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J. Math. Anal. Appl. 332 (2007) 1308–1313

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

A simple proof of inequalities of integrals of composite functions ☆

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Received 13 June 2006

Available online 8 December 2006

Submitted by R.P. Agarwal

Abstract

In this paper we give a simple proof of inequalities of integrals of functions which are the composition of nonnegative continuous convex functions on a vector space $\mathbf{R}^m$ and vector-valued functions in a weakly compact subset of a Banach vector space generated by m L^p_μ-spaces for $1 \leqslant p < +\infty$. Also, the same inequalities hold if these vector-valued functions are in a weakly* compact subset of a Banach vector space generated by m L^∞_μ-spaces instead.

Keywords: Convex functions; Measure spaces; Lebesgue integrals; Banach spaces

1. Introduction

Both convexity of functions and characteristics of weakly compact sets are important in the study of extremum problems and integral estimates in many areas of applied mathematics. The basic results about the continuity and differentiability of convex functions appear in the book of Rockafellar [3] while many characteristics of weakly compact sets in Banach spaces in the books of Benedetto [2] and Yosida [5]. Of interest and importance is the study of integral estimates of

☆ This work was partially supported by NSFC 10271121, 10511120278 and 10611120371, and sponsored by SRF for ROCS, SEM.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.11.007

a function which is the composition of a convex function on a vector space and a vector-valued function in a weakly compact subset in a Banach space since this kind of composite function often appears in many research fields such as compensated compactness methods (e.g., in [1,4]). Therefore it is necessary to give the inequalities of integrals of the composite functions for our solving many problems in applied mathematics.

Throughout this paper, $\mathbf{R}$ denotes the real number system, $\mathbf{R}^n$ is the usual vector space of real n-tuples $x = (x_1, x_2, \dots, x_n)$, μ is a nonnegative Lebesgue measure of $\mathbf{R}^n$, $L^p_\mu(\mathbf{R}^n)$ represents a Banach space where each measurable function $u(x)$ has the following norm

$$\|u(x)\|_p = \left(\int\limits_{\mathbf{R}^n} |u(x)|^p \, d\mu \right)^{\frac{1}{p}} \tag{1}$$

for any $p \in [1, +\infty)$, $(L^p_\mu(\mathbf{R}^n))^m$ denotes a Banach vector space where each measurable vector-valued function has m components in $L^p_\mu(\mathbf{R}^n)$, $L^\infty_\mu(\mathbf{R}^n)$ represents a Banach space where each measurable function $u(x)$ has the following norm

$$\|u(x)\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbf{R}^n} |u(x)| \quad \Big(\text{or say } \|u(x)\|_\infty = \inf_{\substack{E \subseteq \mathbf{R}^n \\ \mu(E^C)=0}} \max_{x \in E} |u(x)|\Big) \tag{2}$$

where E^C represents the complement set of E in $\mathbf{R}^n$, and $(L^\infty_\mu(\mathbf{R}^n))^m$ denotes a Banach vector space where each measurable vector-valued function has m components in $L^\infty_\mu(\mathbf{R}^n)$. A sequence $\{u_i\}_{i=1}^{+\infty}$ is called weakly convergent to u in $L^p_\mu(\mathbf{R}^n)$ as $i \to +\infty$ for $1 < p < +\infty$, if the following equality holds

$$\lim_{i \to +\infty} \int\limits_{\mathbf{R}^n} u_i v \, d\mu = \int\limits_{\mathbf{R}^n} u v \, d\mu \tag{3}$$

