T_{m+1}(k)2^{-2(m+1)} - k2^{-2m}| \geqslant (\tfrac{3}{2} CK \log_* K)^{1/2} m^{1/2} 2^{-m} \right\}$$
$$\leqslant 2(K2^{2m})^{1-C},$$

where $\log_*(x) = \max(1, \log x)$.

Not surprisingly, this and the refinement property (5) imply the uniform closeness of two consecutive approximations of BM if m is large enough.

Lemma 3. *For any $K > 0, C > 1$, and $m \geqslant m_1(C)$, we have*

$$\boldsymbol{P}\left\{ \max_{0 \leqslant k2^{-2m} \leqslant K} |B_{m+1}(k2^{-2m}) - B_m(k2^{-2m})| \geqslant K^{1/4} (\log_* K)^{3/4} m 2^{-m/2} \right\}$$
$$\leqslant 3(K2^{2m})^{1-C}.$$

This lemma ensures the a.s. uniform convergence of the RW approximations on compact intervals and it is clear that the limit process is the Wiener process (BM) with continuous sample paths almost surely.

Theorem 1. *The RW approximation $B_m(t)$ ($t \geqslant 0$, $m = 0, 1, 2, \ldots$) a.s. uniformly converges to a Wiener process $W(t)$ ($t \geqslant 0$) on any compact interval $[0, K]$, $K > 0$. For any $K > 0$, $C \geqslant 3/2$, and for any $m \geqslant m_2(C)$, we have*

$$\boldsymbol{P}\left\{ \max_{0 \leqslant t \leqslant K} |W(t) - B_m(t)| \geqslant K^{1/4} (\log_* K)^{3/4} m 2^{-m/2} \right\} \leqslant 6(K2^{2m})^{1-C}.$$

The results quoted above correspond to Lemmas 2–4 and Theorem 3 in Szabados (1996). We mention that the statements presented here are given in somewhat sharper forms, but they can be read easily from the proofs in the above reference.

3. A pathwise approximation of fractional Brownian motion

An almost surely convergent pathwise construction of fBM was given by Carmona and Coutin (1998) representing fBM as a linear functional of an infinite dimensional Gaussian process. Another pathwise construction was given by Decreusefond and Üstünel (1998,1999) which converges in the L^2 sense. This construction uses discrete approximations of the moving average representation of fBM (2), based on deterministic partitions of the time axis. More exactly, (2) is substituted by an integral over the compact interval $[0,t]$, but with a more complicated kernel containing a hypergeometric function too.

The approximation of fBM discussed here will also be a discrete version of the moving average representation (2) of fBM, but dyadic partitions are taken on the spatial axis of BM and so one gets random partitions on the time axis. This is asymptotically a Skorohod-type embedding of nested RWs into BM. As a result, instead of integral we have sum, and BM is substituted by the nested, refining sequence of its RW approximations discussed in the previous section. Since (2) contains two-sided BM, we need two such sequences: one for the right and one for the left half-axis. From now on, we are going to use the following notations: $m \geqslant 0$ is an integer, $\Delta t = 2^{-2m}$, $t_x = x\,\Delta t$ $(x \in \mathbb{R})$. Introducing the kernel

$$h(s,t) = \frac{1}{\Gamma(H+\frac{1}{2})}[(t-s)^{H-1/2} - (-s)_+^{H-1/2}] \quad (s \leqslant t), \tag{7}$$

the *mth approximation of fBM* by definition is $B_m^{(H)}(0)=0$, and for positive integers k,

$$\begin{aligned} B_m^{(H)}(t_k) &= \sum_{r=-\infty}^{k-1} h(t_r,t_k)\,[B_m(t_r+\Delta t) - B_m(t_r)] \\ &= \frac{2^{-2Hm}}{\Gamma(H+\frac{1}{2})} \sum_{r=-\infty}^{k-1} [(k-r)^{H-1/2} - (-r)_+^{H-1/2}]\tilde{X}_m(r+1), \end{aligned} \tag{8}$$

where the convention $0^{H-1/2}=0$ is applied even for negative exponents.

$B_m^{(H)}$ is *well-defined*, since the "infinite part"

$$\sum_{r=-\infty}^{-1} [(k-r)^{H-1/2} - (-r)^{H-1/2}]\tilde{X}_m(r+1) =: \sum_{r=-\infty}^{-1} Y_{k,-r}$$

converges a.s. to a random variable Z_k by Kolmogorov's "three-series theorem": $\boldsymbol{E}(Y_{k,v}) = 0$ and

$$\sum_{v=1}^{\infty} \mathbf{Var}(Y_{k,v}) = \sum_{v=1}^{\infty} v^{2H-1}\left[\left(1+\frac{k}{v}\right)^{H-1/2} - 1\right]^2 \sim \sum_{v=1}^{\infty} \frac{\text{const}}{v^{3-2H}} < \infty.$$

It is useful to write $B_m^{(H)}$ in another form applying a discrete version of integration by parts. Starting with (8) and rearranging it according to $B_m(t_r)$, one obtains for $k \geqslant 1$ that

$$B_m^{(H)}(t_k) = \sum_{r=-\infty}^{k} \frac{h(t_r-\Delta t,t_k) - h(t_r,t_k)}{\Delta t} B_m(t_r)\Delta t. \tag{9}$$

This way we have got a discrete version of

$$W^{(H)}(t) = \frac{-1}{\Gamma(H+\frac{1}{2})} \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}s}[(t-s)^{H-1/2} - (-s)_+^{H-1/2}]W(s)\,\mathrm{d}s, \tag{10}$$

which is what one obtains from (2) using a formal integration by parts (cf. Lemma 5 below).

To support the above definition we show that $B_m^{(H)}$ has properties analogous to the characterizing properties of fBM in a discrete setting.

(a) $B_m^{(H)}$ is *centered* (clear from its definition) and has *stationary increments*: If k_0 and k are non-negative integers, then (substituting $u = r - k_0$)

$$B_m^{(H)}(t_{k_0} + t_k) - B_m^{(H)}(t_{k_0})$$

$$= \frac{2^{-2Hm}}{\Gamma(H+\frac{1}{2})} \left\{ \sum_{r=0}^{k_0+k-1} (k_0 + k - r)^{H-1/2} \tilde{X}_m(r+1) - \sum_{r=0}^{k_0} (k_0 - r)^{H-1/2} \tilde{X}_m(r+1) \right.$$

$$\left. + \sum_{r=-\infty}^{-1} [(k_0 + k - r)^{H-1/2} - (k_0 - r)^{H-1/2}] \tilde{X}_m(r+1) \right\}$$

$$= \frac{2^{-2Hm}}{\Gamma(H+\frac{1}{2})} \sum_{u=-\infty}^{k-1} [(k-u)^{H-1/2} - (-u)_+^{H-1/2}] \tilde{X}_m(k_0 + u + 1) \overset{\mathrm{d}}{=} B_m^{(H)}(t_k).$$

(b) $B_m^{(H)}$ is approximately *self-similar* in the following sense: If $a = 2^{2m_0}$, where m_0 is an integer, $m_0 \geqslant -m$, then for any k non-negative integer for which ka is also an integer one has that

$$a^{-H} B_m^{(H)}(ak2^{-2m}) = \frac{a^{-H} 2^{-2Hm}}{\Gamma(H+\frac{1}{2})} \sum_{r=-\infty}^{ak-1} [(ak - r)^{H-1/2} - (-r)_+^{H-1/2}] \tilde{X}_m(r+1)$$

$$= \frac{2^{-2H(m+m_0)}}{\Gamma(H+\frac{1}{2})} \sum_{r=-\infty}^{k2^{2m_0}-1} [(k2^{2m_0} - r)^{H-1/2} - (-r)_+^{H-1/2}] \tilde{X}_m(r+1)$$

$$\overset{\mathrm{d}}{=} B_{m+m_0}^{(H)}(k2^{-2m}).$$

On the other hand, Lemma 4 (and Theorem 2) below show that $B_m^{(H)}$ and $B_{m+1}^{(H)}$ (and $B_{m+n}^{(H)}$) are uniformly close with arbitrary large probability on any compact interval if m is large enough (when $H > \frac{1}{4}$). It could be proved in a similar fashion that for $a = j$, where $j \geqslant 0$ is an arbitrary integer, $2^{2n} \leqslant j \leqslant 2^{2(n+1)}$ with an integer $n \geqslant 0$, the finite dimensional distributions of

$$a^{-H} B_m^{(H)}(ak2^{-2m}) = \frac{2^{-H(2m+\log_2 j)}}{\Gamma(H+\frac{1}{2})} \sum_{r=-\infty}^{jk-1} [(jk - r)^{H-1/2} - (-r)_+^{H-1/2}] \tilde{X}_m(r+1)$$

can be made arbitrarily close to the finite dimensional distributions of $B_{m+n}^{(H)}$ if m is large enough. Consequently, $B_m^{(H)}$ is arbitrarily close to self-similar for any dyadic $a = j2^{2m_0}$ if m is large enough.

(c) For any $0 < t_1 < \cdots < t_n$, the limit distribution of the vector

$$(B_m^{(H)}(t_1^{(m)}), B_m^{(H)}(t_2^{(m)}), \ldots, B_m^{(H)}(t_n^{(m)}))$$

as $m \to \infty$ is *Gaussian*, where $t_j^{(m)} = \lfloor t_j 2^{2m} \rfloor 2^{-2m}$, $1 \leqslant j \leqslant n$. This fact follows from Theorem 2 (based on Lemma 5) below that states that the process $B_m^{(H)}$ almost surely converges to the Gaussian process $W^{(H)}$ on compact intervals.

4. Convergence of the approximation to fBM

At first it will be shown that two consecutive approximations of fBM defined by (8), or equivalently by (9), are uniformly close if m is large enough, supposing $H > \frac{1}{4}$. Apparently, the above RW approximation of BM is not good enough to have convergence for $H \leqslant \frac{1}{4}$.

When proving convergence, a large deviation inequality similar to Lemma 1 will play an important role. If $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables, $\boldsymbol{P}\{X_k = \pm 1\} = \frac{1}{2}$, and $S = \sum_r a_r X_r$, where not all $a_r \in \mathbb{R}$ are zero and $\mathbf{Var}(S) = \sum_r a_r^2 < \infty$, then

$$\boldsymbol{P}\{|S| \geqslant x(\mathbf{Var}(S))^{1/2}\} \leqslant 2\mathrm{e}^{-x^2/2} \quad (x \geqslant 0), \tag{11}$$

(see, e.g. Stroock, 1993, p. 33). The summation above may extend either to finitely many or to countably many terms.

