Theoretical Computer Science 12 (1980) 229–254
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UNIQUE FIXED POINTS VS. LEAST FIXED POINTS

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Communicated by M. Nivat
Received April 1979

Abstract. The aim of this paper is to compare two approaches to the semantics of programming languages: the least fixed point approach, and the unique fixed point approach. Briefly speaking, we investigate here the problem of existence of extensions of algebras with the unique fixed point property to ordered algebras with the least fixed point property, that preserve the fixed point solutions. We prove that such extensions always exist, the construction of a free extension is given. It is also shown that in some cases there is no 'faithful' extension, i.e. some elements of a carrier are always collapsed.

0. Introduction

There are known two approaches to the semantics of programming languages, which are using fixed points. Historically the first approach is based on the existence of *least* fixed points of ω-continuous mappings defined on ω-complete posets (cf. [2, 8, 9,]). The second approach, dealing with *unique* fixed points of certain maps, was originated by Elgot [4]. While the first approach seems to be more intuitive, due to the possibility of generating those fixed points by increasing sequences of 'finite pieces of information', nevertheless there exist structures, where the order relation is somehow unnatural and superfluous, and where the unique-fixed-point approach can be applied, as this is the case, for example, with trees (cf. [3]).

Those two approaches have been studied independently. The original motivation for this paper is to compare them in the sense we are going to describe now briefly.

The least-fixed-point approach is represented here by regular algebras. These are, roughly speaking, all those algebras with ordered carriers where one can get least solution of an algebraic system of equations by taking the least upper bound of the ω-chain of iterations. The notion of a regular algebra has been introduced in [10]. The equivalence of this notion to the notion of a rational algebraic theory (cf. [1]) is shown in [12]. The application of regular algebras to the semantics of nondeterministic recursive procedures, where the ω-continuous algebras cannot be applied, the reader may find in [13].

The unique-fixed-point approach is represented here by iterative algebras. Informally, these are all algebras with the property that every 'ideal' (i.e. nontrivial) algebraic system of fixed point equations has a unique solution. The notion of an iterative algebra is introduced in this paper, and it is shown here that it corresponds to the notion of iterative algebraic theory.

Here we would like to justify our choice of algebras rather than algebraic theories. Beside of author's preference of algebras there are some sound reasons for our choice. It seems that despite of a strong tendency to work within a language of category theory there is still a remarkable number of mathematicians better understanding (or prefering) results formulated in algebraic terms rather than in categorical ones. Secondly, and perhaps in connection with the previous argument, it seems that the methods of proof applicable to a single structure, as this is the case in this paper, are more transparent when using a traditional language of algebra rather than that of category theory. (We are not discussing here a useful role of category theory in the process of generalizing and/or comparing results.)

On the other hand, having clear relationships between the algebraic structures we are dealing in this paper with and their categorical counterparts, it is a routine matter to reformulate our results in terms of categorical notions.

Let $k, n \in \omega$, and let $p = (p_0, \ldots, p_{n-1})$ be a vector of $n + k$-ary polynomial symbols (i.e. finite trees) over the signature Σ. Assume moreover that none of p_i's is a variable (i.e. p is ideal). In a given iterative Σ-algebra $\boldsymbol{A}$, for every vector of parameters $a \in \boldsymbol{A}^k$ one may solve the system of equations

$$x_0 = p_{0_{\boldsymbol{A}}}(x_0, \ldots, x_{n-1}, a), \ldots, x_{n-1} = p_{n-1_{\boldsymbol{A}}}(x_0, \ldots, x_{n-1}, a) \tag{0.1}$$

getting the unique solution: $(p_{\boldsymbol{A}})^+(a)$. Similarly, in a given regular Σ-algebra $\boldsymbol{B}$, for every vector of parameters $b \in \boldsymbol{B}^k$ one may solve the system

$$x_0 = p_{0_{\boldsymbol{B}}}(x_0, \ldots, x_{n-1}, b), \ldots, x_{n-1} = p_{n-1_{\boldsymbol{B}}}(x_0, \ldots, x_{n-1}, b) \tag{0.2}$$

getting the least solution: $(p_{\boldsymbol{B}})^\nabla(b)$.

