JOURNAL OF ALGEBRA 28, 311–318 (1974)

A Note on Z-Planes

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Communicated by Erwin Kleinfeld

Received March 24, 1972

1. INTRODUCTION

A Z-plane is a finite affine plane π with a collineation group G satisfying:

(1) G acts as a rank 3 collineation group on π;

(2) if $\bar{\pi}$ is the extension of π to a projective plane in the usual way then G permutes the points of the line l_∞ at infinity in two orbits: $\{\mathscr{U}, \mathscr{V}\}$ and $l_\infty - \{\mathscr{U}, \mathscr{V}\}$.

The points $\mathscr{U}, \mathscr{V}$ are called the *special points* of π with respect to G.

Kallaher [6] has shown that a Z-plane is a translation plane and hence it has order p^r for some prime p. In [8] Kallaher and Ostrom showed that if either (i) $r = 3$ or (ii) $r > 4$ and $p^r \neq 2^6$, then π is a generalized André plane. In this article we extend this result by proving:

THEOREM A. *Let π be a Z-plane of order p^r, p a prime. If*

(i) $r = 4$, *or*

(ii) $r = 2$, $p \neq 5, 11, 19, 23, 29, 59$, *or a Mersenne prime, then π is a generalized André plane.*

Now $p^r - 1$, p a prime, has a prime p-primitive divisor except when $p^r = 2^6$ or $r = 2$ and p is a Mersenne prime [2]. Thus, using the result from [8] referred to above, we have the corollary:

COROLLARY B. *Let π be a Z-plane of order p^r, p a prime. If $p^r \neq 5^2, 11^2, 19^2, 23^2, 29^2, 59^2$, and p^r has a prime p-primitive divisor, then π is a generalized André plane.*

Before proving Theorem A we shall give a few examples to show in what

* Supported in part by NSF Grant #27101.

ways Theorem A and Corollary B are best possible. These examples are the planes coordinatized by the irregular nearfields [3, pp. 230–231] of which none are generalized André planes [4].

The irregular nearfields of orders 5^2, 11^2, 23^2, 29^2, 59^2 show that these orders have to be excluded. The irregular nearfield of order 7^2 shows the necessity of an assumption excluding the order 7^2. This is accomplished by assuming $p^r - 1$ has a prime p-primitive divisor. The reason for excluding 19^2 is simply that the proof does not apply to this order. We should also mention that the assumption about a p-primitive divisor is crucial since both in this paper and in [8], Ostrom's characterization of generalized André planes is used.

We assume that the reader is familiar with the theory of projective and affine planes (see [3]). More importantly, we assume the reader is also familiar with the article [8]. For not only will we use its terminology and results, but our proofs in this note will just be extensions of certain proofs in [8].

2. Preliminaries

In this section we prove three results which are similar to results given in Section 2 of [8]. We start with

Lemma 2.1. *Let G be a solvable fixed-point-free group of linear transformations acting on $V_r(p)$ and let u be a prime p-primitive divisor of $p^r - 1$. If $u \mid |G|$ and $u \neq 3$ then we can identify $V_r(p)$ with the additive group of $GF(p^r)$ in such a way that $G \subset T(p^r)$.*

Proof. Let $\rho \in G$ be an element of order u. By Theorem 18.2 of Passman [9, p. 196] ρ belongs to a normal subgroup $\bar{G}$ of G with $\bar{G}$ a Z-group. Corollary 2.1.1 of [8] tells us that $\bar{G} \subset T(p^r)$ and by Lemma 2.4 of [8] $\langle\rho\rangle$ is a characteristic cyclic subgroup of $\bar{G}$. $\bar{G}$ normal in G implies $\langle\rho\rangle$ is normal in G. Since $\langle\rho\rangle$ is irreducible [8, Lemma 2.2], $G \subset T(p^r)$ by Proposition 19.8 of Passman [9, p. 244].

Theorem 2.1. *Let π be a translation plane of order p^r coordinatized by a quasifield $(Q, +, \cdot)$. Let H be a solvable group of $((\infty), y = 0)$-homologies. If $|H|$ is divisible by u, where u is a prime p-primitive divisor of $p^r - 1$ and $u \neq 3$, then the additive group of Q can be identified with the additive group of $GF(p^r)$ in such a way that H is isomorphic to a subgroup of $T(p^r)$.*

Proof. The set of points $(0, y)$ on $x = 0$ is a vector space isomorphic to $(Q, +)$ and is of dimension r over $GF(p)$. H acts as a fixed-point-free group on $x = 0$. Furthermore, H is isomorphic to the group it induces on $x = 0$. The theorem thus follows from Lemma 2.1.