As a corollary, if $S_1, S_2, \ldots, S_N$ are arbitrary sums of the above type, one can get the following analog of Lemma 1. For any $C > 1$ and $N \geqslant 1$,

$$\boldsymbol{P}\left\{\max_{1 \leqslant k \leqslant N} |S_k| \geqslant (2C\log N)^{1/2} \max_{1 \leqslant k \leqslant N} (\mathbf{Var}(S_k))^{1/2}\right\}$$

$$\leqslant \sum_{k=1}^{N} \boldsymbol{P}\left\{|S_k| \geqslant (2C\log N\,\mathbf{Var}(S_k))^{1/2}\right\} \leqslant 2N\mathrm{e}^{-C\log N} = 2N^{1-C}. \tag{12}$$

Lemma 4. *For any $H \in (\frac{1}{4}, 1)$, $K > 0$, $C \geqslant 3$, and $m \geqslant m_3(C)$, we have*

$$\boldsymbol{P}\left\{\max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)| \geqslant \alpha(H,K) m 2^{-\beta(H)m}\right\} \leqslant 8(K2^{2m})^{1-C},$$

where $t_k = k2^{-2m}$ for $k \geqslant 0$ integers, $\beta(H) = \min(2H - \frac{1}{2}, \frac{1}{2})$ and

$$\alpha(H,K) = \frac{(\log_* K)^{1/2}}{\Gamma(H+\frac{1}{2})}\left[\frac{|H-\frac{1}{2}|}{(1-H)^{1/2}} + (\log_* K)^{1/4}(8K^{1/4} + 36|H-\tfrac{1}{2}|K^{H-1/4})\right]$$

if $H \in (\frac{1}{4}, \frac{1}{2})$,

$$\alpha(H,K) = \frac{(\log_* K)^{1/2}}{\Gamma(H+\frac{1}{2})}\left[\frac{|H-\frac{1}{2}|}{(1-H)^{1/2}} + (\log_* K)^{1/4}(5 + 312|H-\tfrac{1}{2}|)K^{H-1/4}\right]$$

if $H \in (\frac{1}{2}, 1)$. (The case $H = \frac{1}{2}$ is described by Lemma 3.)

Proof. The proof is long, but elementary. Introduce the following abbreviations: $\Delta B_m(t) = B_m(t + \Delta t) - B_m(t)$, $\Delta B_{m+1}(t) = B_{m+1}(t + \frac{1}{4}\Delta t) - B_{m+1}(t)$. Using (8) and then substituting $u = 4r + j$, one gets that

$$B_{m+1}^{(H)}(t_k) = B_{m+1}^{(H)}(4k2^{-2(m+1)})$$

$$= \frac{2^{-2H(m+1)}}{\Gamma(H+\frac{1}{2})} \sum_{u=-\infty}^{4k-1} [(4k-u)^{H-1/2} - (-u)_+^{H-1/2}]\tilde{X}_{m+1}(u+1)$$

$$= \frac{2^{-2Hm-1}}{\Gamma(H+\frac{1}{2})} \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} \left[\left(k - r - \frac{j}{4}\right)^{H-1/2}\right.$$

$$- \left(-r - \frac{j}{4}\right)_+^{H-1/2}\Bigg] \tilde{X}_{m+1}(4r+j+1)$$

$$= \frac{1}{\Gamma(H+\frac{1}{2})} \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} [(t_k - t_{r+j/4})^{H-1/2} - (-t_{r+j/4})_+^{H-1/2}] \Delta B_{m+1}(t_{r+j/4}).$$

So, subtracting and adding a suitable "intermediate" term, one arrives at

$$\Gamma\left(H+\frac{1}{2}\right)[B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)]$$

$$= \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} [(t_k - t_{r+j/4})^{H-1/2} - (-t_{r+j/4})_+^{H-1/2}] \Delta B_{m+1}(t_{r+j/4})$$

$$- [(t_k - t_r)^{H-1/2} - (-t_r)_+^{H-1/2}] \tfrac{1}{4} \Delta B_m(t_r)$$

$$= \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} \{[(t_k - t_{r+j/4})^{H-1/2} - (-t_{r+j/4})_+^{H-1/2}]$$

$$- [(t_k - t_r)^{H-1/2} - (-t_r)_+^{H-1/2}]\} \Delta B_{m+1}(t_{r+j/4})$$

$$+ \sum_{r=-\infty}^{k-1} \sum_{j=0}^{3} [(t_k - t_r)^{H-1/2} - (-t_r)_+^{H-1/2}][\Delta B_{m+1}(t_{r+j/4}) - \tfrac{1}{4}\Delta B_m(t_r)]$$

$$=: (Z_{m,k} + Y_{m,k} + V_{m,k} + U_{m,k}). \tag{13}$$

Here we introduced the following notations:

$$Z_{m,k} = \sum_{r=0}^{k-1} \sum_{j=0}^{3} [(t_k - t_{r+j/4})^{H-1/2} - (t_k - t_r)^{H-1/2}] \Delta B_{m+1}(t_{r+j/4})$$

$$= 2^{-2Hm-1} \sum_{r=0}^{k-1} \sum_{j=0}^{3} [(k - r - j/4)^{H-1/2} - (k-r)^{H-1/2}] \tilde{X}_{m+1}(4r+j+1) \tag{14}$$

and

$$Y_{m,k} = \sum_{r=0}^{k-1} (t_k - t_r)^{H-1/2} \sum_{j=0}^{3} [\Delta B_{m+1}(t_{r+j/4}) - \tfrac{1}{4} \Delta B_m(t_r)]$$

$$= \sum_{r=0}^{k-1} (t_k - t_r)^{H-1/2} \{[B_{m+1}(t_{r+1}) - B_{m+1}(t_r)] - [B_m(t_{r+1}) - B_m(t_r)]\}$$

$$= \sum_{r=0}^{k} [(t_k - t_{r-1})^{H-1/2} - (t_k - t_r)^{H-1/2}][B_{m+1}(t_r) - B_m(t_r)], \tag{15}$$

applying "summation by parts" in the last row, as in (9). Similarly, we introduced the following notations for the corresponding "infinite parts" in (13) (using $v=-r$):

$$V_{m,k} = \sum_{v=1}^{\infty}\sum_{j=0}^{3}[(k+v-j/4)^{H-1/2}-(v-j/4)^{H-1/2}-(k+v)^{H-1/2}+v^{H-1/2}] \times 2^{-2Hm-1}\tilde{X}_{m+1}(-4v+j+1), \tag{16}$$

and

$$\begin{aligned} U_{m,k} &= \sum_{v=1}^{\infty}[(t_k+t_v)^{H-1/2}-t_v^{H-1/2}]\sum_{j=0}^{3}[\Delta B_{m+1}(-t_{v-j/4})-\tfrac{1}{4}\Delta B_m(-t_v)] \\ &= \sum_{v=1}^{\infty}[(t_k+t_v)^{H-1/2}-t_v^{H-1/2}] \\ &\quad \times\{[B_{m+1}(-t_{v-1})-B_{m+1}(-t_v)]-[B_m(-t_{v-1})-B_m(-t_v)]\} \\ &= \sum_{v=1}^{\infty}[(t_k+t_{v+1})^{H-1/2}-t_{v+1}^{H-1/2}-(t_k+t_v)^{H-1/2}+t_v^{H-1/2}] \\ &\quad \times[B_{m+1}(-t_v)-B_m(-t_v)]. \end{aligned} \tag{17}$$

The maxima of $Z_{m,k}, Y_{m,k}, V_{m,k}$ and $U_{m,k}$ can be estimated separately:

$$\max_{0\leqslant t_k\leqslant K}|B_{m+1}^{(H)}(t_k)-B_m^{(H)}(t_k)| \leqslant \frac{1}{\Gamma(H+\frac{1}{2})}\left(\max_k|Z_{m,k}|+\max_k|Y_{m,k}|+\max_k|V_{m,k}|+\max_k|U_{m,k}|\right), \tag{18}$$

where each maximum on the right-hand side is taken for $1\leqslant k\leqslant K2^{2m}$ and one can suppose that $K2^{2m}\geqslant 1$, that is, $\Delta t\leqslant K$, since otherwise the maximal difference in (18) is zero.

(a) *The maximum of* $Z_{m,k}$: In the present case the large deviation inequality (11), or rather, its corollary (12) is applied. By (14),

$$\begin{aligned} \mathbf{Var}(Z_{m,k}) &= 2^{-4Hm-2}\sum_{r=0}^{k-1}\sum_{j=0}^{3}[(k-r-j/4)^{H-1/2}-(k-r)^{H-1/2}]^2 \\ &= 2^{-4Hm-2}\sum_{r=0}^{k-1}\sum_{j=0}^{3}(k-r)^{2H-1}\left[\left(1-\frac{j}{4(k-r)}\right)^{H-1/2}-1\right]^2. \end{aligned}$$

The term in brackets can be estimated using a binomial series with $0\leqslant j\leqslant 3$, $k-r\geqslant 1$:

$$\begin{aligned} \left|\left(1-\frac{j}{4(k-r)}\right)^{H-1/2}-1\right| &= \left|\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}(-1)^s\left(\frac{j}{4(k-r)}\right)^s\right| \\ &\leqslant \left|H-\frac{1}{2}\right|\frac{j}{4(k-r)}\left(1-\frac{j}{4(k-r)}\right)^{-1} \leqslant \left|H-\frac{1}{2}\right|\frac{j}{4(k-r)}\left(1-\frac{j}{4}\right)^{-1}. \end{aligned}$$

Thus

$$\sum_{j=0}^{3}\left[\left(1-\frac{j}{4(k-r)}\right)^{H-1/2}-1\right]^2 \leqslant \left(H-\frac{1}{2}\right)^2\frac{91}{9}\frac{1}{(k-r)^2}.$$

Also, if $k\geqslant 1$, one has

$$\sum_{r=0}^{k-1}(k-r)^{2H-3}<1+\int_0^{k-1}(k-x)^{2H-3}\,\mathrm{d}x\leqslant\frac{3}{2}\frac{1}{1-H}.$$

Then for any $k\geqslant 1$ it follows that

$$\mathbf{Var}(Z_{m,k})\leqslant 2^{-4Hm}\left(H-\frac{1}{2}\right)^2\frac{273}{72}\frac{1}{1-H}.$$

Hence taking $N=K2^{2m}$ and $C>1$ in (12), one obtains that

$$\boldsymbol{P}\left\{\max_{1\leqslant k\leqslant N}|Z_{m,k}|\geqslant\left(\frac{273}{72}\right)^{1/2}|H-\tfrac{1}{2}|(1-H)^{-1/2}2^{-2Hm}(2C\log N)^{1/2}\right\}$$

$$\leqslant\sum_{k=1}^{N}\boldsymbol{P}\{|Z_{m,k}|\geqslant(2C\log N\,\mathbf{Var}(Z_{m,k}))^{1/2}\}\leqslant 2N^{1-C}.$$

Since

$$\log N=\log(K2^{2m})\leqslant(1+\log 4)m\log_* K\leqslant 2.5m\log_* K, \tag{19}$$

one obtains the following result:

$$\max_{1\leqslant k\leqslant K2^{2m}}|Z_{m,k}|\leqslant 5|H-\tfrac{1}{2}|(1-H)^{-1/2}(C\log_* K)^{1/2}m^{1/2}2^{-2Hm}, \tag{20}$$

with the exception of a set of probability at most $2(K2^{2m})^{1-C}$, where $m\geqslant 1$, $K>0$ and $C>1$ are arbitrary.