In both cases it gives rise to functions: $(p_{\boldsymbol{A}})^+ : \boldsymbol{A}^k \to \boldsymbol{A}^n$, and $(p_{\boldsymbol{B}})^\nabla : \boldsymbol{B}^k \to \boldsymbol{B}^n$.

The following definition is a basic one for this paper. An iterative Σ-algebra $\boldsymbol{A}$ is said to *admit a regular extension* iff there is a regular algebra $\boldsymbol{A}_{\mathrm{R}}$ and a map $\varphi_{\boldsymbol{A}} : \boldsymbol{A} \to \boldsymbol{A}_{\mathrm{R}}$ such that:

for any vector p of polynomial symbols as above,

$$\varphi_{\boldsymbol{A}}(p_{\boldsymbol{A}})^+ = (p_{\boldsymbol{A}_{\mathrm{R}}})^\nabla \varphi_{\boldsymbol{A}},$$

for any regular Σ-algebra $\boldsymbol{B}$ and for any map $f : \boldsymbol{A} \to \boldsymbol{B}$ satisfying

$$f(p_{\boldsymbol{A}})^+ = (p_{\boldsymbol{B}})^\nabla f, \quad \text{for all } p\text{'s},$$

there is exactly one regular homomorphism

$$f^* : \boldsymbol{A}_{\mathrm{R}} \to \boldsymbol{B} \quad \text{with } f^* \varphi_{\boldsymbol{A}} = f;$$

f^* is a regular homomorphism which means: for all polynomial vectors p's

$$f^*(p_{\boldsymbol{A}_{\mathrm{R}}})^\nabla = (p_{\boldsymbol{B}})^\nabla f^*.$$

In (0.3) and (0.4) we use the same notation for a function and for its extension to vectors (componentwise).

Condition (0.3) states that $\varphi_{\mathbf{A}}$ 'translates' results, i.e. interpretation of the unique solution of fixed point equations with parameters, in the iterative algebra $\mathbf{A}$ is the same as least solution of the same system with interpreted parameters in the regular algebra $\mathbf{A}_{\mathbf{R}}$. Condition (0.4) states that the construction $\mathbf{A} \to (\varphi_{\mathbf{A}}, \mathbf{A}_{\mathbf{R}})$ is universal (cf. [7]).

The aim of this paper is to prove the *existence* and provide the *construction* of a *regular extension* for *every iterative algebra.* It is shown that every iterative algebra admits exactly one (up to isomorphism) regular extension. It is also shown that there exists an iterative algebra $\mathbf{A}$ which cannot be 'freely embedded' into any regular algebra, i.e. there is no regular extension $(\varphi_{\mathbf{A}}, \mathbf{A}_R)$ with injective $\varphi_{\mathbf{A}}$.

In the author's opinion the above-mentioned construction is quite hard to carry due to unsatisfactory development of combinatorial methods for infinite trees.

The paper is divided into seven sections. Sections 1 to 4 are of preliminary character. They collect basic definitions and results to be used in the sequel. Section 1 fixes some notations and definitions. In Section 2 and 3 we introduce the notion of an iterative algebra, show its connection with Elgot's iterative theories, and give a construction of free iterative theories by using the result of Ginali [5] relating free iterative algebras with free rational theories. In Section 4 we state some basic results on regular algebras, that are used in Section 6. In Section 5 we formulate the problem which was described above. Section 6 contains the solution of this problem. Some open problems are stated in Section 7.

1. Preliminary notations and definitions

1.1. Suppose X is a set, n-vectors over X (or simply vectors), i.e. elements of X^n will be treated as functions from $n = \{0, \ldots, n-1\}$ into X. If $x \in X^n$ and $i < n$, then x_i denotes the ith component of x. The map that associates with each vector its ith component is called the ith projection and will be denoted by e_i^n. We denote by ω the set of all finite ordinals.