THEOREM 2.2. *Let π be a translation plane of order p^r coordinatized by the quasifield $(Q, +, \cdot)$, let G be the Q-autotopism group of π, let H_x and H_y be the groups of homologies of G with axis $y = 0$ and $x = 0$, respectively, and let $H = \langle H_x, H_y \rangle$. If S is a Sylow u-subgroup of H, where u is a prime p-primitive divisor of $p^r - 1$, then S is normal in G. If $u \neq 3$, $u \mid |H_x|$ and $u \mid |H_y|$, and both H_x and H_y are solvable, then G is solvable and the permutation groups G_y and G_x induced on $y = 0$ and $x = 0$ respectively are isomorphic to subgroups of $T(p^r)$.*

Proof. Using Lemma 2.1 and Theorem 2.1 above in place of Lemma 2.5 and Theorem 2.2, respectively, of [8] the proof of Theorem 2.3 in [8] carries over and proves this theorem.

Remark. This theorem has a geometric interpretation similar to that of Theorem 2.3 in [8].

3. THE BASIC TOOL

In this section we give the tool (Theorem 3.1) used in the investigation of the case $r = 2$. This line of thought is easily generalized and seems to be of value for investigating square-order planes (see [7]). We start with the following result:

LEMMA 3.1. *Let G be a nonsolvable subgroup of $PSL(2, p)$, p a prime, and assume $(p + 1) \mid |G|$. Either*

(a) $G = PSL(2, p)$, *or*

(b) $G = PSL(2, 5)$ *and* $p = 5, 11, 19, 29$, *or* 59.

Proof. Since $PSL(2, 2)$ and $PSL(2, 3)$ are solvable, G nonsolvable implies $p > 3$. By a theorem of L. E. Dickson [5, p. 213] the only nonsolvable subgroups of $PSL(2, p)$ are (a) $PSL(2, p)$ or (b) $PSL(2, 5)$ with $p = 5$ or $p^2 - 1 \equiv 0 \pmod 5$. In case (b) $|PSL(2, 5)| = 60$ and $(p + 1) \mid |G|$ gives $(p + 1) \mid 60$. This gives the possibilities for p.

LEMMA 3.2. *Let G be a nonsolvable subgroup of $PGL(2, p)$, p a prime, and assume $(p + 1) \mid |G|$. Either*

(a) *G contains $PSL(2, p)$ and $|G| = k[\frac{1}{2}p(p^2 - 1)]$, $k \mid 2$, or*

(b) *G contains $PSL(2, 5)$ and $|G| = 60k$, $k \mid 2$, and $p = 5, 11, 19, 29$, or 59.*

Proof. $PSL(2, p)$ is a normal subgroup of $PGL(2, p)$ of index 2. Thus $G \cap PSL(2, p)$ is a normal subgroup of G of index $k \mid 2$. Also, since $PGL(2, p)/PSL(2, p)$ is solvable, $G \cap PSL(2, p)$ is nonsolvable if G is. The result is therefore a consequence of Lemma 3.1.

LEMMA 3.3. *Let G be a non-solvable group of linear transformations on a vector space V of dimension 2 over $GF(p)$, p a prime. If G contains the group K of scalar linear transformations and if G is transitive on $V - \{0\}$, then either*

(a) $|G| = (1/2)kp(p^2 - 1)(p - 1)$ *with $k \mid 2$ and G contains $SL(2, p)$,*

or

(b) $|G| = 60k(p - 1)$ *with $k \mid 2$, $p = 5, 11, 19, 29,$ or 59, and G contains $SL(2, 5)$.*

Proof. Let $\bar{G} = G/K$. Then $\bar{G}$ is a subgroup of $PGL(2, p)$. The result follows from Lemma 3.2 since the transitivity hypothesis implies $(p + 1) \mid |\bar{G}|$.

