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Pricing perpetual options using Mellin transforms

Radha Panini, Ram P. Srivastav*

Department of Applied Mathematics and Statistics, SUNY at Stony Brook, Stony Brook, NY 11794, USA

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Abstract

We have derived the expression for the free boundary and price of an American perpetual put as the limit of a finite-lived option.

Keywords: American put option; Perpetual option; Free boundary

1. Introduction

The pricing of American options has been the subject of extensive research in the last three decades. There is no known closed form solution and many numerical and analytic approximations have been proposed. However, in the special case of a perpetual option, Samuelson [1] derived a closed-form expression for the price of the American perpetual warrant and Carr and Faguet [2] used this expression to derive the price of a finite-lived option.

In this note, we derive the same expressions as Samuelson [1] for the free boundary and price of a perpetual put using Mellin transform techniques. Our expression for the price is derived as a steady-state solution (solving as a limiting case of the time to maturity tending to infinity) to the non-homogenous Black–Scholes equation, rather than as a solution to the 'static' problem (where the option price is assumed to be independent of time).

* Corresponding author.
E-mail address: rsrivastav@notes.cc.sunysb.edu (R.P. Srivastav).

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2. Price of an American put

For a non-dividend paying stock, it has been shown by Panini and Srivastav [3], that the value of a finite-lived American put may be expressed as an inverse Mellin transform as follows:

The non-homogenous Black–Scholes equation for the price of an American put $P(S,t)$ is

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}+rS\frac{\partial P}{\partial S}-rP=f(S,t),\qquad 0<t<T,\qquad 0<S<\infty, \tag{2.1}$$

where

$$f=f(S,t)=\begin{cases}-rK & \text{if } 0<S\le S^*(t),\\ 0 & \text{if } S>S^*(t),\end{cases} \tag{2.2}$$

with the final time condition

$$P(S,T)=\theta(S)=(K-S)^+. \tag{2.3}$$

Note that the free boundary $S^*(t)$ also depends on the expiry time T. Strictly, we should write $S^*(t,T)$.

The Mellin transform of (2.1) yields

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}t}+\left(\frac{\sigma^2}{2}(w^2+w)-rw-r\right)\hat{P}=\hat{f}(w,t), \tag{2.4}$$

where

$$\hat{f}(w,t)=\int_0^{S^*(t)}-rKS^{w-1}\,\mathrm{d}S=\frac{-rK}{w}(S^*(t))^w. \tag{2.5}$$

Eq. (2.4) is a linear ordinary differential equation of first order which is solved for $\hat{P}$. Inverting the Mellin transform we get the relation

$$\begin{aligned}P(S,t)&=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\hat{\theta}(w)\mathrm{e}^{\frac{1}{2}\sigma^2 q(w)(T-t)}S^{-w}\,\mathrm{d}w\\ &\quad+\frac{rK}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}S^{-w}\int_t^T\frac{(S^*(x))^w}{w}\mathrm{e}^{\frac{1}{2}\sigma^2 q(w)(x-t)}\,\mathrm{d}x\,\mathrm{d}w\end{aligned} \tag{2.6}$$

where $\mathrm{Re}(w)>0$,

$$q(w)=(w+1)(w-k_1)\quad\text{with}\quad k_1=\frac{2r}{\sigma^2}, \tag{2.7}$$

and $\hat{\theta}(w)$ is the Mellin transform of the payoff function $\theta(S)$.

The unknown free boundary $S^*(t)$ is determined using the "smooth pasting" conditions:

$$P(S^*(t),t)=K-S^*(t), \tag{2.8}$$

$$\left.\frac{\partial P}{\partial S}\right|_{S=S^*}=-1. \tag{2.9}$$

Note that for (2.6) to hold as $T\to\infty$, it is necessary that $\mathrm{Re}(q(w))<0$, i.e. $0<\mathrm{Re}(w)<k_1$.

The first term in (2.6) is the price of a European put, which by using the Black–Scholes formula and the call-put parity, can be written as

$$P_0(S,t)=K\mathrm{e}^{-r(T-t)}(1-N(d_2))-S(1-N(d_1)), \tag{2.10}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right], \tag{2.11}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{2.12}$$

and N is the Gaussian distribution function.