(b) *The maximum of* $Y_{m,k}$: By its definition (15),

$$\max_{1\leqslant k\leqslant K2^{2m}}|Y_{m,k}|\leqslant\max_{0\leqslant t_r\leqslant K}|B_{m+1}(t_r)-B_m(t_r)|$$

$$\times\max_{0\leqslant t_k\leqslant K}\sum_{0\leqslant t_r\leqslant t_k}|(t_k-t_{r-1})^{H-1/2}-(t_k-t_r)^{H-1/2}|.$$

The first factor, the maximal difference between two consecutive approximations of BM appearing here can be estimated by Lemma 3. For the second factor one can apply a binomial series:

$$\sum_{r=0}^{k}|(t_k-t_{r-1})^{H-1/2}-(t_k-t_r)^{H-1/2}|$$

$$=2^{-m(2H-1)}\left\{1+|2^{H-1/2}-1|+\sum_{r=0}^{k-2}(k-r)^{H-1/2}\left|\left(1+\frac{1}{k-r}\right)^{H-1/2}-1\right|\right\}$$

$$=2^{-m(2H-1)}\left\{1+|2^{H-1/2}-1|+\sum_{r=0}^{k-2}(k-r)^{H-1/2}\left|\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}\frac{1}{(k-r)^s}\right|\right\}$$

$$\leqslant 2^{-m(2H-1)}\left\{1+|2^{H-1/2}-1|+\sum_{r=0}^{k-2}(k-r)^{H-1/2}\frac{|H-\frac{1}{2}|}{k-r}\right\}.$$

Since for $H \neq \frac{1}{2}$

$$\sum_{r=0}^{k-2}(k-r)^{H-3/2}\leqslant\int_0^{k-1}(k-x)^{H-3/2}\,\mathrm{d}x=\frac{1-k^{H-1/2}}{\frac{1}{2}-H},$$

it follows for any $m\geqslant 0$ that

$$\max_{1\leqslant k\leqslant K2^{2m}}\sum_{r=0}^{k}|(t_k-t_{r-1})^{H-1/2}-(t_k-t_r)^{H-1/2}|$$

$$\leqslant 2^{-m(2H-1)}\max_{1\leqslant k\leqslant K2^{2m}}\{1+|2^{H-1/2}-1|+|1-k^{H-1/2}|\}$$

$$\leqslant\begin{cases}2^{-2m(H-1/2)}(3-2^{H-1/2})\leqslant 3\cdot 2^{-2m(H-1/2)} & \text{if } 0<H<\frac{1}{2},\\ 2^{-2m(H-1/2)}(2^{H-1/2}-1)+K^{H-1/2}\leqslant(2K)^{H-1/2} & \text{if } \frac{1}{2}<H<1.\end{cases}$$

(In the last row we used that here $2^{-2m}\leqslant K$.)

Combining this with Lemma 3, we obtain the result

$$\max_{1\leqslant k\leqslant K2^{2m}}|Y_{m,k}|\leqslant\begin{cases}3K^{1/4}(\log_* K)^{3/4}m2^{-2m(H-1/4)} & \text{if } \frac{1}{4}<H<\frac{1}{2},\\ 2^{H-1/2}K^{H-1/4}(\log_* K)^{3/4}m2^{-m/2} & \text{if } \frac{1}{2}<H<1.\end{cases}\tag{21}$$

with the exception of a set of probability at most $3(K2^{2m})^{1-C}$, where $K>0$, $C>1$ are arbitrary, and $m\geqslant m_1(C)$. Thus in the case $0<H<\frac{1}{2}$ we have only a partial result: the relative weakness of the above-described RW approximation of BM causes that apparently we have no convergence for $0<H\leqslant\frac{1}{4}$.

(c) *The maximum of* $V_{m,k}$: Here one can use the same idea as in part (a), including the application of the corollary (12) of the large deviation principle. We begin with (16),

$$\mathbf{Var}(V_{m,k})=2^{-4Hm-2}\sum_{v=1}^{\infty}\sum_{j=0}^{3}\left[\left(k+v-\frac{j}{4}\right)^{H-1/2}-(k+v)^{H-1/2}\right.$$

$$\left.-\left(v-\frac{j}{4}\right)^{H-1/2}+v^{H-1/2}\right]^2$$

$$=2^{-4Hm-2}\sum_{v=1}^{\infty}\sum_{j=0}^{3}\left\{(k+v)^{H-1/2}\left[\left(1-\frac{j}{4(k+v)}\right)^{H-1/2}-1\right]\right.$$

$$\left.-v^{H-1/2}\left[\left(1-\frac{j}{4v}\right)^{H-1/2}-1\right]\right\}^2.$$

As in (a), now we use binomial series for the expressions in brackets ($k \geqslant 1$, $0 \leqslant j \leqslant 3$, $v \geqslant 1$):

$$A = (k+v)^{H-1/2}\left[\left(1-\frac{j}{4(k+v)}\right)^{H-1/2} - 1\right]$$

$$= (k+v)^{H-1/2}\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}(-1)^s\left(\frac{j}{4(k+v)}\right)^s$$

and

$$B = v^{H-1/2}\left[\left(1-\frac{j}{4v}\right)^{H-1/2} - 1\right] = v^{H-1/2}\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}(-1)^s\left(\frac{j}{4v}\right)^s.$$

Then A and B have the same sign and $0 \leqslant A_1 \leqslant |A| \leqslant |B| \leqslant B_2$, where

$$A_1 = (k+v)^{H-1/2}\left|H-\frac{1}{2}\right|\frac{j}{4(k+v)} = \left|H-\frac{1}{2}\right|\frac{j}{4}(k+v)^{H-3/2}$$

and

$$B_2 = v^{H-1/2}\left|H-\frac{1}{2}\right|\frac{j}{4v}\left(1-\frac{j}{4v}\right)^{-1} \leqslant \left|H-\frac{1}{2}\right|\frac{j}{4-j}\,v^{H-3/2}.$$

Hence

$$(A-B)^2 \leqslant A_1^2 + B_2^2 \leqslant \left(H-\frac{1}{2}\right)^2\left[\left(\frac{j}{4}\right)^2(k+v)^{2H-3} + \left(\frac{j}{4-j}\right)^2 v^{2H-3}\right].$$

Since for any $k \geqslant 0$,

$$\sum_{v=1}^{\infty}(k+v)^{2H-3} < 1 + \int_1^{\infty}(k+x)^{2H-3}\,\mathrm{d}x = 1 + \frac{(k+1)^{2H-2}}{2-2H} < \frac{3}{2}\frac{1}{1-H}$$

it follows that

$$\mathbf{Var}(V_{m,k}) \leqslant 2^{-4Hm-2}\left(H-\frac{1}{2}\right)^2\sum_{v=1}^{\infty}\sum_{j=0}^{3}\left[\left(\frac{j}{4}\right)^2(k+v)^{2H-3} + \left(\frac{j}{4-j}\right)^2 v^{2H-3}\right]$$

$$\leqslant \frac{791}{192}\frac{(H-\frac{1}{2})^2}{1-H}2^{-4Hm}.$$

Applying corollary (12) of the large deviation inequality with $N = K2^{2m}$ one obtains that

$$\boldsymbol{P}\left\{\max_{1\leqslant k\leqslant N}|V_{m,k}| \geqslant \left(\frac{791}{192}\right)^{1/2}|H-\tfrac{1}{2}|(1-H)^{-1/2}2^{-2Hm}(2C\log N)^{1/2}\right\}$$

$$\leqslant \sum_{k=1}^{N}\boldsymbol{P}\{|V_{m,k}| \geqslant (2C\log N\,\mathbf{Var}(V_{m,k}))^{1/2}\} \leqslant 2N^{1-C}.$$

Hence using (19) one gets the result

$$\max_{1\leqslant k\leqslant K2^{2m}}|V_{m,k}| \leqslant 5|H-\tfrac{1}{2}|(1-H)^{-1/2}(C\log_* K)^{1/2}m^{1/2}2^{-2Hm}, \tag{22}$$

with the exception of a set of probability at most $2(K2^{2m})^{1-C}$, where $m \geqslant 1$, $K > 0$ and $C > 1$ are arbitrary.

(d) *The maximum of* $U_{m,k}$: We divide the half line into intervals of length L, where $L \geqslant 4K$. For definiteness, choose $L = 4K$. Apart from this, this part will be similar to part (b). In the sequel we use the convention that when the lower limit of a summation is a real number x, the summation starts at $\lceil x \rceil$, and similarly, if the upper limit is y, the summation ends at $\lfloor y \rfloor$. By (17),

$$\begin{aligned} |U_{m,k}| &\leqslant \sum_{j=1}^{\infty} \sum_{(j-1)L < t_v \leqslant jL} |(t_k + t_{v+1})^{H-1/2} - (t_k + t_v)^{H-1/2} - t_{v+1}^{H-1/2} + t_v^{H-1/2}| \\ &\quad \times |B_{m+1}(-t_v) - B_m(-t_v)| \\ &\leqslant \sum_{j=1}^{\infty} \max_{(j-1)L < t_v \leqslant jL} |B_{m+1}(-t_v) - B_m(-t_v)| (\Delta t)^{H-1/2} \\ &\quad \times \sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} |(k+v+1)^{H-1/2} - (k+v)^{H-1/2} - (v+1)^{H-1/2} + v^{H-1/2}|. \end{aligned} \tag{23}$$

Lemma 3 gives an upper bound for the maximal difference between two consecutive approximations of BM if $j \geqslant 1$ is an arbitrary fixed value:

$$\begin{aligned} &\max_{(j-1)L < t_v \leqslant jL} |B_{m+1}(-t_v) - B_m(-t_v)| \\ &\quad \leqslant (jL)^{1/4} (\log_*(jL))^{3/4} m 2^{-m/2} \\ &\quad \leqslant \begin{cases} L^{1/4} (\log_* L)^{3/4} m 2^{-m/2} & \text{if } j = 1, \\ 2 j^{1/4} (\log_* j)^{3/4} L^{1/4} (\log_* L)^{3/4} m 2^{-m/2} & \text{if } j \geqslant 2, \end{cases} \end{aligned} \tag{24}$$

with the exception of a set of probability at most $3(jL2^{2m})^{1-C}$, where $C > 1$ is arbitrary and $m \geqslant m_1(C)$. This implies for any $C \geqslant 3$ and $m \geqslant m_1(C)$ that the above inequality (24) holds simultaneously for all $j = 1, 2, 3, \ldots$ with the exception of a set of probability at most

$$3(L2^{2m})^{1-C} \sum_{j=1}^{\infty} j^{1-C} < 3(L2^{2m})^{1-C} \frac{\pi^2}{6} < (K2^{2m})^{1-C}. \tag{25}$$