Let π be a finite affine translation plane of order p^s with T its group of translations. If G is a collineation group of π then $G = G_{\mathcal{O}}(G \cap T)$, $\mathcal{O}$ an affine point of π (usually taken to be the origin in a coordinate system for π). We call $G_{\mathcal{O}}$ a *translation complement* (of π). The group K of $(\mathcal{O}, l_\infty)$-homologies of π is called the *kernel of* π. If $(Q, +, \cdot)$ is a (left) quasifield coordinatizing π then the set $\bar{K}$ consisting of those elements $a \in Q$ satisfying

$$(c + d)a = ca + da, \qquad (cd)a = c(da), \qquad \text{for all} \quad c, d \in Q,$$

form a field under the operations of Q. Note that $\bar{K}$ contains $GF(p)$ since $\bar{K}$ contains 1. André [1] proved that there is an isomorphism of K onto the multiplicative group of $\bar{K}$. Thus traditionally, $\bar{K}$ has also been called the *kernel of* π. In Theorem 3.1, we will use the word in both senses.

André [1] has shown that a translation complement is a group of semilinear transformations of π considered as a vector space over $\bar{K}$. In this setting the group K corresponds to the scalar transformations and the dimension of π as a vector space over $\bar{K}$ is $2r/s$ where π has order p^r and $\bar{K} = GF(p^s)$. We can now prove:

THEOREM 3.1. *Let π be a finite translation plane of order p^2, p a prime, and kernel $\bar{K} = GF(p)$. Let G be a nonsolvable translation complement fixing a line l and transitive on $l - \{\mathcal{O}\}$. Assume also that G contains the group K of $(\mathcal{O}, l_\infty)$-homologies of π. If H is the group of collineations of G fixing l pointwise, then either*

(a) $|G/H| = (1/2)kp(p^2 - 1)(p - 1)$ *with $k \mid 2$ and G/H contains a subgroup isomorphic to $SL(2, p)$, or*

(b) $|G/H| = 60k(p-1)$ *with* $k \mid 2, p = 5, 11, 19, 29,$ *or* 59, *and* G/H *contains a subgroup isomorphic to* $SL(2, 5)$.

Proof. G can be considered as a group of semilinear transformations on the 4-dimensional vector space π over $GF(p)$. The line l is a 2-dimensional vector subspace and hence the group G/H induced on l by G is a group of semi-linear transformations on l as a vector space over $GF(p)$. In a vector space over $GF(p)$ every semilinear transformation is a linear transformation. Also K a subgroup of G implies G/H contains the scalar linear transformations. The theorem now follows from Lemma 3.3.

We close this section with:

LEMMA 3.4. *Let G be a subgroup of $GL(2, p)$, p a prime greater than 3, and assume G contains $SL(2, p)$. If H is a normal subgroup of G, then one of the following holds*:

(a) $H \cap SL(2, p)$ *has order* 1,

(b) $H \cap SL(2, p)$ *has order* 2,

(c) H *contains* $SL(2, p)$.

Proof. Let $\bar{H} = H \cap SL(2, p)$ and consider $\bar{H}\eta$, where η is the natural homomorphism of $SL(2, p)$ onto $PSL(2, p)$. The kernel N of η has order 2. H normal in G implies $\bar{H}$ is normal in $SL(2, p)$ and hence $\bar{H}\eta$ is normal in $PSL(2, p)$. But $p > 3$ implies $PSL(2, p)$ is simple. Hence $\bar{H}\eta$ has order 1 or $\bar{H}\eta = PSL(2, p)$. If $\bar{H}\eta$ has order 1, then either (a) or (b) holds. If $\bar{H}\eta = PSL(2, p)$, then $\bar{H}$ contains p-elements. Since $SL(2, p)$ is generated by its p-elements which are all conjugate (Passman [9], Proposition 13.5, p. 115), $\bar{H} = SL(2, p)$ and case (c) must hold.

4. PROOF OF THEOREM A

Let π be a Z-plane of order p^r with rank 3 collineation group G and let $\mathscr{U}$ and $\mathscr{V}$ be the special points of π with respect to G. Since π is a translation plane [6] we coordinatize π with a quasifield $(Q, +, \cdot)$ such that $\mathscr{U} = (0)$, $\mathscr{V} = (\infty)$. Let $\mathscr{O} = (0, 0)$ and $\mathscr{I} = (1, 1)$. Then $G = TG_{\mathscr{O}}$, where T is the group of translations of π. Since T fixes 1_∞ pointwise G and $G_{\mathscr{O}}$ have the same action on 1_∞.