1.2. If X, Y, Z are sets and $f: X \to Y$, $g: Y \to Z$ are functions, then the composition of f and g will be denoted by $gf: X \to Z$.

Each map $f: X \to Y$ determines for any $n < \omega$ a map $\tilde{f}: x^n \to Y^n, \tilde{f}(x) = (f(x_0), \ldots, f(x_{n-1}))$. Observe, that using notations from 1.1 one gets

$$\tilde{f}(x) = fx \quad \text{for any } x \in X^n.$$

Usually we will not introduce a special symbol for the map $\tilde{f}$ and we will denote it simply by f. Thus $f(x)$ means either the value of f at x, if $x \in X$, or the vector $(f(x_0), \ldots, f(x_{n-1}))$ if $x \in X^n$.

1.3. If Σ is a signature (i.e. a ranked alphabet) and $n < \omega$, then by $T_\Sigma(n)$ we denote the Σ-algebra of all *finite terms* over Σ, with at most n variables. A vector $p \in T_\Sigma(n)^k$ is called an *ideal* vector of terms if none of its components is a variable.

Finite terms over Σ with at most n variables will be called sometimes n-ary *Σ-polynomial symbols.*

1.4. A category T is an *algebraic theory* (cf. [6]) if $\mathrm{Ob}(T) = \omega$, for each $n \in \omega$ there are *basic morphisms* $\{e_0^n, \ldots, e_{n-1}^n\} \subseteq T(1, n)$. For each n, k there is defined a *source tupling* operation:

$$\alpha_0, \ldots, \alpha_{n-1} \in T(1, k) \mapsto (\alpha_0, \ldots, \alpha_{n-1}) \in T(n, k).$$

Moreover, the above notions are supposed to satisfy the following axioms for any $n, k \in \omega$, $\alpha \in T(n, k)$, $\alpha_0, \ldots, \alpha_{n-1} \in T(1, k)$:

$$(e_0^n\alpha, \ldots, e_{n-1}^n\alpha) = \alpha, \tag{1.4.1}$$

$$\alpha(e_0^k, \ldots, e_{k-1}^k) = \alpha, \tag{1.4.2}$$

$$e_i^n(\alpha_0, \ldots, \alpha_{n-1}) = \alpha_i \quad \text{for all } i < n. \tag{1.4.3}$$

We compose morphisms in algebraic theories in converse order to that for functions, i.e. if $\alpha : n \to k$, $\beta : k \to p$, then $\alpha\beta : n \to p$ denotes the composition of α and β.

The above conditions give for each $n \in \omega$ the unique morphism $O_n : O \to n$.

1.5. An algebraic theory T is said to be *ideal* (cf. [4]) if for every $n \in \omega$ and for any non-base morphism $\alpha \in T(1, n)$, $\alpha\beta$ is non-base for every mosphism $\beta \in T(n, k)$, $k \in \omega$.

A morphism $\alpha \in T(n, k)$ is said to be *ideal* if each of its components is an ideal morphism.

1.6. An ideal theory T is said to be *iterative* (cf. [4]) if for any $n, k \in \omega$, and for any ideal mosphism $\alpha \in T(n, n+k)$, the equation

$$x = \alpha(x, e_0^k, \ldots, e_{k-1}^k) \tag{1.6.1}$$

has unique solution in $(T(n, k)$. The solution of (1.6.1) will be denoted by α^+.

1.7. A *theory morphism* F between iterative algebraic theories T_1, T_2 is a functor between underlying categories which is an identity on objects and preserves: basic mosphisms, the property of 'being ideal morphism', and $^+$ operation.