For the other major factor in (23) binomial series are applied as above, with $m \geqslant 0$, $k \geqslant 1$, and $v \geqslant 1$:

$$\begin{aligned} A &= (k+v+1)^{H-1/2} - (k+v)^{H-1/2} = (k+v)^{H-1/2} \left[\left(1 + \frac{1}{k+v}\right)^{H-1/2} - 1 \right] \\ &= (k+v)^{H-1/2} \sum_{s=1}^{\infty} \binom{H-\frac{1}{2}}{s} \frac{1}{(k+v)^s}, \end{aligned}$$

and for $v \geqslant 2$:

$$B = (v+1)^{H-1/2} - v^{H-1/2} = v^{H-1/2}\left[\left(1+\frac{1}{v}\right)^{H-1/2} - 1\right]$$

$$= v^{H-1/2}\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}\frac{1}{v^s},$$

while $B = 2^{H-1/2} - 1$ when $v = 1$. Then A and B have the same sign, $0 \leqslant A_1 \leqslant |A| \leqslant |B| \leqslant B_2$, and so $|A - B| \leqslant B_2 - A_1$, where

$$A_1 = (k+v)^{H-3/2}\left|H-\frac{1}{2}\right| - (k+v)^{H-5/2}\tfrac{1}{2}\left|H-\frac{1}{2}\right|(\tfrac{3}{2} - H)$$

and $B_2 = v^{H-3/2}|H - \frac{1}{2}|$.

Thus if the second major factor in (23) is denoted by $C_{m,k,j}$, we obtain for any $j \geqslant 1$ that

$$C_{m,k,j} = (\Delta t)^{H-1/2}\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} |(k+v+1)^{H-1/2} - (k+v)^{H-1/2}$$

$$- (v+1)^{H-1/2} + v^{H-1/2}|$$

$$\leqslant \left|H-\frac{1}{2}\right|(\Delta t)^{H-1/2}\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} v^{H-3/2} - (k+v)^{H-3/2}$$

$$+ \tfrac{1}{2}(\tfrac{3}{2} - H)(k+v)^{H-5/2}.$$

For $H \neq \frac{1}{2}$ one can get the estimates for $j = 1$:

$$\sum_{v=1}^{L2^{2m}} v^{H-3/2} < 1 + \int_1^{L2^{2m}} x^{H-3/2}\,\mathrm{d}x = \frac{(\Delta t)^{1/2-H}}{H-\frac{1}{2}}L^{H-1/2} + \frac{H-\frac{3}{2}}{H-\frac{1}{2}},$$

and for $j \geqslant 2$:

$$\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} v^{H-3/2} < \int_{(j-1)L2^{2m}}^{jL2^{2m}} x^{H-3/2}\,\mathrm{d}x$$

$$= \frac{(\Delta t)^{1/2-H}}{H-\frac{1}{2}}[(jL)^{H-1/2} - ((j-1)L)^{H-1/2}],$$

further, for any $j \geqslant 1$,

$$\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} (k+v)^{H-3/2} > \int_{(j-1)L2^{2m}+1}^{jL2^{2m}+1} (k+x)^{H-3/2}\,\mathrm{d}x$$

$$= \frac{(\Delta t)^{1/2-H}}{H-\frac{1}{2}}[(t_{k+1} + jL)^{H-1/2} - (t_{k+1} + (j-1)L)^{H-1/2}],$$

and also for any $j \geqslant 1$,

$$\sum_{v=(j-1)L2^{2m}+1}^{jL2^{2m}} (k+v)^{H-5/2} < \int_{(j-1)L2^{2m}}^{jL2^{2m}} (k+x)^{H-5/2}\,\mathrm{d}x$$

$$= \frac{(\Delta t)^{3/2-H}}{\frac{3}{2}-H}[(t_k+(j-1)L)^{H-3/2}-(t_k+jL)^{H-3/2}].$$

Denote the sign of a real number x by ε_x (0 if $x=0$). When $j=1$, it follows that

$$C_{m,k,1} \leqslant \varepsilon_{H-1/2}\left\{L^{H-1/2}\left[1-\left(1+\frac{t_{k+1}}{L}\right)^{H-1/2}+\left(1+\frac{t_{k+1}-L}{L}\right)^{H-1/2}\right]\right.$$

$$\left.+(H-\tfrac{3}{2})(\Delta t)^{H-1/2}\right\}+\left|H-\frac{1}{2}\right|\frac{\Delta t}{2}L^{H-3/2}$$

$$\times\left[\left(1+\frac{t_k-L}{L}\right)^{H-3/2}-\left(1+\frac{t_k}{L}\right)^{H-3/2}\right],$$

and similarly, when $j \geqslant 2$,

$$C_{m,k,j} \leqslant \varepsilon_{H-1/2}[(jL)^{H-1/2}-((j-1)L)^{H-1/2}-(t_{k+1}+jL)^{H-1/2}$$

$$+(t_{k+1}+(j-1)L)^{H-1/2}]+\left|H-\frac{1}{2}\right|\frac{\Delta t}{2}[(t_k+(j-1)L)^{H-3/2}-(t_k+jL)^{H-3/2}]$$

$$=\varepsilon_{H-1/2}(jL)^{H-1/2}\left[1-\left(1-\frac{1}{j}\right)^{H-1/2}-\left(1+\frac{t_{k+1}}{jL}\right)^{H-1/2}\right.$$

$$\left.+\left(1+\frac{t_{k+1}-L}{jL}\right)^{H-1/2}\right]+\left|H-\frac{1}{2}\right|\frac{\Delta t}{2}(jL)^{H-3/2}$$

$$\times\left[\left(1+\frac{t_k-L}{jL}\right)^{H-3/2}-\left(1+\frac{t_k}{jL}\right)^{H-3/2}\right].$$

Applying binomial series here again, first we get when $j \geqslant 2$ that

$$\varepsilon_{H-1/2}\left[1-\left(1-\frac{1}{j}\right)^{H-1/2}\right]$$

$$=\varepsilon_{H-1/2}\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}\frac{(-1)^{s+1}}{j^s}$$

$$\leqslant\left|H-\frac{1}{2}\right|\frac{1}{j}+\frac{1}{2}\left|H-\frac{1}{2}\right|\left(\frac{3}{2}-H\right)\frac{1}{j^2}\left(1-\frac{1}{j}\right)^{-1}$$

$$\leqslant\left|H-\frac{1}{2}\right|\frac{1}{j}+\left|H-\frac{1}{2}\right|\left(\frac{3}{2}-H\right)\frac{1}{j^2},$$

since each term of the series is positive. Furthermore, with any $j \geqslant 1$,

$$\varepsilon_{H-1/2}\left[\left(1-\frac{L-t_{k+1}}{jL}\right)^{H-1/2}-\left(1+\frac{t_{k+1}}{jL}\right)^{H-1/2}\right]$$

$$=\varepsilon_{H-1/2}\sum_{s=1}^{\infty}\binom{H-\frac{1}{2}}{s}\frac{(-1)^s}{(jL)^s}[(L-t_{k+1})^s-(-t_{k+1})^s]$$

$$\leqslant -\left|H-\frac{1}{2}\right|\frac{1}{j},$$

since each term of the series is negative: $L = 4K \geqslant 2t_{k+1}$, and the term in brackets is not larger than $2(L-\Delta t)^s$. Finally,

$$\left(1-\frac{L-t_k}{jL}\right)^{H-3/2}-\left(1+\frac{t_k}{jL}\right)^{H-3/2}$$

$$=\sum_{s=1}^{\infty}\binom{H-\frac{3}{2}}{s}\frac{(-1)^s}{(jL)^s}[(L-t_k)^s-(-t_k)^s]$$

$$\leqslant\left(\frac{3}{2}-H\right)\frac{4}{3j}\left(1-\frac{L-\Delta t}{jL}\right)^{-1}\leqslant\left(\frac{3}{2}-H\right)\frac{4L}{3j\,\Delta t},$$

since each term of the series is positive and the term in brackets is not larger than $\frac{4}{3}(L-\Delta t)^s$. Thus when $j \geqslant 2$ it follows for any $m \geqslant 0$, $k \geqslant 1$ that

$$C_{m,k,j}\leqslant (jL)^{H-1/2}\left|H-\frac{1}{2}\right|\left(\frac{3}{2}-H\right)\frac{1}{j^2}+\left|H-\frac{1}{2}\right|\frac{\Delta t}{2}(jL)^{H-3/2}\left(\frac{3}{2}-H\right)\frac{4L}{3j\,\Delta t}$$

$$\leqslant a_H|H-\tfrac{1}{2}|L^{H-1/2}j^{H-5/2}\quad\text{where } a_H=\begin{cases}\frac{5}{2} & \text{if } 0<H<\frac{1}{2},\\ \frac{5}{3} & \text{if } \frac{1}{2}<H<1.\end{cases}$$

In a similar manner, when $j = 1$ one can get for any $m \geqslant 0$, $k \geqslant 1$ that

$$C_{m,k,1}\leqslant \varepsilon_{H-1/2}L^{H-1/2}-|H-\tfrac{1}{2}|L^{H-1/2}+\varepsilon_{H-1/2}(H-\tfrac{3}{2})(\Delta t)^{H-1/2}$$

$$+\left|H-\frac{1}{2}\right|\frac{\Delta t}{2}L^{H-3/2}\left(\frac{3}{2}-H\right)\frac{4L}{3\Delta t}$$

$$=\varepsilon_{H-1/2}(\tfrac{3}{2}-H)[\tfrac{2}{3}HL^{H-1/2}-(\Delta t)^{H-1/2}]$$

$$\leqslant\begin{cases}\frac{3}{2}(\Delta t)^{H-1/2} & \text{if } 0<H<\frac{1}{2},\\ \frac{3}{8}L^{H-1/2} & \text{if } \frac{1}{2}<H<1.\end{cases}$$

Then combine these results with (24) and (25) in (23). Using

$$\sum_{j=2}^{\infty} j^{1/4}(\log_* j)^{3/4} j^{H-5/2} < \int_1^{\infty} x^{H-9/4}\log_* x\,\mathrm{d}x$$

$$=\int_1^{e} x^{H-9/4}\,\mathrm{d}x+\int_e^{\infty} x^{H-9/4}\log x\,\mathrm{d}x$$

$$= \left(\frac{5}{4} - H\right)^{-1} + \left(\frac{5}{4} - H\right)^{-2} e^{H-5/4}$$

$$< \begin{cases} 2.5 & \text{if } 0 < H < \frac{1}{2}, \\ 16.5 & \text{if } \frac{1}{2} < H < 1, \end{cases} \tag{26}$$

one can get the result of part (d). Consider first the case $\frac{1}{2} < H < 1$:

$$\max_{1 \leqslant k \leqslant K2^{2m}} |U_{m,k}|$$

$$\leqslant L^{1/4}(\log_* L)^{3/4} m 2^{-m/2} \frac{3}{8} L^{H-1/2} + 33 L^{1/4}(\log_* L)^{3/4} m 2^{-m/2} \frac{5}{3} |H - \tfrac{1}{2}| L^{H-1/2}$$

$$\leqslant (3 + 312|H - \tfrac{1}{2}|) K^{H-1/4} (\log_* K)^{3/4} m 2^{-m/2}, \tag{27}$$

for any $C \geqslant 3$ and $m \geqslant m_1(C)$ with the exception of a set of probability at most $(K2^{2m})^{1-C}$. (Recall that $L = 4K$.)