If A is the Q-autotopism group of π, let $G' = G_{\mathscr{O}} \cap A$. Then G' is a normal subgroup of $G_{\mathscr{O}}$ with index 2. Let H_x and H_y be the group of homologies of $G_{\mathscr{O}}$ with axis $\mathscr{O}\mathscr{U}$ and axis $\mathscr{O}\mathscr{V}$, respectively. Then H_x and H_y are normal in G' since G' fixes both $\mathscr{O}\mathscr{U}$ and $\mathscr{O}\mathscr{V}$. Also G' is transitive on the points different than $\mathscr{O}$ which lie on $\mathscr{O}\mathscr{U}$ and G' is transitive on the points of

$\mathcal{OV}$ different than $\mathcal{O}$. These two lines are the axes $y = 0$ and $x = 0$ respectively in our coordinate system.

Note that every homology of H_x has center (∞) and every homology of H_y has center (0). Note also that $H_x \cong H_y$ since $G_{\mathcal{O}}$ contains a collineation interchanging $y = 0$ and $x = 0$.

LEMMA 4.1. *Let π be a Z-plane of order p^r with rank 3 collineation group G and assume $p^r - 1$ has a prime p-primitive divisor u. H_x has a Sylow u-subgroup S_x whose order is u^a where $u^a \| (p^r - 1)$. Also H_y has a Sylow u-subgroup S_y whose order is u^a.*

Proof. The proof of Lemma 4.1 in [8] proves the first assertion; for in that proof the hypothesis $u \neq 3, 5$ is not used. The second statement follows from the fact that $H_x \cong H_y$.

LEMMA 4.2. *If in addition to the hypothesis of Lemma 4.1 we assume $u \neq 3$ and H_x (thus also H_y) is solvable, then $G_{\mathcal{O}}$ has an abelian Sylow u-subgroup $S = S_x \otimes S_y$ of order u^{2a}.*

Proof. As in the proof of Lemma 4.2 of [8] it is sufficient to prove S_x and S_y are cyclic. Consider G' (see the beginning of this section). If G_y' is the permutation group induced on $y = 0$ by G', then G' is isomorphic to a subgroup of $T(p^r)$ by Theorem 2.2. S_y induces a Sylow u-subgroup S_y' of G_y' isomorphic to S_y since $G_y' \cong G'/H_x$ and $S_y \cap H_x$ consists only of the identity. Since the Sylow u-subgroups of G_y' are cyclic (Lemma 2.4 of [8]) S_y', and hence S_y, is cyclic. Similarly S_x is cyclic.

LEMMA 4.3. *Under the hypothesis of Lemma 4.2 the Sylow u-subgroup S is normal in $G_{\mathcal{O}}$.*

Proof. If we use Theorem 2.2 in place of Theorem 2.3 in [8], the proof of Lemma 4.3 in [8] gives a proof of Lemma 4.3.

We can now prove Theorem A. Because of Theorem 4.1 on p. 175 of [5] we may restrict ourselves to the cases where the only prime p-primitive divisors of $p^r - 1$ are 3 and/or 5.

Assume first that $r = 4$. By Birkhoff and Vandiver [2] $p^4 - 1$ always has a prime p-primitive divisor u. We claim that $u \neq 3$. For 3 to be a p-primitive divisor it necessarily must divide $p^2 + 1$. Now either $p \equiv 0 (\text{mod } 3)$, $p \equiv 1 (\text{mod } 3)$, or $p \equiv 2 (\text{mod } 3)$. Thus either $p^2 + 1 \equiv 1 (\text{mod } 3)$ or $p^2 + 1 \equiv 2 (\text{mod } 3)$ and hence for all p, $3 \nmid p^2 + 1$. Thus we may restrict ourselves to the case where 5 is the only prime p-primitive divisor of $p^4 - 1$.

We next show that H_x (and thus also H_y) must be solvable. Assume not. H_x nonsolvable implies, by Theorem 18.6 of Passman [9], that H_x contains a normal subgroup $\bar{H}$ with $[H_x : \bar{H}] = 1$ or 2 and $\bar{H} \cong SL(2, 5) \times M$ with $|M|$ prime to 2, 3, 5. If $5^a \| (p^4 - 1)$, then $5^a \mid |\bar{H}|$ by Lemma 4.1 and therefore $5 \| (p^4 - 1)$. Since every other prime dividing $p^2 + 1$ must divide $p^2 - 1$ (for otherwise it would be a p-primitive divisor of $p^4 - 1$) and g.c.d. $(p^2 + 1, p^2 - 1) = 2$, we must have $p^2 + 1 = 5.2^i$ for some $i \geqslant 1$. If $p = 2k + 1$, $k \geqslant 1$, then $5.2^i = p^2 + 1 = 2(2k^2 + 2k + 1)$. Hence $i = 1$ and $k = 1$. Therefore $p = 3$.

