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Localization of the complex zeros of parametrized families of polynomials

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Abstract

Let $P_n(x) = x^m + p_{m-1}(n)x^{m-1} + \cdots + p_1(n)x + p_m(n)$ be a parametrized family of polynomials of a given degree with complex coefficients $p_k(n)$ depending on a parameter $n \in \mathbf{Z}_{\geq 0}$. We use Rouché's theorem to obtain approximations to the complex roots of $P_n(x)$. As an example, we obtain approximations to the complex roots of the quintic polynomials $P_n(x) = x^5 + nx^4 - (2n+1)x^3 + (n+2)x^2 - 2x + 1$ studied by A. M. Schöpp.

Keywords: Polynomials; Roots

1. Introduction

Usually, real roots of parametrized families of polynomials with real coefficients are easy to locate by looking at sign changes. Here, we explain how to locate complex zeros of such families. By Rouché's theorem, if f and g are entire and $|f(z) - g(z)| < |g(z)|$ for $|z - a| = r$, then f and g have the same number of zeros in the open disc with center at a and radius $r > 0$ if they are counted according to their multiplicities (see Rudin (1987)). Let

$$P(x) = x^m + p_{m-1}x^{m-1} + \cdots + p_1x + p_0 = \prod_{k=1}^{m}(x - \rho^{(k)})$$

and

$$Q(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_1x + q_0 = \prod_{k=1}^{m}(x - \theta^{(k)})$$

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be two polynomials of the same degree $m \geq 1$ with complex coefficients and, to simplify, assume that the $\theta^{(k)}$ are pairwise distinct and that $q_0 \neq 0$ (i.e., that none of the $\theta^{(k)}$ is equal to zero). If $|z - \theta^{(1)}| = r|\theta^{(1)}|$ with

$$r < \frac{\min\limits_{2 \leq k \leq m} |\theta^{(1)} - \theta^{(k)}|}{2|\theta^{(1)}|}, \tag{1}$$

then $Q(z) = (z - \theta^{(1)}) \prod_{k=2}^{m} (z - \theta^{(1)} + \theta^{(1)} - \theta^{(k)})$ yields

$$|Q(z)| \geq r|\theta^{(1)}| \left(\min_{2 \leq k \leq m} |\theta^{(1)} - \theta^{(k)}| - r|\theta^{(1)}| \right)^{m-1}$$

and

$$|Q(z)| > 2^{m-1} r |\theta^{(1)}| \prod_{k=2}^{m} |\theta^{(1)} - \theta^{(k)}|,$$

and also we have

$$|P(z) - Q(z)| \leq (\max_{0 \leq k \leq m-1} |p_k - q_k|) \sum_{k=0}^{m-1} |\theta^{(1)}|^k (1 + r)^k.$$

By Rouché's theorem, if

$$m 2^{2m-1} \frac{(\max\limits_{0 \leq k \leq m-1} |p_k - q_k|)(\max(1, |\theta^{(1)}|)^{m-1}}{|\theta^{(1)}| \prod\limits_{k=2}^{m} |\theta^{(1)} - \theta^{(k)}|} \leq r \leq 1, \tag{2}$$

then one of the roots of $P(x)$, say $\rho^{(1)}$, satisfies

$$|\rho^{(1)} - \theta^{(1)}| < r|\theta^{(1)}|.$$

2. Parametrized families of polynomials

In particular, let $P_n(x) = x^m + p_{m-1}(n)x^{m-1} + \cdots + p_1(n)x + p_0(n)$ (with n a sufficiently large positive integer) be an explicit parametrized family of complex monic polynomials of a given degree $m \geq 1$.