In the second case when $0 < H < \frac{1}{2}$ the above method apparently gives convergence here (just like in part (b)) only when $\frac{1}{4} < H < \frac{1}{2}$:

$$\max_{1 \leqslant k \leqslant K2^{2m}} |U_{m,k}|$$

$$\leqslant L^{1/4}(\log_* L)^{3/4} m 2^{-m/2} \tfrac{3}{2} (\Delta t)^{H-1/2} + 5 L^{1/4}(\log_* L)^{3/4} m 2^{-m/2} \frac{5}{2} |H - \tfrac{1}{2}| L^{H-1/2}$$

$$\leqslant 5 K^{1/4} (\log_* K)^{3/4} m 2^{-2m(H-1/4)}) + 36 |H - \tfrac{1}{2}| K^{H-1/4} (\log_* K)^{3/4} m 2^{-m/2}, \tag{28}$$

for any $C \geqslant 3$ and $m \geqslant m_1(C)$ with the exception of a set of probability at most $(K2^{2m})^{1-C}$.

Now one can combine the results of parts (a)–(d), see (18), (20), (21), (22), (27), (28), to obtain the statement of the lemma. Remember that the rate of convergence in parts (a) and (c) is faster than the one in parts (b) and (d). Particularly, observe that there is a factor m in (b) and (d) which has a counterpart $m^{1/2}$ in (a) and (c). Since in the statement of this lemma we simply replaced the faster converging factors by the slower converging ones, the constant multipliers in (a) and (c) can be ignored if m is large enough. □

It is simple to extend formula (9) of the mth approximation $B_m^{(H)}$ of fBM to real arguments t by linear interpolation, just like in the case of the mth approximation $B_m(t)$ of ordinary BM; see, e.g. in Szabados (1996). So let $m \geqslant 0$ and $k \geqslant 0$ be integers, $\gamma \in [0, 1]$, and define

$$B_m^{(H)}(t_{k+\gamma}) = \gamma B_m^{(H)}(t_{k+1}) + (1-\gamma) B_m^{(H)}(t_k)$$

$$= \frac{1}{\Gamma(H + \frac{1}{2})} \sum_{r=-\infty}^{k} [(t_k - t_{r-1})^{H-1/2} - (t_k - t_r)^{H-1/2}] B_m(t_{r+\gamma})$$

$$+ [(-t_r)_+^{H-1/2} - (-t_{r-1})_+^{H-1/2}] B_m(t_r). \tag{29}$$

Then the resulting *continuous parameter approximations of fBM* $B_m^{(H)}(t)$ $(t \geqslant 0)$ have continuous, piecewise linear sample paths. With this definition we are ready to state a main result of this paper.

Theorem 2. *For any* $H \in (\frac{1}{4}, 1)$, *the sequence* $B_m^{(H)}(t)$ $(t \geqslant 0, m = 0, 1, 2, \ldots)$ *a.s. uniformly converges to a fBM* $W^{(H)}(t)$ $(t \geqslant 0)$ *on any compact interval* $[0, K]$, $K > 0$. *If* $K > 0$, $C \geqslant 3$, *and* $m \geqslant m_4(C)$, *it follows that*

$$\boldsymbol{P}\left\{ \max_{0 \leqslant t \leqslant K} |W^{(H)}(t) - B_m^{(H)}(t)| \geqslant \frac{\alpha(H,K)}{(1 - 2^{-\beta(H)})^2} m 2^{-\beta(H) m} \right\} \leqslant 9(K 2^{2m})^{1-C},$$

where $\alpha(H,K)$ *and* $\beta(H) = \min(2H - \frac{1}{2}, \frac{1}{2})$ *are the same as in Lemma* 4. (*The case* $H = \frac{1}{2}$ *is described by Theorem* 1.)

Proof. At first we consider the maximum of $|B_{m+1}^{(H)}(t) - B_m^{(H)}(t)|$ for real $t \in [0, K]$. Lemma 4 gives an upper bound D_m for their maximal difference at vertices with $t = t_k = k\,\Delta t$:

$$\max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)| \leqslant D_m,$$

except for an event of probability at most $8(K2^{2m})^{1-C}$. Since both $B_{m+1}^{(H)}(t)$ and $B_m^{(H)}(t)$ have piecewise linear sample paths, their maximal difference must occur at vertices of the sample paths. Let M_m denote the maximal increase of $B_m^{(H)}$ between pairs of points t_k, t_{k+1} in $[0, K]$:

$$\max_{0 \leqslant t_k \leqslant K} |B_m^{(H)}(t_{k+1}) - B_m^{(H)}(t_k)| \leqslant M_m,$$

except for an event of probability at most $2(K2^{2m})^{1-C}$, cf. (31) below. A sample path of $B_{m+1}^{(H)}(t)$ makes four steps on any interval $[t_k, t_{k+1}]$. To compute its maximal deviation from D_m it is enough to estimate its change between the midpoint and an endpoint of such an interval, at two steps from both the left and right endpoints:

$$\max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{(H)}(t_{k \pm 1/2}) - B_{m+1}^{(H)}(t_k)| \leqslant 2M_{m+1},$$

except for an event of probability at most $2(K2^{2(m+1)})^{1-C}$. Hence

$$\begin{aligned}
&\max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{(H)}(t_{k+1/2}) - B_m^{(H)}(t_{k+1/2})| \\
&\quad = \max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{(H)}(t_{k+1/2}) - \tfrac{1}{2}(B_m^{(H)}(t_k) + B_m^{(H)}(t_{k+1}))| \\
&\quad \leqslant \max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{(H)}(t_k) - B_m^{(H)}(t_k)| + \max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{(H)}(t_{k \pm 1/2}) - B_{m+1}^{(H)}(t_k)| \\
&\quad \leqslant D_m + 2M_{m+1},
\end{aligned}$$

except for an event of probability at most $(8 + 2^{3-2C})\,(K2^{2m})^{1-C}$. The explanation above shows that at the same time this gives the upper bound we were looking for

$$\max_{0 \leqslant t \leqslant K} |B_{m+1}^{(H)}(t) - B_m^{(H)}(t)| \leqslant D_m + 2M_{m+1}, \tag{30}$$

except for an event of probability at most $(8 + 2^{3-2C})(K2^{2m})^{1-C}$.

Thus we have to find an upper estimate M_m. For that the large deviation inequality (12) will be used. By (8), the increment of $B_m^{(H)}(t)$ on $[t_k, t_{k+1}]$ is

$$A_{m,k} = |B_m^{(H)}(t_{k+1}) - B_m^{(H)}(t_k)|$$

$$= \frac{2^{-2Hm}}{\Gamma(H+\frac{1}{2})} \sum_{r=-\infty}^{k} [(k+1-r)^{H-1/2} - (k-r)^{H-1/2}]\tilde{X}_m(r+1).$$

Then a similar argument can be used as in the proof of Lemma 4, see, e.g. part (a) there:

$$\frac{\Gamma^2(H+\frac{1}{2})}{2^{-4Hm}} \mathbf{Var}(A_{m,k})$$

$$= \sum_{r=-\infty}^{k} [(k+1-r)^{H-1/2} - (k-r)^{H-1/2}]^2$$

$$= 1 + (2^{H-1/2}-1)^2 + \sum_{r=-\infty}^{k-2} (k-r)^{2H-1} \left[\left(1+\frac{1}{k-r}\right)^{H-1/2} - 1\right]^2$$

$$\leqslant 1 + (2^{H-1/2}-1)^2 + \sum_{r=-\infty}^{k-2} (k-r)^{2H-1} \left(H-\frac{1}{2}\right)^2 \frac{1}{(k-r)^2}$$

$$\leqslant 1 + \left(H-\frac{1}{2}\right)^2 + \left(H-\frac{1}{2}\right)^2 \frac{1}{2-2H} \leqslant \tfrac{5}{2}(H-\tfrac{1}{2})^2(1-H)^{-1}.$$

Hence taking $N = K2^{2m}$ and $C>1$ in (12), and using (19) too, one obtains for $m \geqslant 1$ that

$$M_m = \max_{1 \leqslant k \leqslant K2^{2m}} |A_{m,k}|$$

$$\leqslant \frac{5/\sqrt{2}}{\Gamma(H+\frac{1}{2})} |H-\tfrac{1}{2}|(1-H)^{-1/2}(C \log_* K)^{1/2} m^{1/2} 2^{-2Hm}, \tag{31}$$

with the exception of a set of probability at most $2(K2^{2m})^{1-C}$, where $K>0$ and $C>1$ are arbitrary.

Then substituting this and Lemma 4 into (30), it follows that when $K>0$, $C \geqslant 3$, and $m \geqslant m_4(C)$,

$$\max_{0 \leqslant t \leqslant K} |B_{m+1}^{(H)}(t) - B_m^{(H)}(t)| \leqslant \alpha(H,K) m 2^{-\beta(H)m} \tag{32}$$

except for an event of probability at most $8.125(K2^{2m})^{1-C}$ where $\alpha(H,K)$ and $\beta(H)$ are the same as in Lemma 4. Remember that the rate of convergence in (31), just like in parts (a) and (c) of the proof of Lemma 4, is faster than the one in parts (b) and (d) of that proof. Apart from constant multipliers, the result of (31) has the same form as the results of (a) and (c) there. Since in the statement of this theorem we simply replaced the faster converging factors by the slower converging ones, the constant multipliers of (31) can be ignored if m is large enough. This is why the $\alpha(H,K)$ defined by Lemma 4 is suitable here too.