for all $v \in L^q_\mu(\mathbf{R}^n)$ where $q = \frac{p}{p-1}$ and $1 < p < +\infty$. A sequence $\{u_i\}_{i=1}^{+\infty}$ is called weakly convergent to u in $L^1_\mu(\mathbf{R}^n)$ as $i \to +\infty$, if the equality (3) holds for all $v \in L^\infty_\mu(\mathbf{R}^n)$. A sequence $\{u_i\}_{i=1}^{+\infty}$ is called weakly* convergent to u in $L^\infty_\mu(\mathbf{R}^n)$ as $i \to +\infty$, if the equality (3) holds for all $v \in L^1_\mu(\mathbf{R}^n)$. A sequence $\{u_i = (u_{1i}, u_{2i}, \dots, u_{mi})\}_{i=1}^{+\infty}$ is called weakly convergent to $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m)$ in a Banach vector space $(L^p_\mu(\mathbf{R}^n))^m$ as $i \to +\infty$ for $1 \leqslant p < +\infty$, if $\{u_{ji}\}_{i=1}^{+\infty}$ is weakly convergent to $\hat{u}_j$ in $L^p_\mu(\mathbf{R}^n)$ as $i \to +\infty$ for all $j = 1, 2, \dots, m$ and $1 \leqslant p < +\infty$. If $\{u_{ji}\}_{i=1}^{+\infty}$ is weakly* convergent to $\hat{u}_j$ in $L^\infty_\mu(\mathbf{R}^n)$ as $i \to +\infty$ for all $j = 1, 2, \dots, m$, a sequence $\{u_i = (u_{1i}, u_{2i}, \dots, u_{mi})\}_{i=1}^{+\infty}$ is called weakly* convergent to $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m)$ in a Banach vector space $(L^\infty_\mu(\mathbf{R}^n))^m$ as $i \to +\infty$.

Let $f(x)$ be a function whose values are real or $\pm\infty$ and whose domain is a subset S of $\mathbf{R}^m$. $f(x)$ is called a convex function on S if the set $\{(x, y) \mid x \in S,\ y \in \mathbf{R},\ y \geqslant f(x)\}$ is convex as a subset of $\mathbf{R}^{m+1}$. Then it is easily known that $f(x)$ is convex from S to $(-\infty, +\infty]$ if and only if

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k) \leqslant \lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_k f(x_k) \tag{4}$$

whenever S is a convex subset of $\mathbf{R}^m$, $x_i \in S$ $(i = 1, 2, \dots)$, $\lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \dots, \lambda_k \geqslant 0, \lambda_1 + \lambda_2 + \dots + \lambda_k = 1$. This is called Jensen's inequality as $S = \mathbf{R}^m$.

Inequalities of integrals of functions which are the composition of convex functions and vector-valued functions in a weakly* compact subset of $(L^\infty_\mu(\mathbf{R}^n))^m$ are shown in [4] (see Theorem 3 in Section 3). In [1], if a special nonnegative convex function is considered and a weakly convergent sequence is constrained in $(L^1_\mu(\mathbf{R}^n))^2$, similar integrals of their composite functions are obtained (see Example 2 in Section 2). Therefore in this paper, by Fatou's lemma, we are going to give inequalities of integrals of functions which are the composition of nonnegative continuous convex functions on a vector space $\mathbf{R}^m$ and vector-valued functions in a weakly compact subset of a Banach vector space generated by m L^p_μ-spaces for $1 \leqslant p < +\infty$. Also, the same inequalities hold if these vector-valued functions are in a weakly* compact subset of a Banach vector space generated by m L^∞_μ-spaces instead.

The plan of this paper is as follows. In Section 2, we obtain inequalities of integrals of the composite functions $f(u)$ where u is a limit of a weakly convergent sequence $\{u_i\}_{i=1}^{+\infty}$ in $(L^p_\mu(\mathbf{R}^n))^m$ for $1 \leqslant p < +\infty$ and $f = f(x)$ is a nonnegative convex function from $\mathbf{R}^m$ to $\mathbf{R}$. A similar result for a weakly* convergent sequence is shown in Section 3.

2. Inequalities for weakly convergent sequences

The basic concepts have been introduced in the previous section. In this section we show inequalities of integrals of functions which are the composition of nonnegative continuous convex functions on a vector space $\mathbf{R}^m$ and vector-valued functions in a weakly compact subset of a Banach vector space generated by m L^p_μ-spaces for $1 \leqslant p < +\infty$. That is the following

Theorem 1. *Suppose that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly converges in $(L^p_\mu(\mathbf{R}^n))^m$ to u for $1 \leqslant p < +\infty$ as $i \to +\infty$, where m and n are two positive integers. If $f(x)$ is a nonnegative continuous convex function from $\mathbf{R}^m$ to $\mathbf{R}$, then*

$$\varliminf_{i\to+\infty} \int_\Omega f(u_i)\,d\mu \geqslant \int_\Omega f(u)\,d\mu \tag{5}$$

for any measurable set $\Omega \subseteq \mathbf{R}^n$.