The second integral in (2.6), which we shall denote as $P_1(S,t)$, is the premium for the opportunity of early exercise offered by the American put.

3. Free boundary for a perpetual put

In this section we use the second smooth pasting condition to derive an expression for the free boundary of the perpetual put. Note that at any time t, there is infinite time to maturity, and therefore the free boundary for the perpetual put is constant, i.e. $S^*_\infty(t) = S^*_\infty$ for all t.

The smooth pasting condition (2.8) for a perpetual option can be written as

$$-1 = \left.\frac{\partial P_0}{\partial S}\right|_{S=S^*_\infty} + \left.\frac{\partial P_1}{\partial S}\right|_{S=S^*_\infty} \qquad \text{as} \qquad T \to \infty. \tag{3.1}$$

We begin with the P_0 term. Differentiating (2.10), we get

$$\left.\frac{\partial P_0}{\partial S}\right|_{S=S^*_\infty} = -(1 - N(\hat{d}_1)), \tag{3.2}$$

where

$$\hat{d}_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S^*_\infty}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]. \tag{3.3}$$

As $T \to \infty$, $\hat{d}_1 \to \infty$ and therefore

$$\left.\frac{\partial P_0}{\partial S}\right|_{S=S^*_\infty} \to 0. \tag{3.4}$$

Now consider the P_1 term,

$$\frac{\partial P_1}{\partial S} = -\frac{rK}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{1}{S}\left(\int_t^T \left(\frac{S}{S^*(x)}\right)^{-w} \mathrm{e}^{\frac{1}{2}\sigma^2 q(w)(x-t)}\,\mathrm{d}x\right)\mathrm{d}w. \tag{3.5}$$

The limit $T \to \infty$ yields

$$\frac{\partial P_1}{\partial S} = -\frac{rK}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{1}{S}\left(\int_t^\infty \left(\frac{S}{S^*_\infty}\right)^{-w} \mathrm{e}^{\frac{1}{2}\sigma^2 q(w)(x-t)}\,\mathrm{d}x\right)\mathrm{d}w. \tag{3.6}$$

Therefore,

$$\left.\frac{\partial P_1}{\partial S}\right|_{S=S^*_\infty} = \frac{rK}{2\pi \mathrm{i}}\frac{2}{\sigma^2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{1}{S^*_\infty}\frac{1}{(w+1)(w-k_1)}\,\mathrm{d}w. \tag{3.7}$$

Since $0 < \mathrm{Re}(w) < k_1$, application of Cauchy's residue theorem leads to

$$\left.\frac{\partial P_1}{\partial S}\right|_{S=S^*_\infty} = -\frac{K}{S^*_\infty}\frac{k_1}{k_1+1}. \tag{3.8}$$

Eqs. (3.1), (3.4) and (3.8) together give

$$S^*_\infty = \frac{k_1}{k_1+1}K. \tag{3.9}$$

4. Price of a perpetual put

We use the value of S^*_∞ from (3.9) to derive an expression for the price of a perpetual put $P_\infty(S,t)$.

Note that the price of a perpetual European put is zero, since it can never be exercised. Therefore, taking the limit $T \to \infty$ in (2.6), the price of the American perpetual put for $S > S^*_\infty$, is given by

$$P_\infty(S,t) = \frac{rK}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\frac{S}{S^*_\infty}\right)^{-w}\frac{1}{w}\left(\int_t^\infty \mathrm{e}^{\frac{1}{2}\sigma^2 q(w)(x-t)}\,\mathrm{d}x\right)\mathrm{d}w, \tag{4.1}$$

where $\mathrm{Re}(q(w)) < 0$. Integrating the time variable leads to

$$P_\infty(S,t) = -\frac{rK}{2\pi \mathrm{i}}\frac{2}{\sigma^2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\frac{S}{S^*_\infty}\right)^{-w}\frac{1}{w(w+1)(w-k_1)}\,\mathrm{d}w. \tag{4.2}$$

As before, since $0 < \mathrm{Re}(w) < k_1$, we can apply the residue theorem to get

$$P_\infty(S,t) = (K - S^*_\infty)\left(\frac{S}{S^*_\infty}\right)^{-\frac{2r}{\sigma^2}} \quad \text{for} \quad S > S^*_\infty. \tag{4.3}$$

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