In the second part of the proof we compare $B_m^{(H)}(t)$ to $B_{m+j}^{(H)}(t)$, where $j \geqslant 1$ is an arbitrary integer. If $K > 0$, $C \geqslant 3$, and $m \geqslant m_4(C)$, then (32) implies that

$$\max_{0 \leqslant t \leqslant K} |B_{m+j}^{(H)}(t) - B_m^{(H)}(t)| \leqslant \sum_{k=m}^{m+j-1} \max_{0 \leqslant t \leqslant K} |B_{k+1}^{(H)}(t) - B_k^{(H)}(t)|$$

$$\leqslant \sum_{k=m}^{\infty} \alpha(H,K) k 2^{-\beta(H)k} \leqslant \frac{\alpha(H,K)}{(1-2^{-\beta(H)})^2} m 2^{-\beta(H)m}.$$

Hence one can get that

$$\boldsymbol{P}\left\{ \sup_{j \geqslant 1} \max_{0 \leqslant t \leqslant K} |B_{m+j}^{(H)}(t) - B_m^{(H)}(t)| \geqslant \frac{\alpha(H,K)}{(1-2^{-\beta(H)})^2} m 2^{-\beta(H)m} \right\}$$

$$\leqslant \sum_{k=m}^{\infty} 8.125 (K 2^{2k})^{1-C} \leqslant 9 (K 2^{2m})^{1-C}. \tag{33}$$

By the Borel–Cantelli lemma this implies that with probability 1, the sample paths of $B_m^{(H)}(t)$ converge uniformly to a process $W^{(H)}(t)$ on any compact interval $[0,K]$. Then $W^{(H)}(t)$ has continuous sample paths, and inherits the properties of $B_m^{(H)}(t)$ described in Section 3: it is a centered, self-similar process with stationary increments. As Lemma 5 below implies, the process $(W^{(H)}(t)\colon t \geqslant 0)$ so defined is Gaussian. Therefore, $W^{(H)}(t)$ is an fBM and by (33) the convergence rate of the approximation is the one stated in the theorem. □

The aim of the next lemma to show that integration by parts is essentially valid for (2) representing $W^{(H)}(t)$, resulting in a formula similar to (10). Then it follows that $(W^{(H)}(t)\colon t \geqslant 0)$ can be stochastically arbitrarily well approximated by a linear transform of the Gaussian process $(W(t)\colon t \geqslant 0)$, so it is also Gaussian.

Lemma 5. *Let $W^{(H)}(t)$ be the process whose existence is proved in Theorem 2 above for $H \in (\frac{1}{4}, 1)$, or, by a modified construction, in Theorem 3 below for any $H \in (0,1)$. Then for any $t > 0$ and $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ we have*

$$\boldsymbol{P}\left\{ \left| W^{(H)}(t) - W_\delta^{(H)}(t) - \frac{\delta^{H-1/2} W(t-\delta)}{\Gamma(H+\frac{1}{2})} \right| \geqslant \varepsilon \right\} \leqslant \varepsilon, \tag{34}$$

where

$$W_\delta^{(H)}(t) := -\int_{[-1/\delta, -\delta] \cup [0, t-\delta]} h_s'(s,t)\, W(s)\, \mathrm{d}s, \tag{35}$$

and $h(s,t)$ is defined by (7). ($W_\delta^{(H)}(t)$ is almost surely well-defined pathwise as an integral of a continuous function.)

The lemma shows that as $\delta \to 0+$, $W_\delta^{(H)}(t)$ stochastically converges to $W^{(H)}(t)$ when $H > \frac{1}{2}$, while $W_\delta^{(H)}(t)$ has a singularity given by the extra term in (34) when $H < \frac{1}{2}$. (If $H = \frac{1}{2}$ then $W_\delta^{(H)}(t) = 0$ and the lemma becomes trivial.)

Proof. Fix $t>0$ and $\varepsilon>0$ and take any δ, $0<\delta\leqslant t$. Let us introduce the notation (cf. (9))

$$B_{m,\delta}^{(H)}(t_{(m)})=\sum_{t_r\in I_{m,\delta}}\frac{h(t_r-\Delta t,t_{(m)})-h(t_r,t_{(m)})}{\Delta t}B_m(t_r)\Delta t, \tag{36}$$

where

$$I_{m,\delta}=\left(\left(\frac{-1}{\delta}\right)_{(m)},-\delta_{(m)}\right]\cup(0,t_{(m)}-\delta_{(m)}]$$

and the abbreviation $s_{(m)}=\lfloor s2^{2m}\rfloor 2^{-2m}$ is used for $s=t,\delta$, and $-1/\delta$ (an empty sum being zero by convention). Then we get the inequality

$$\begin{aligned}
&\left|W^{(H)}(t)-W_\delta^{(H)}(t)-\frac{\delta^{H-1/2}W(t-\delta)}{\Gamma(H+\frac{1}{2})}\right|\\
&\quad\leqslant|W^{(H)}(t)-B_m^{(H)}(t_{(m)})|\\
&\qquad+\left|B_m^{(H)}(t_{(m)})-B_{m,\delta}^{(H)}(t_{(m)})-\frac{\delta_{(m)}^{H-1/2}B_m(t_{(m)}-\delta_{(m)}+\Delta t)}{\Gamma(H+\frac{1}{2})}\right|\\
&\qquad+|B_{m,\delta}^{(H)}(t_{(m)})-W_\delta^{(H)}(t)|\\
&\qquad+\left|\frac{\delta_{(m)}^{H-1/2}B_m(t_{(m)}-\delta_{(m)}+\Delta t)}{\Gamma(H+\frac{1}{2})}-\frac{\delta^{H-1/2}W(t-\delta)}{\Gamma(H+\frac{1}{2})}\right|.
\end{aligned}\tag{37}$$

First we have to estimate the second term on the right-hand side as $\delta\to 0+$, uniformly in m (this requires the longest computation):

$$B_m^{(H)}(t_{(m)})-B_{m,\delta}^{(H)}(t_{(m)})-\frac{\delta_{(m)}^{H-1/2}B_m(t_{(m)}-\delta_{(m)}+\Delta t)}{\Gamma(H+\frac{1}{2})}=:E_{m,\delta}+F_{m,\delta}+G_{m,\delta},$$

where

$$\begin{aligned}
E_{m,\delta}=&\sum_{t_{(m)}-\delta_{(m)}<t_r\leqslant t_{(m)}}\frac{h(t_r-\Delta t,t_{(m)})-h(t_r,t_{(m)})}{\Delta t}B_m(t_r)\Delta t\\
&-\frac{\delta_{(m)}^{H-1/2}B_m(t_{(m)}-\delta_{(m)}+\Delta t)}{\Gamma(H+\frac{1}{2})},
\end{aligned}$$

$$F_{m,\delta}=\sum_{-\delta_{(m)}<t_r\leqslant 0}\frac{h(t_r-\Delta t,t_{(m)})-h(t_r,t_{(m)})}{\Delta t}B_m(t_r)\Delta t$$

and

$$G_{m,\delta}=\sum_{-\infty<t_r\leqslant(-1/\delta)_{(m)}}\frac{h(t_r-\Delta t,t_{(m)})-h(t_r,t_{(m)})}{\Delta t}B_m(t_r)\Delta t.$$

Then "summation by parts" shows that

$$E_{m,\delta}=\sum_{t_{(m)}-\delta_{(m)}<t_r<t_{(m)}}h(t_r,t_{(m)})[B_m(t_{r+1})-B_m(t_r)].$$

(This is the point where the extra term in the definition of $E_{m,\delta}$ is needed.) Thus

$$\begin{aligned}\mathbf{Var}(\Gamma(H+\tfrac{1}{2})E_{m,\delta}) &= \sum_{t_{(m)}-\delta_{(m)}<t_r<t_{(m)}} (t_{(m)}-t_r)^{2H-1}2^{-2m} \\ &= t_{(m)}^{2H-1} \sum_{t_{(m)}-\delta_{(m)}<t_r<t_{(m)}} \left(1-\frac{t_r}{t_{(m)}}\right)^{2H-1} \Delta t \\ &\leqslant t_{(m)}^{2H-1} \int_{t_{(m)}-\delta_{(m)}}^{t_{(m)}} \left(1-\frac{u}{t_{(m)}}\right) \mathrm{d}u \\ &= \frac{\delta_{(m)}^{2H}}{2H} \leqslant \frac{\delta^{2H}}{2H},\end{aligned}$$

for any $m \geqslant 0$. Then by the large deviation inequality (11), for any $m \geqslant 0$ and for any $C>0$,

$$\boldsymbol{P}\left\{ |E_{m,\delta}| \geqslant (2C\log_*(1/\delta))^{1/2} \frac{\delta^H}{\Gamma(H+\frac{1}{2})(2H)^{1/2}} \right\} \leqslant 2\delta^C. \tag{38}$$

Similarly as above, the definition of $F_{m,\delta}$ can be rewritten using "summation by parts" that gives

$$F_{m,\delta} = \sum_{-\delta_{(m)}<t_r<0} h(t_r,t_{(m)})[B_m(t_{r+1})-B_m(t_r)] + h(-\delta_{(m)},t_{(m)})B_m(-\delta_{(m)}+\Delta t).$$

The definition of $F_{m,\delta}$ shows that it is equal to zero whenever $\delta<\Delta t$. Therefore when giving an upper bound for its variance it can be assumed that $\delta \geqslant \Delta t$. Thus

$$\begin{aligned}&\mathbf{Var}(\Gamma(H+\tfrac{1}{2})F_{m,\delta}) \\ &= \sum_{0<t_v<\delta_{(m)}} [(t_{(m)}+t_v)^{H-1/2} - t_v^{H-1/2}]^2 \Delta t \\ &\quad + [(t_{(m)}+\delta_{(m)})^{H-1/2} - \delta_{(m)}^{H-1/2}]^2(\delta_{(m)}-\Delta t) \\ &\leqslant t_{(m)}^{2H-1} \sum_{0<t_v<\delta_{(m)}} \left[\left(1+\frac{t_v}{t_{(m)}}\right)^{2H-1} + \left(\frac{t_v}{t_{(m)}}\right)^{2H-1}\right] \Delta t \\ &\quad + [(t_{(m)}+\delta_{(m)})^{2H-1} + \delta_{(m)}^{2H-1}]\delta_{(m)} \\ &\leqslant t_{(m)}^{2H-1} \int_0^{\delta_{(m)}} \left[\left(1+\frac{u}{t_{(m)}}\right)^{2H-1} + \left(\frac{u}{t_{(m)}}\right)^{2H-1}\right] \mathrm{d}t + 2t_{(m)}^{2H-1}\delta_{(m)} + \delta_{(m)}^{2H} \\ &= \frac{t_{(m)}^{2H}}{2H}\left[\left(1+\frac{\delta_{(m)}}{t_{(m)}}\right)^{2H} + \left(\frac{\delta_{(m)}}{t_{(m)}}\right)^{2H} - 1\right] + 2t_{(m)}^{2H-1}\delta_{(m)} + \delta_{(m)}^{2H} \\ &\leqslant \frac{3}{2}t_{(m)}^{2H-1}\delta_{(m)} + \frac{\delta_{(m)}^{2H}}{2H} + 2t_{(m)}^{2H-1}\delta_{(m)} + \delta_{(m)}^{2H} \leqslant \frac{7}{2}t^{2H-1}\delta + \frac{3}{2H}\delta^{2H}.\end{aligned}$$