Remark 1. For $m = 1$, Theorem 1 can be also written as follows. Suppose that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly converges in $L^p_\mu(\mathbf{R}^n)$ to u for $1 \leqslant p < +\infty$ as $i \to +\infty$, where n is a positive integer. If $f(x)$ is a nonnegative continuous convex function from $\mathbf{R}$ to $\mathbf{R}$, then the inequality (5) holds.

Remark 2. Let S denote a convex subset of $\mathbf{R}$ and S^m the usual vector space of real m-tuples $x = (x_1, x_2, \ldots, x_m)$ where $x_i \in S$ $(i = 1, 2, \ldots, m)$. Then S^m is also convex. Suppose that $\{u_i\}_{i=1}^{+\infty}$ weakly converges in $(L^p_\mu(\mathbf{R}^n))^m$ to u for $1 \leqslant p < +\infty$ as $i \to +\infty$. Assume that all the values of $\{u_i\}_{i=1}^{+\infty}$ and of u belong to S^m and that $f(x)$ is a nonnegative continuous convex function from S^m to $\mathbf{R}$. Then the inequality (5) also holds.

Example 1. Assume that $\{u_i\}_{i=1}^{+\infty}$ is a nonnegative sequence which weakly converges in $L^1_\mu(\mathbf{R}^n)$ to u. Then

$$\varliminf_{i\to+\infty} \int_\Omega u_i\,d\mu \geqslant \int_\Omega u\,d\mu \tag{6}$$

for any measurable set $\Omega \subseteq \mathbf{R}^n$. This is obviously a special case of Remark 2 when $m = 1$ and $f(x) = x$ is defined in $[0, +\infty]$.

Example 2. Step 2 of the theorem proof of DiPerna and Lions in [1] shows the following result that

$$\varliminf_{i\to+\infty}\int_{\mathbf{R}^n} F(a_i,b_i)\,d\mu \geqslant \int_{\mathbf{R}^n} F(a,b)\,d\mu$$

if two positive sequences $\{a_i\}_{i=1}^{+\infty}$ and $\{b_i\}_{i=1}^{+\infty}$ weakly converge in $L^1_\mu(\mathbf{R}^n)$ to a and b, respectively, where $F(x,y)=(x-y)\log(\frac{x}{y})$ for $x>0$ and $y>0$. Obviously, since $F(x,y)$ is a nonnegative continuous convex function defined in $(0,+\infty)\times(0,+\infty)$, DiPerna and Lions' result is a special example of Remark 2 for the case of $p=1$ and $m=2$.

We are below going to give a simple proof of Theorem 1. In order to prove Theorem 1, we first recall the following result (see [2]):

Lemma 1. *Given a measure space* $(X,\mathcal{A},\mu)$, $1\leqslant p<+\infty$, *and* $\{u_n,u\colon n=1,2,\ldots\}\subseteq L^p_\mu(X)$. *Assume* $u_n\to u$ *weakly. Then there is a subsequence* $\{u_{n_k}\colon k=1,2,\ldots\}$ *whose arithmetic means* $\frac{1}{m}\sum_{k=1}^m u_{n_k}$ *converge in the* $L^p_\mu(X)$*-topology to* u.

Banach and Saks only proved Lemma 1 for the $1<p<+\infty$ cases; the result for $L^1_\mu(X)$ was shown by Szlenk in 1965. Using Lemma 1, we can easily prove Theorem 1.

Proof of Theorem 1. Put $\alpha_i=\int_\Omega f(u_i)\,d\mu$ $(i=1,2,\ldots)$ and $\alpha=\varliminf_{i\to+\infty}\int_\Omega f(u_i)\,d\mu$ for any measurable set $\Omega\subseteq\mathbf{R}^n$. Then there exists a subsequence of $\{\alpha_i\}_{i=1}^{+\infty}$ such that this subsequence, denoted without loss of generality by $\{\alpha_i\}_{i=1}^{+\infty}$, converges to α as $i\to+\infty$.