1. As in the case that $P_n(x) = x^5 + nx^4 - (2n + 1)x^3 + (n + 2)x^2 - 2x + 1$ developed in the next section, suppose that using any algorithm or software for computing complex roots of polynomials we can find an explicit parametrized family of monic polynomials $Q_n(x) = x^m + q_{m-1}(n)x^{m-1} + \cdots + q_1(n)x + q_0(n)$ of degree m and known pairwise distinct complex roots $\theta_n^{(k)}$, $1 \leq k \leq m$, such that

$$\max_{0 \leq k \leq m-1} |p_k(n) - q_k(n)| \leq C_1 n^{-\alpha}$$

for some $\alpha > 0$ and some $C_1 > 0$ (i.e., the coefficients of $Q_n(x)$ become closer to those of $P_n(x)$ as $n \to \infty$).

2. Suppose that for n large enough we have

$$\min_{2\le k\le m} |\theta_n^{(1)} - \theta_n^{(k)}| \ge C_2 n^{\beta},$$

$$\prod_{k=2}^{m} |\theta_n^{(1)} - \theta_n^{(k)}| \ge C_3 n^{\gamma}$$

and

$$C_4 n^{\delta} \le |\theta_n^{(1)}| \le C_5 n^{\delta}$$

for some β, γ and δ, and $C_2 > 0$, $C_3 > 0$, $C_4 > 0$ and $C_5 \ge 1$ (recall that we assume that the $\theta_n^{(k)}$'s are known beforehand).

3. Finally, suppose that the $Q_n(x)$'s are chosen so that α is large enough to satisfy

$$\alpha > -\beta - \gamma + (m-1)\max(\delta, 0) \quad \text{and} \quad \alpha > -\gamma - \delta + (m-1)\max(\delta, 0). \tag{3}$$

(Notice that β, γ and δ do not depend much on how close to the $\rho_n^{(k)}$'s are the $\theta_n^{(k)}$'s, whereas α can be constructed as large as desired by finding the $\theta_n^{(k)}$'s close enough to the $\rho_n^{(k)}$'s. See the example in the next section.)

Then, we take

$$r = C_6 n^{-\alpha-\gamma-\delta+(m-1)\max(\delta,0)}$$

(with $C_6 = m2^{2m-1}C_1 C_5^{(m-1)\max(\delta,0)}/C_3C_4$), which will satisfy (1) and (2) for n large enough. We deduce approximations to one of the complex roots of $P_n(x)$, say of $\rho_n^{(1)}$, for they satisfy

$$|\rho_n^{(1)} - \theta_n^{(1)}| \le C_6 n^{-\alpha-\gamma-\delta+(m-1)\max(\delta,0)} |\theta_n^{(1)}|$$

and

$$\rho_n^{(1)} = \theta_n^{(1)} + O(n^{-\alpha-\gamma+(m-1)\max(\delta,0)}). \tag{4}$$

The point is that no matter how we guessed approximations $\theta_n^{(k)}$ to the complex roots of a parametrized family of monic polynomials $P_n(x)$ of a given degree, provided that these approximations are indeed close enough to the complex roots $\rho_n^{(k)}$ of these polynomials, then we can readily deduce proved approximations to these complex roots $\rho_n^{(k)}$. We just have to use any software (e.g., Maple) to determine α.

3. An example

As in Schöpp (2006, Proof of Lemma 4.4), consider the polynomials

$$P_n(x) = x^5 + nx^4 - (2n+1)x^3 + (n+2)x^2 - 2x + 1,$$

where $n > 0$ is a positive integer. These $P_n(x)$ are $\mathbf{Q}$-irreducible and have one real root $\rho_n = \rho_n^{(1)}$ and four complex roots $\rho_n^{(2)}$, $\rho_n^{(3)} = \overline{\rho_n^{(2)}}$, $\rho_n^{(4)}$ and $\rho_n^{(5)} = \overline{\rho_n^{(4)}}$. The core of Schöpp's paper is a long proof of his Lemma 4.4, in which he gives approximations to these $\rho_n^{(k)}$'s, to deduce, in his Theorem 4.1, a system of fundamental units for the quintic orders $\mathbf{Z}[\rho_n]$. Here, on the basis of the method outlined in our introduction, we will readily obtain better results than his. To begin with, using any software for computing approximations to these complex roots for various values of n (of the form $n = 10^{2k}$ with $k \geq 1$), e.g. see Cohen (1993, Section 3.6.3) or Ralston (1965, Section 8.10), or using Newton's method, we suspect that for n large, good approximations to these complex roots are