But if $p = 3$ then $p^4 - 1 = 80$ and $|H_x| \mid 80$ since H_x is a fixed-point-free group of linear transformations on the vextor space $x = 0$ of order $3^4 = 81$. This contradicts the fact that $120 \mid |H_x|$ (since $SL(2, 5)$ is a subgroup of H_x). Hence H_x is solvable.

To complete the proof of this case we just use the proof of Theorem 4.1 in [8, p. 175], replacing Lemmas 4.1 to 4.3 of that article with Lemmas 4.1 to 4.3.

We now consider the case $r = 2$. Without loss of generality we may assume G contains the group of $(\mathcal{O}, 1_\infty)$-homologies of π since they fix each line through $\mathcal{O}$ and we may assume $p > 2$. The group G_y' (see the proof of Lemma 4.2) is a group of linear transformations on the vector space $y = 0$ of dimension 2 over $GF(p)$. Since $|G_y'| = (1/2)(p^2 - 1)^2 n |H_x|^{-1}$ and $|GL(2, p)| = p(p^2 - 1)(p - 1)$ we must have $(p + 1)/2 \mid |H_x|$.

We will first show that G_y' is solvable. If it is nonsolvable, then we can apply Theorem 3.1 to G' and H_x, l being the axis $y = 0$. Since the values $p = 5$, 11, 19, 29, and 59 are excluded, we have $G'/H_x = G_y'$ contains $SL(2, p)$ and $|G_y'| = (1/2)\, kp(p^2 - 1)(p - 1)$ with $k \mid 2$.

The group H_y of homologies with axis $x = 0$ induces a normal subgroup H_y' of G_y' since H_y is normal in G' (see the beginning of this section) and $(p + 1)/2 \mid |H_y'|$ since $H_y' \cong H_y \cong H_x$. Hence $|H_y'| = (1/2)(p + 1)l$ for some integer $l \geqslant 1$. We can thus apply Lemma 3.4 to G_y' and H_y'. If $\bar{H} = H_y' \cap SL(2, p)$ has order 1, then $|H_y'\, SL(2, p)| = |H_y'|\, |SL(2, p)| = (1/4)\, p(p^2 - 1)(p + 1)l$ divides $|G_y'| = (1/2)\, kp(p^2 - 1)(p - 1)$, since H_y' normal in G_y' implies $H_y' SL(2, p)$ is a subgroup of G_y'. This implies $(p + 1) \mid 2k(p - 1)$. Now $k \mid 2$ and g.c.d. $(p + 1, p - 1) = 2$ gives $(p + 1) \mid 4k = 4$ or 8. Hence p must be either 3 or 7. But p cannot be 3 or 7 since in both cases $p^2 - 1$ does not have a p-primitive divisor. Thus case (a) of Lemma 3.4 does not hold.

If case (b) of Lemma 3.4 holds, then $|\bar{H}| = 2$ and

$$|H_y' SL(2, p)| = |H_y|\, |SL(2, p)|\, |\bar{H}|^{-1} = (1/8)\, p(p^2 - 1)(p + 1)l$$

divides $(1/2)\, kp(p^2 - 1)(p - 1)$. Thus $(p + 1) \mid 4k(p - 1)$, $(p + 1) \mid 8k = 8$ or 16. Again this says $p = 3$ or 7 which cannot be. Hence case (b) cannot hold either. Thus case (c) of Lemma 3.4 holds.

But then $p \mid | H_y' |$ and this is a contradiction. For H_y' is a fixed-point-free group of linear transformations on the vector space $y = 0$ and thus $| H_y' | \mid p^2 - 1$. Thus G_y' is solvable and hence also H_y'.

H_x is also solvable since $H_y' \cong H_y \cong H_x$. Since $G_y' = G'/H_x$, G' is solvable and, since G' has index 2 in $G_{\mathcal{O}}$, $G_{\mathcal{O}}$ is solvable. Thus $G = TG_{\mathcal{O}}$ is solvable. The theorem follows from Theorem 4.2, p. 176 in [8].

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