1.8. If Σ is a signature and X is a set, then by $\Sigma(X)$ we denote the extension of Σ by 'constants' from X, i.e. $\Sigma(X)$ is a signature with $\Sigma(X)_0 = \Sigma_0 \sqcup X$ ($\sqcup$ denotes disjoint set-union) and $\Sigma(X)_n = \Sigma_n$ for $n > 0$.

1.9. Suppose Σ is a signature. By a *Σ-tree* t we mean a partial function $t:\omega^* \multimap \bigcup_{n<\omega} \Sigma_n$ (here ω^* denotes the set of all finite words over ω and Λ the empty word), such that for all $w \in \omega^*$, $i \in \omega$:

$$\text{if } wi \in \mathrm{Dom}(t), \text{ then } w \in \mathrm{Dom}(t); \tag{1.9.1}$$

$$\text{if } w \in \mathrm{Dom}(t) \text{ and } t(w) \in \Sigma_n, \text{ then } wi \in \mathrm{Dom}(t) \text{ for all } i<n; \tag{1.9.2}$$

$$\Lambda \in \mathrm{Dom}(t). \tag{1.9.3}$$

A tree t is *finite* if $\mathrm{Dom}(t)$ is finite. In particular T_Σ is the set of all finite Σ-trees. Denote by T_Σ^∞ the set of all Σ-trees.

1.10. Let $t \in T_\Sigma^\infty$, and $w \in \mathrm{Dom}(t)$. By $t \restriction w$ we denote a subtree of t determined by the path w, i.e.

$$\mathrm{Dom}(t \restriction w) = \{u \in \omega^*: wu \in \mathrm{Dom}(t)\}, \tag{1.10.1}$$

$$t \restriction w(u) = t(wu) \quad \text{for } u \in \mathrm{Dom}(t \restriction w). \tag{1.10.2}$$

A tree t is said to be of *finite index* if $\{t \restriction w: w \in \mathrm{Dom}(t)\}$ is finite. Trees of finite index are called sometimes *regular trees.*

Now we describe tree substitution. If $n \in \omega$ and $t \in T_{\Sigma(n)}^\infty$ (i.e. t is a Σ-tree with n variables), then for $p \in (T_\Sigma^\infty)^n$, $t[p]$ is a Σ-tree defined as follows:

$$\mathrm{Dom}(t[p]) = \mathrm{Dom}(t) \cup \bigcup_{i<n} \{wv: t(w) = i, v \in \mathrm{Dom}(p_i)\}, \tag{1.10.3}$$

$$t[p](u) = \begin{cases} t(u), & \text{if } u \in \mathrm{Dom}(t) \text{ and } t(u) \notin n, \\ p_i(v), & \text{if } u = wv, t(w) = i, v \in \mathrm{Dom}(p_i). \end{cases} \tag{1.10.4}$$

1.11. On the set ω^* we define a partial order $\leqslant$ by:

$$w \leqslant u \text{ iff } u = wv \text{ for some } v \in \omega^*.$$

We shall also write $w < u$ to indicate that $w \leqslant u$ and $w \neq u$.

1.12. If $(P, \leqslant)$ is a poset (i.e. partially ordered set), then we extend the partial order $\leqslant$ to subsets of $P: X \leqslant Y$ iff for any $x \in X$ there is $y \in Y$ such that $x \leqslant y$. In particular, if $X \subseteq P$ and $a \in P$, then $X \leqslant a$ means that a is an upper bound of X.

The same convention we use for arbitrary binary relations.

If $R \subseteq P \times P$, and $X, Y \subseteq P$, then XRY iff for any $x \in X$ there is a $y \in Y$ with xRy.

1.13. If $R \subseteq X \times X$ is an equivalence relation on X, and $a \in X$, then $|a|_R = \{x \in X: (a, x) \in R\}$. We shall frequently omit the subscript R if it will be clear from the context what relation R is meant. Since $|\ |_R: X \to X/R = \{|x|_R: x \in X\}$ is a function, $|A|_R$ will denote the set $\{|a|_R: a \in A\}$, for any set $A \subseteq X$.