So by the large deviation inequality (11), for any $m \geqslant 0$ and for any $C > 0$,

$$\boldsymbol{P}\left\{|F_{m,\delta}| \geqslant \left(2C\log_*\left(\frac{1}{\delta}\right)\right)^{1/2} \frac{(\frac{7}{2}t^{2H-1}\delta + \frac{3}{2H}\delta^{2H})^{1/2}}{\Gamma(H+\frac{1}{2})}\right\} \leqslant 2\delta^C. \tag{39}$$

Proceeding in a similar way with $G_{m,\delta}$, one obtains that

$$G_{m,\delta} = \sum_{-\infty < t_r < (-1/\delta)_{(m)}} h(t_r, t_{(m)})[B_m(t_{r+1}) - B_m(t_r)]$$

$$- h\left(\left(\frac{-1}{\delta}\right)_{(m)}, t_{(m)}\right) B_m\left(\left(\frac{-1}{\delta}\right)_{(m)}\right).$$

Hence

$$\begin{aligned}
&\mathbf{Var}(\Gamma(H+\tfrac{1}{2})G_{m,\delta}) \\
&= \sum_{-(-1/\delta)_{(m)} < t_v < \infty} [(t_{(m)} + t_v)^{H-1/2} - t_v^{H-1/2}]^2 \Delta t \\
&\quad + \left[\left(t_{(m)} - \left(\frac{-1}{\delta}\right)_{(m)}\right)^{H-1/2} - \left(-\left(\frac{-1}{\delta}\right)_{(m)}\right)^{H-1/2}\right]^2 \left(-\left(\frac{-1}{\delta}\right)_{(m)}\right) \\
&\leqslant \sum_{-(-1/\delta)_{(m)} < t_v < \infty} t_v^{2H-1}\left(H - \frac{1}{2}\right)^2 \left(\frac{t_{(m)}}{t_v}\right)^2 \Delta t \\
&\quad + \left(-\left(\frac{-1}{\delta}\right)_{(m)}\right)^{2H-1} \left(H - \frac{1}{2}\right)^2 \left(\frac{t_{(m)}}{-(-1/\delta)_{(m)}}\right)^2 \left(-\left(\frac{-1}{\delta}\right)_{(m)}\right) \\
&= \left(H - \frac{1}{2}\right)^2 t_{(m)}^2 \left\{\sum_{-(-1/\delta)_{(m)} < t_v < \infty} t_v^{2H-3}\,\Delta t + \left(-\left(\frac{-1}{\delta}\right)_{(m)}\right)^{2H-2}\right\} \\
&\leqslant \left(H - \frac{1}{2}\right)^2 t_{(m)}^2 \left\{\int_{-(-1/\delta)_{(m)}}^{\infty} u^{2H-3}\,\mathrm{d}u + \left(-\left(\frac{-1}{\delta}\right)_{(m)}\right)^{2H-2}\right\} \\
&\leqslant \frac{3(H-\frac{1}{2})^2}{2(1-H)} t^2 \delta^{2-2H}.
\end{aligned}$$

So again by the large deviation inequality (11), for any $m \geqslant 0$ and for any $C > 0$,

$$\boldsymbol{P}\left\{|G_{m,\delta}| \geqslant (2C\log_*(1/\delta))^{1/2} \frac{|H - \frac{1}{2}|}{\Gamma(H+\frac{1}{2})} \left(\frac{\frac{3}{2}}{1-H}\right)^{1/2} t\delta^{1-H}\right\} \leqslant 2\delta^C. \tag{40}$$

Combining (38)–(40), it follows that there exists a $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ and for any $m \geqslant 0$,

$$\boldsymbol{P}\left\{\left|B_m^{(H)}(t_{(m)}) - B_{m,\delta}^{(H)}(t_{(m)}) - \frac{\delta_{(m)}^{H-1/2} B_m(t_{(m)} - \delta_{(m)} + \Delta t)}{\Gamma(H+\frac{1}{2})}\right| \geqslant \frac{\varepsilon}{4}\right\} \leqslant \frac{\varepsilon}{4}.$$

After the second term on the right-hand side of (37) we turn to the third term. Take now any $\delta \in (0, \delta_0)$. Since $h(s,t)$ has continuous partial derivative w.r.t. s on the intervals $[-1/\delta, -\delta]$ and $[\delta, t-\delta]$ and by Theorem 1, B_m a.s. uniformly converges to the Wiener process W on these intervals, comparing (35) and (36) shows that with this δ there exists an m such that

$$\boldsymbol{P}\left\{|B_{m,\delta}^{(H)}(t_{(m)}) - W_{\delta}^{(H)}(t)| \geqslant \frac{\varepsilon}{4}\right\} \leqslant \frac{\varepsilon}{4}.$$

Theorem 1 also implies that m can be chosen so that for the fourth term in (37) one similarly has

$$\boldsymbol{P}\left\{\left|\frac{\delta_{(m)}^{H-1/2} B_m(t_{(m)} - \delta_{(m)} + \Delta t)}{\Gamma(H+\frac{1}{2})} - \frac{\delta^{H-1/2} W(t-\delta)}{\Gamma(H+\frac{1}{2})}\right| \geqslant \frac{\varepsilon}{4}\right\} \leqslant \frac{\varepsilon}{4}.$$

Finally, Theorem 2 (or, with a modified construction, Theorem 3 below) guarantees that m can be chosen so that the first term in (37) satisfies the same inequality:

$$\boldsymbol{P}\left\{|W^{(H)}(t) - B_m^{(H)}(t)| \geqslant \frac{\varepsilon}{4}\right\} \leqslant \frac{\varepsilon}{4}.$$

The last four formulae together prove the lemma. □

5. Improved construction using the KMT approximation

Parts (b) and (d) of the proof of Lemma 4 gave worse rate of convergence than parts (a) and (c), in which the rates can be conjectured to be best possible. The reason for this is clearly the relatively weaker convergence rate of the RW approximation of ordinary BM, that was used in parts (b) and (d), but not in parts (a) and (c). It is also clear from there that using the best possible KMT approximation instead would eliminate this weakness and would give hopefully the best possible rate here too. The price one has to pay for this is the intricate and "future-dependent" procedure by which the KMT method constructs suitable approximating RWs from BM.

The result we need from Komlós et al. (1975, 1976) is as follows. Suppose that one wants to define an i.i.d. sequence $X_1, X_2, \ldots$ of random variables with a given distribution so that the partial sums are as close to BM as possible. Assume that $\boldsymbol{E}(X_k) = 0$, $\mathbf{Var}(X_k) = 1$ and the moment generating function $\mathbf{E}(e^{uX_k}) < \infty$ for $|u| \leqslant u_0$, $u_0 > 0$. Let $S(k) = X_1 + \cdots + X_k$, $k \geqslant 1$ be the partial sums. If BM $W(t)$ $(t \geqslant 0)$ is given, then for any $n \geqslant 1$ there exists a sequence of conditional quantile transformations applied to $W(1), W(2), \ldots, W(n)$ so that one obtains the desired partial sums $S(1), S(2), \ldots, S(n)$ and the difference between the two sequences is the smallest possible:

$$\boldsymbol{P}\left\{\max_{0 \leqslant k \leqslant n} |S(k) - W(k)| > C_0 \log n + x\right\} < K_0 e^{-\lambda x}, \tag{41}$$

for any $x > 0$, where C_0, K_0, λ are positive constants that may depend on the distribution of X_k, but not on n or x. Moreover, λ can be made arbitrarily large by choosing a large enough C_0. Taking $x = C_0 \log n$ here one obtains

$$\boldsymbol{P}\left\{\max_{0 \leqslant k \leqslant n} |S(k) - W(k)| > 2C_0 \log n\right\} < K_0 n^{-\lambda C_0}, \tag{42}$$

where $n \geqslant 1$ is arbitrary.

Fix an integer $m \geqslant 0$, and introduce the same notations as in previous sections: $\Delta t = 2^{-2m}$, $t_x = x\,\Delta t$. Then multiply the inner inequality in (42) by 2^{-m} and use self-similarity (1) of BM (with $H = \frac{1}{2}$) to obtain a shrunken RW $B_m^*(t_k) = 2^{-m} S_m(k)$ $(0 \leqslant k \leqslant K2^{2m})$ from the corresponding dyadic values $W(t_k)$ $(0 \leqslant k \leqslant K2^{2m})$ of BM by a sequence of conditional quantile transformations so that

$$\max_{0 \leqslant t_k \leqslant K} |B_m^*(t_k) - W(t_k)| \leqslant 2C_0 2^{-m} \log_*(K2^{2m}) \leqslant 5C_0 \log_* K\, m2^{-m}, \tag{43}$$

with the exception of a set of probability smaller than $K_0(K2^{2m})^{-\lambda C_0}$, for any $m \geqslant 1$ and $K > 0$. [Here (19) was used too.] Then (43) implies for the difference of two consecutive approximations that

$$\boldsymbol{P}\left\{ \max_{0 \leqslant t_k \leqslant K} |B_{m+1}^*(t_k) - B_m^*(t_k)| > 10C_0 \log_* K\, m2^{-m} \right\} < 2K_0(K2^{2m})^{-\lambda C_0} \tag{44}$$

for any $m \geqslant 1$ and $K > 0$. This is exactly what we need to improve the rates of convergence in parts (b) and (d) of Lemma 4.

Substitute these KMT approximations $B_m^*(t_r)$ into definition (8) or (9) of $B_m^{(H)}(t_k)$. This way one can obtain faster converging approximations $B_m^{*(H)}$ of fBM. Then everything above in Sections 3 and 4 are still valid, except that one can use the improved formula (44) instead of Lemma 3 at parts (b) and (d) in the proof of Lemma 4. This way, instead of (21) one gets

$$\max_{1 \leqslant k \leqslant K2^{2m}} |Y_{m,k}| \leqslant \begin{cases} 23C_0 \log_* K\, m2^{-2Hm} & \text{if } 0 < H < \frac{1}{2}, \\ 15C_0 \log_* K\, K^{H-1/2} m2^{-m} & \text{if } \frac{1}{2} < H < 1, \end{cases} \tag{45}$$

for any $m \geqslant 1$, except for a set of probability smaller than $2K_0(K2^{2m})^{-\lambda C_0}$.