Since $u_i\to u$ weakly in $(L^p_\mu(\mathbf{R}^n))^m$ for $1\leqslant p<+\infty$, it can be shown by Lemma 1 that there exists a subsequence $\{u_{i_j}\colon j=1,2,\ldots\}$ such that $\frac{1}{k}\sum_{j=1}^k u_{i_j}\to u$ in $(L^p_\mu(\mathbf{R}^n))^m$ for $1\leqslant p<+\infty$ as $k\to+\infty$. Thus there exists a subsequence of $\{\frac{1}{k}\sum_{j=1}^k u_{i_j}\colon k=1,2,\ldots\}$ such that this subsequence (also denoted without loss of generality by $\{\frac{1}{k}\sum_{j=1}^k u_{i_j}\colon k=1,2,\ldots\}$) satisfies that, as $k\to+\infty$,

$$\frac{1}{k}\sum_{j=1}^k u_{i_j}\to u \quad \text{a.e. in } \mathbf{R}^n. \tag{7}$$

On the other hand, since $f(x)$ is a nonnegative continuous convex function from $\mathbf{R}^m$ to $\mathbf{R}$, we have

$$f\left(\frac{1}{k}\sum_{j=1}^k u_{i_j}\right)\leqslant\frac{1}{k}\sum_{j=1}^k f(u_{i_j}). \tag{8}$$

By (8) and Fatou's lemma, it follows that

$$\int_\Omega \varliminf_{k\to+\infty} f\left(\frac{1}{k}\sum_{j=1}^k u_{i_j}\right)d\mu\leqslant\varliminf_{k\to+\infty}\frac{1}{k}\sum_{j=1}^k\int_\Omega f(u_{i_j})\,d\mu. \tag{9}$$

Combining (7) and (9), we can know that

$$\int_\Omega f(u)\,d\mu\leqslant\varliminf_{k\to+\infty}\frac{1}{k}\sum_{j=1}^k\int_\Omega f(u_{i_j})\,d\mu, \tag{10}$$

or equivalently,

$$\int_{\Omega} f(u)\,d\mu \leqslant \varliminf_{k\to+\infty} \frac{1}{k}\sum_{j=1}^{k}\alpha_{i_j} \tag{11}$$

which gives (5) since $\alpha_i \to \alpha$ as $i \to +\infty$. This completes our proof. □

3. Inequalities for weakly* convergent sequences

In the previous section we have given inequalities of integrals of the composite functions for weakly convergent sequences in $(L^p_\mu(\mathbf{R}^n))^m$ for $1 \leqslant p < +\infty$. A similar result for weakly* convergent sequences in $(L^\infty_\mu(\mathbf{R}^n))^m$ can be also obtained below in this section.

Using the process of the proof of Theorem 1, we can prove the following theorem.

Theorem 2. *Assume that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly* converges in $(L^\infty_\mu(\mathbf{R}^n))^m$ to u as $i \to +\infty$, where m and n are two positive integers. If $f(x)$ is a nonnegative continuous convex function from $\mathbf{R}^m$ to $\mathbf{R}$, then the inequality* (5) *also holds for any measurable set $\Omega \subseteq \mathbf{R}^n$.*

Proof. Put $\Omega_R = \Omega \cap \{w\colon |w| < R,\ w \in \mathbf{R}^n\}$. Then Ω_R is a bounded set in $\mathbf{R}^n$ for any fixed positive real number R. Since $u_i \to u$ weakly* in $(L^\infty_\mu(\mathbf{R}^n))^m$, $u_i \to u$ weakly* in $(L^\infty_\mu(\Omega_R))^m$. Hence, by $L^\infty(\Omega_R) \subset L^1(\Omega_R)$, it can be easily known that $u_i \to u$ weakly in $(L^1_\mu(\Omega_R))^m$. Then, using the process of the proof of Theorem 1, we can get

$$\varliminf_{i\to+\infty} \int_{\Omega_R} f(u_i)\,d\mu \geqslant \int_{\Omega_R} f(u)\,d\mu. \tag{12}$$

It follows from the nonnegativity of the convex function f that

$$\varliminf_{i\to+\infty} \int_{\Omega} f(u_i)\,d\mu \geqslant \int_{\Omega_R} f(u)\,d\mu. \tag{13}$$