$$\rho_n^{(1)} \approx \theta_n^{(1)} := -n - 2 + \frac{2}{n}$$

(recall also that real roots are easy to locate by looking at sign changes, as in Schöpp (2006, Proof of Lemma 4.4)),

$$\rho_n^{(2)} \approx \theta_n^{(2)} := \frac{\mathrm{i}}{\sqrt{n}}$$

(you may also observe that $P_n(x)$ must have complex roots of small absolute value, and these roots should be close to the ones of $(n+2)x^2 - 2x + 1$) and

$$\rho_n^{(4)} \approx \theta_n^{(4)} := 1 - \frac{1}{n} + \frac{\mathrm{i}}{\sqrt{n}}$$

(you may also notice that the sum of the five complex roots $\rho_n^{(k)}$ is equal to $-n$ and their product to -1 to guess this third approximation from the two previous ones). Thanks to these approximations, we have the following table:

$(m = 4)$	β	γ	δ	Condition on α (see (3))	Approximation (see (4))
$\theta_n^{(1)}$	1	4	1	$\alpha > -1$	$\rho_n^{(1)} = \theta_n^{(1)} + O(n^{-\alpha})$
$\theta_n^{(2)}$	$-1/2$	$1/2$	$-1/2$	$\alpha > 0$	$\rho_n^{(2)} = \theta_n^{(2)} + O(n^{-\alpha-1/2})$
$\theta_n^{(4)}$	$-1/2$	$1/2$	0	$\alpha > 0$	$\rho_n^{(4)} = \theta_n^{(4)} + O(n^{-\alpha-1/2})$

Using these approximations, we have

$$\begin{aligned} Q_n(x) &= \left(x + n + 2 - \frac{2}{n}\right)\left(x - \frac{\mathrm{i}}{\sqrt{n}}\right)\left(x + \frac{\mathrm{i}}{\sqrt{n}}\right) \\ &\quad \times \left(x - 1 + \frac{1}{n} - \frac{\mathrm{i}}{\sqrt{n}}\right)\left(x - 1 + \frac{1}{n} + \frac{\mathrm{i}}{\sqrt{n}}\right) \end{aligned} \tag{5}$$

and $n^4(P_n(x) - Q_n(x)) = -(8n^3 - 3n^2)x^3 + (3n^3 - 4n^2 + 2n)x^2 + (n^3 - 7n^2 + 3n)x - (n^3 - 3n^2 + 4n - 2)$, i.e., the coefficients of $P_n(x) - Q_n(x)$ are $O(\frac{1}{n})$. Hence, here $\alpha = 1$, and we obtain that

$$\rho_n^{(1)} = n - 2 + O\left(\frac{1}{n}\right),$$

$$\rho_n^{(2)} = \frac{\mathrm{i}}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right)$$

and

$$\rho_n^{(4)} = 1 - \frac{1}{n} + \frac{\mathrm{i}}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right),$$

which implies Points (iii)–(vi) of Schöpp (2006, Lemma 4.4) for n effectively large enough. Now, construct the polynomial $Q_n(x)$ by using the following suspected better approximations (which can also be deduced from the previous ones by applying Newton's method):

$$\rho_n^{(1)} \approx \theta_n^{(1)} := -n - 2 + \frac{2}{n} - \frac{8}{n^2},$$
$$\rho_n^{(2)} \approx \theta_n^{(2)} := \frac{1}{2n^2} + \left(\frac{1}{\sqrt{n}} - \frac{1}{2n\sqrt{n}}\right)\mathrm{i}$$

and

$$\rho_n^{(4)} \approx \theta_n^{(4)} := 1 - \frac{1}{n} + \frac{7}{2n^2} + \left(\frac{1}{\sqrt{n}} - \frac{5}{2n\sqrt{n}}\right)\mathrm{i}.$$