Also by (44), instead of (24) and (25) one has the improved inequalities:

$$\max_{(j-1)L < t_v \leqslant jL} |B_{m+1}^*(-t_v) - B_m^*(-t_v)| \leqslant \begin{cases} 10C_0 \log_* L\, m2^{-m} & \text{if } j = 1, \\ 14C_0 \log_* j \log_* L\, m2^{-m} & \text{if } j \geqslant 2, \end{cases} \tag{46}$$

with the exception of a set of probability smaller than $2K_0(jL2^{2m})^{-\lambda C_0}$, where $m \geqslant 1$. If C_0 is chosen large enough so that $\lambda C_0 \geqslant 2$, then (46) holds simultaneously for all $j = 1, 2, 3, \ldots$ except for a set of probability smaller than

$$2K_0(L2^{2m})^{-\lambda C_0} \sum_{j=1}^{\infty} j^{-\lambda C_0} < \tfrac{1}{4} K_0(K2^{2m})^{-\lambda C_0}. \tag{47}$$

(Remember that we chose $L = 4K$ in part (d) of the proof of Lemma 4.) Then using this in part (d) of Lemma 4, instead of (26) one needs the estimate

$$\sum_{j=2}^{\infty} j^{H-5/2} \log_* j < \int_1^{\infty} x^{H-5/2} \log_* x \, dx < \begin{cases} 1.5 & \text{if } 0 < H < \frac{1}{2}, \\ 4.5 & \text{if } \frac{1}{2} < H < 1. \end{cases}$$

Then instead of (27) and (28), the improved results are as follows. First, in the case $\frac{1}{2} < H < 1$ one has

$$\max_{1 \leqslant k \leqslant K2^{2m}} |U_{m,k}|$$

$$\leqslant 10C_0 \log_* L\, m2^{-m}\frac{3}{8}L^{H-1/2} + 14C_0 \log_* L\, m2^{-m}\frac{5}{3}|H - \tfrac{1}{2}|L^{H-1/2}(4.5)$$

$$\leqslant (18 + 502|H - \tfrac{1}{2}|)C_0 \log_* K\, K^{H-1/2} m2^{-m} \tag{48}$$

for any $m \geqslant 1$ and C_0 large enough so that $\lambda C_0 \geqslant 2$, except for a set of probability smaller than given by (47). Now in the case $0 < H < \frac{1}{2}$ it follows that

$$\max_{1 \leqslant k \leqslant K2^{2m}} |U_{m,k}|$$

$$\leqslant 10C_0 \log_* L\, m2^{-m}\tfrac{3}{2}(\Delta t)^{H-1/2} + 14C_0 \log_* L\, m2^{-m}\frac{5}{2}|H - \tfrac{1}{2}|L^{H-1/2}(1.5)$$

$$\leqslant C_0 \log_* K\, m(36 \cdot 2^{-2Hm} + 126\,|H - \tfrac{1}{2}|K^{H-1/2}2^{-m}) \tag{49}$$

for any $m \geqslant 1$ and C_0 large enough so that $\lambda C_0 \geqslant 2$, except for a set of probability smaller than given by (47).

As a result, there is convergence for any $H \in (0,1)$. Since the KMT approximation itself has best possible rate for approximating ordinary BM by RW, it can be conjectured that the resulting convergence rates in the next lemma and theorem are also best possible (apart from constant multipliers) for approximating fBM by moving averages of a RW.

Lemma 6. *For any* $H \in (0,1)$, $m \geqslant 1$, $K > 0$, $C > 1$, *and* C_0 *large enough, we have*

$$\boldsymbol{P}\left\{\max_{0 \leqslant t_k \leqslant K} |B_{m+1}^{*(H)}(t_k) - B_m^{*(H)}(t_k)| \geqslant \alpha^* m2^{-\beta^*(H)m}\right\}$$

$$\leqslant 4(K2^{2m})^{1-C} + 3K_0(K2^{2m})^{-\lambda C_0},$$

where $t_k = k2^{-2m}$, $\beta^*(H) = \min(2H, 1)$, $\alpha^* = \alpha^*(H, K, C, C_0)$,

$$\alpha^* = \frac{(\log_* K)^{1/2}}{\Gamma(H + \frac{1}{2})}\left[10C^{1/2}\frac{|H - \frac{1}{2}|}{(1-H)^{1/2}} + C_0(\log_* K)^{1/2}(59 + 126|H - \tfrac{1}{2}|K^{H-1/2})\right]$$

if $H \in (0, \frac{1}{2})$,

$$\alpha^* = \frac{(\log_* K)^{1/2}}{\Gamma(H + \frac{1}{2})}\left[10C^{1/2}\frac{|H - \frac{1}{2}|}{(1-H)^{1/2}} + C_0(\log_* K)^{1/2}(33 + 502|H - \tfrac{1}{2}|)K^{H-1/2}\right]$$

if $H \in (\frac{1}{2}, 1)$, *and the constants* λ, C_0 *and* K_0 *are defined by the KMT approximation* (41) *with* C_0 *chosen so large that* $\lambda C_0 \geqslant 2$. [*The case* $H = \frac{1}{2}$ *is described by* (44).]

Proof. Combine the results of parts (a) and (c) in the proof of Lemma 4 and the improved inequalities above, that is, apply (18), (20), (45), (22), (48), and (49). Here too, we simply replace the faster converging factors by the slower converging ones, but the constant multipliers of faster converging terms cannot be ignored, since the lemma is stated for any $m \geqslant 1$. □

Now we can extend the improved approximations of fBM to real arguments by linear interpolation, in the same way as we did with the original approximations, see (29). This way we get continuous parameter approximations $B_m^{*(H)}(t)$, $(t \geqslant 0)$ for $m = 0, 1, 2, \ldots$, with continuous, piecewise linear sample paths. Now we can state the second main result of this paper.

Theorem 3. *For any* $H \in (0,1)$, *the sequence* $B_m^{*(H)}(t)$ $(t \geqslant 0, m = 0,1,2,\ldots)$ *a.s. uniformly converges to a fBM* $W^{(H)}(t)$ $(t \geqslant 0)$ *on any compact interval* $[0,K]$, $K > 0$. *If* $m \geqslant 1$, $K > 0$, $C \geqslant 2$, *and* C_0 *is large enough, it follows that*

$$\boldsymbol{P}\left\{\max_{0 \leqslant t \leqslant K} |W^{(H)}(t) - B_m^{*(H)}(t)| \geqslant \frac{\bar{\alpha}^*}{(1 - 2^{-\beta^*(H)})^2} m 2^{-\beta^*(H)m}\right\}$$

$$\leqslant 6(K2^{2m})^{1-C} + 4K_0(K2^{2m})^{-\lambda C_0}$$

with

$$\bar{\alpha}^* = \alpha^* + \frac{10}{\Gamma(H + \frac{1}{2})} C^{1/2} (\log_* K)^{1/2} |H - \tfrac{1}{2}| (1 - H)^{-1/2},$$

where α^* *and* $\beta^*(H)$ *are the same as in Lemma* 6. (*In other words, in the definition of* α^* *in Lemma* 6 *the constant multiplier* 10 *has to be changed to* 20 *here.*) *The constants* λ, C_0, K_0 *are defined by the KMT approximation* (41) *with* C_0 *chosen so large that* $\lambda C_0 \geqslant 2$. [*The case* $H = \frac{1}{2}$ *is described by* (43).]

Proof. The proof can follow the line of the proof of Theorem 2 with one exception: the constant multipliers in (31) and consequently in (30) cannot be ignored here. This is why the multiplier α^* of Lemma 6 had to be modified in the statement of the theorem. □

It can be conjectured that the best rate of approximation of fBM by moving averages of simple RWs is $O(N^{-H} \log N)$, where N is the number of points considered. Though it seems quite possible that definition of $B_m^{*(H)}(t)$ above, see (8) with the KMT approximations $B_m^*(t_r)$, supplies this rate of convergence for any $H \in (0,1)$, but in Theorem 3 we were able to prove this rate only when $H \in (0, \frac{1}{2})$. A possible explanation could be that in parts (b) and (d) of Lemma 4 we separated the maxima of the kernel and the “integrator” parts.

As a result, the convergence rate we were able to prove when $\frac{1}{2} < H < 1$ is the same $O(N^{-1/2} \log N)$ that the original KMT approximation (43) gives for ordinary BM, where $N = K2^{2m}$, though in this case the sample paths of fBM are smoother than that of BM. (See, e.g. Decreusefond and Üstünel, 1998.) On the other hand, the obtained convergence rate is worse than this, but still thought to be the best possible, $O(N^{-H} \log N)$, when $0 < H < \frac{1}{2}$, which heuristically can be explained by the more zigzagged sample paths of fBM in this case.

References

Carmona, P., Coutin, L., 1998. Fractional Brownian motion and the Markov property. Elect. Comm. Probab. 3, 95–107.

Decreusefond, L., Üstünel, A.S., 1998. Fractional Brownian Motion: Theory and Applications. Systèmes Différentiels Fractionnaires, ESAIM Proceedings 5, Paris, pp. 75–86.

Decreusefond, L., Üstünel, A.S., 1999. Stochastic analysis of the fractional Brownian motion. Potential Anal. 10, 174–214.

Feller, W., 1966. An Introduction to Probability Theory and Its Applications, Vol. II. Wiley, New York.

Knight, F.B., 1961. On the random walk and Brownian motion. Trans. Amer. Math. Soc. 103, 218–228.

Kolmogorov, A.N., 1940. Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. Doklady A.N. S.S.S.R. 26, 115–118.

Komlós, J., Major, P., Tusnády, G., 1975. An approximation of partial sums of independent RV's, and the sample DF. I. Z. Wahrscheinlichkeitstheorie verw. Gebiete 32, 111–131.

Komlós, J., Major, P., Tusnády, G., 1976. An approximation of partial sums of independent RV's, and the sample DF. II. Z. Wahrscheinlichkeitstheorie verw. Gebiete 34, 33–58.

Mandelbrot, B.B., van Ness, J.W., 1968. Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10, 422–437.

Révész, P., 1990. Random Walk in Random and Non-Random Environments. World Scientific, Singapore.

Samko, S.G., Kilbas, A.A., Marichev, O.I., 1993. Fractional Integrals and Derivatives. Gordon & Breach Science, Yverdon.

Skorohod, A.V., 1965. Studies in the Theory of Random Processes. Addison-Wesley, Reading, MA.

Stroock, D.W., 1993. Probability Theory, an Analytic View. Cambridge University Press, Cambridge.

Szabados, T., 1990. A discrete Itô's formula. Coll. Math. Soc. János Bolyai 57. Limit Theorems in Probability and Statistics, Pécs (Hungary) 1989. North-Holland, Amsterdam, pp. 491–502.

Szabados, T., 1996. An elementary introduction to the Wiener process and stochastic integrals. Studia Sci. Math. Hung. 31, 249–297.

Wiener, N., 1921. The average of an analytical functional and the Brownian movement. Proc. Nat. Acad. Sci. U.S.A. 7, 294–298.

Wiener, N., 1923. Differential space. J. Math. Phys. 2, 132–174.