Finally, by Lebesgue dominated convergence theorem, as $R \to +\infty$, (13) implies (5). Our proof is completed. □

Furthermore, in the case of continuous convex functions, using Mazur's lemma [5], we can deduce

Theorem 3. *Assume that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly* converges in $(L^\infty_\mu(\mathbf{R}^n))^m$ to u as $i \to +\infty$, where m and n are two positive integers. If $f(x)$ is a continuous convex function from $\mathbf{R}^m$ to $\mathbf{R}$, then the inequality* (5) *also holds for any bounded measurable set $\Omega \subset \mathbf{R}^n$.*

Since the nonnegativity of the convex function appears in Theorem 2 but it is not required in Theorem 3, the condition in Theorem 3 is obviously weaker than that in Theorem 2. We can also deduce Theorem 2 from Theorem 3. Theorem 3 can be easily proved using the following lemma.

Lemma 2. *Assume $u_n \to u$ weakly in a normed linear space. Then there exists, for any $\epsilon > 0$, a convex combination $\sum_{k=1}^n \lambda_k u_k$ ($\lambda_k \geqslant 0, \sum_{k=1}^n \lambda_k = 1$) of $\{u_k\colon k = 1, 2, \ldots\}$ such that $\|u - \sum_{k=1}^n \lambda_k u_k\| \leqslant \epsilon$ where $\|v\|$ is a norm of v in the space.*

This is called Mazur's lemma. Its proof can be found in the book of Yosida [5].

In fact, Theorem 3 is a part of the results given by Ying [4]. However, we still give its proof below.

Proof of Theorem 3. Put $\alpha_i = \int_\Omega f(u_i)\,d\mu$ $(i = 1, 2, \ldots)$ and $\alpha = \underline{\lim}_{i\to+\infty} \int_\Omega f(u_i)\,d\mu$ for any bounded measurable set Ω. Then there exists a subsequence of $\{\alpha_i\}_{i=1}^{+\infty}$ such that this subsequence, denoted without loss of generality by $\{\alpha_i\}_{i=1}^{+\infty}$, converges to α as $i \to +\infty$.

Since $u_i \to u$ weakly* in $(L_\mu^\infty(\mathbf{R}^n))^m$, $u_i \to u$ weakly* in $(L_\mu^\infty(\Omega))^m$. Hence, by $L^\infty(\Omega) \subset L^1(\Omega)$, it can be easily known that $u_i \to u$ weakly in $(L_\mu^1(\Omega))^m$. It follows from Lemma 2 that, for any natural number j, there exists a convex combination $\sum_{k=j}^{N(j)} \lambda_k u_k$ $(\lambda_k \geqslant 0, \sum_{k=j}^{N(j)} \lambda_k = 1)$ of $\{u_k\colon k = j, j+1, \ldots\}$ such that $\|u - \sum_{k=j}^{N(j)} \lambda_k u_k\|_\infty \leqslant \frac{1}{j}$ where $N(j)$ is a natural number which depends on j and $\{u_k\colon k = j, j+1, \ldots\}$, $\|v\|_\infty$ represents a norm of v in $(L_\mu^\infty(\Omega))^m$. Put $v_j = \sum_{k=j}^{N(j)} \lambda_k u_k$. Then, as $j \to +\infty$, $v_j \to u$ in $(L_\mu^\infty(\Omega))^m$. Since $f(x)$ is continuous, for any given positive ϵ, there exists a natural number N such that

$$f(u) < f(v_j) + \frac{\epsilon}{\operatorname{mes}(\Omega)} \tag{14}$$

for all $j > N$, where $\operatorname{mes}(\Omega)$ represents the measure of Ω. By the convexity of the function $f(x)$, integrating (14) gives

$$\int_\Omega f(u)\,d\mu \leqslant \sum_{k=j}^{N(j)} \lambda_k \int_\Omega f(u_k)\,d\mu + \epsilon, \tag{15}$$

or equivalently,

$$\int_\Omega f(u)\,d\mu \leqslant \sum_{k=j}^{N(j)} \lambda_k \alpha_k + \epsilon. \tag{16}$$

First let $j \to +\infty$ and then $\epsilon \to 0$, (16) gives (5) since $\alpha_k \to \alpha$ as $k \to +\infty$. This completes the proof. □

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