Then, the coefficients of the complicated polynomial $P_n(x) - Q_n(x)$ are $O(\frac{1}{n^2})$ (e.g., use Maple). Hence, now $\alpha = 2$ and we obtain the following result which implies (Schöpp, 2006, Lemma 4.4) for n effectively large enough:

Theorem 1. *For $n \longrightarrow \infty$, the five complex roots $\rho_n^{(1)}$, $\rho_n^{(2)}$, $\rho_n^{(3)} = \overline{\rho_n^{(2)}}$, $\rho_n^{(4)}$ and $\rho_n^{(5)} = \overline{\rho_n^{(4)}}$ of $P_n(x) = x^5 + nx^4 - (2n+1)x^3 + (n+2)x^2 - 2x + 1$ satisfy*

$$\rho_n^{(1)} = -n - 2 + \frac{2}{n} + O\left(\frac{1}{n^2}\right),$$
$$\rho_n^{(2)} = \frac{1}{2n^2} + \left(\frac{1}{\sqrt{n}} - \frac{1}{2n\sqrt{n}}\right)\mathrm{i} + O\left(\frac{1}{n^{5/2}}\right)$$

and

$$\rho_n^{(4)} = 1 - \frac{1}{n} + \frac{7}{2n^2} + \left(\frac{1}{\sqrt{n}} - \frac{5}{2n\sqrt{n}}\right)\mathrm{i} + O\left(\frac{1}{n^{5/2}}\right),$$

where the implied constants in these errors terms are effective and explicit.

4. Explicit results

To show that our method easily leads to explicit results, let us, for example, give a proof based on our ideas of Point (iii) of Schöpp (2006, Lemma 4.4). We use the polynomial $Q_n(x)$ given in (5).

If $|z| = \frac{1}{2\sqrt{n}}$ and $n \geq 2$, we have

$$|Q_n(z)| \geq \left(n + 2 - \frac{1}{2\sqrt{n}} - \frac{2}{n}\right)\left(\left(1 - \frac{1}{n} - \frac{1}{2\sqrt{n}}\right)^2 + \frac{1}{n}\right) \Big/ 4n$$

and

$$|P_n(z) - Q_n(z)| \leq \frac{\frac{8n^3-3n^2}{8n^{3/2}} + \frac{3n^3-4n^2+2n}{4n} + \frac{n^3-7n^2+3n}{2n^{1/2}} + \left(n^3 - 3n^2 + 4n - 2\right)}{n^4}.$$

Hence, $|P_n(z) - Q_n(z)| < |Q_n(z)|$ for $|z| = \frac{1}{2\sqrt{n}}$ and $n \geq 2$. Since $Q_n(z)$ has no complex root inside the open disc $|z| < \frac{1}{2\sqrt{n}}$, neither does $P_n(x)$.

If $|z| = \frac{2}{\sqrt{n}}$ and $n \geq 9$, we have

$$|Q_n(z)| \geq \left(n + 2 - \frac{2}{\sqrt{n}} - \frac{2}{n}\right)\left(\left(1 - \frac{1}{n} - \frac{2}{\sqrt{n}}\right)^2 + \frac{1}{n}\right) \Big/ n$$

and

$$|P_n(z) - Q_n(z)| \leq \frac{8\frac{8n^3-3n^2}{n^{3/2}} + 4\frac{3n^3-4n^2+2n}{n} + 2\frac{n^3-7n^2+3n}{n^{1/2}} + (n^3 - 3n^2 + 4n - 2)}{n^4}.$$

Hence, $|P_n(z) - Q_n(z)| < |Q_n(z)|$ for $|z| = \frac{2}{\sqrt{n}}$ and $n \geq 14$. Since $Q_n(z)$ has two conjugate complex roots inside the open disc $|z| < \frac{2}{\sqrt{n}}$, so does $P_n(x)$.

Hence, we have proved that

$$\frac{1}{2\sqrt{n}} \leq |\rho_n^{(2)}| < \frac{2}{\sqrt{n}} \quad \text{for } n \geq 12.$$

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