1.14. If m and n are finite cardinals, then by $m \times n$ we denote a finite cardinal that corresponds to $m \cdot n$. Thus, in fact, $m \times n = \{I : i < m \cdot n\}$. Sometimes it will be more convenient to treat elements of $m \times n$ as ordered pairs (i, j) with $i < m, j < n$. Therefore we make a convention to treat (i, j) as a number of corresponding ordered pair, under some fixed numbering.

If m, n are finite cardinals, then $m \oplus n$ denotes a finite cardinal that corresponds to $m + n$. Sometimes we will treat $m \oplus n$ as a set $(\{0\} \times m) \cup (\{1\} \times n)$. So in this case we make a convention to treat $(0, i)$ for $i < m$ as i, while $(1, j)$ for $j < n$ as $m + j$.

2. Iterative algebras

2.1. A Σ-algebra $\boldsymbol{A}$ is said to be *iterative* if for any $n, k \in \omega$ and for any ideal $p \in T_\Sigma(n+k)^n$ the following hold:

For any $a \in A^k$ the equation $x = p_{\boldsymbol{A}}(x, a)$ has unique solution in A^n. Denote this solution by $(p_{\boldsymbol{A}})^+(a)$, (2.1.1)

There is $a \in A^k$ such that $a_0 \neq ((p_{\boldsymbol{A}})^+(a))_0$ (2.1.2)

2.2. Remarks. (1) Condition (2.1.1) guarantee that for ideal p the map $(p_{\boldsymbol{A}})^+ : A^k \to A^n$ is well defined and $p_{\boldsymbol{A}}((p_{\boldsymbol{A}})^+(x), x) = (p_{\boldsymbol{A}})^+(x)$ for $x \in A^k$.

(2) Condition (2.1.2) says that $(p_{\boldsymbol{A}})_0^+$ is not a projection.

2.3. With each iterative algebra we associate a set of maps one can get solving (2.1.1) and treating a as a vector of parameters.

Suppose A is an iterative Σ-algebra. A map $f : A^k \to A$ is said to be an (k-ary) *iterative polynomial* if either it is a projection, or if there is n and $p \in T_\Sigma(n+k)^n$ being ideal, with $f = (p_{\boldsymbol{A}})_0^+$. Denote by $k\mathrm{IP}(\boldsymbol{A})$ the set of all k-ary iterative polynomials in $\boldsymbol{A}$. The following result follows immediately from our definitions.

2.4. Proposition. *Suppose* $\boldsymbol{A}, \boldsymbol{B}$ *are iterative* Σ*-algebras. If* $h : \boldsymbol{A} \to \boldsymbol{B}$ *is a* Σ*-homomorphism; Then for any* $n, k \in \omega$ *and for any ideal vector* $p \in T_\Sigma(n+k)^n$ *and for any* $a \in A^k$,

$$h((p_{\boldsymbol{A}})_0^+(a)) = (p_{\boldsymbol{B}})_0^+(h(a)).$$

2.5. With a given iterative algebra $\boldsymbol{A}$ we associate an algebraic theory $I(\boldsymbol{A})$, where for $k \in \omega$, $I(\boldsymbol{A})(1, k) = \mathrm{kIP}(\boldsymbol{A})$. Composition of morphisms is just usual composition of functions, basic morphisms are projections, and source tupling is just an ordinary tupling of functions, i.e.

$$f_0, \ldots, f_{n-1} \mapsto \lambda x \cdot (f_0(x), \ldots, f_{n-1}(x)).$$

One easily checks that iterative polynomials are closed under composition, so $I(\boldsymbol{A})$ really does form an algebraic theory. In fact one may say more about $I(\boldsymbol{A})$.

2.6. Proposition. *If* $\mathbf{A}$ *is an iterative algebra, then* $I(\mathbf{A})$ *is an iterative algebraic theory.*

Proof. One easily checks that $I(\mathbf{A})$ is an ideal theory. Now we are going to prove that $I(\mathbf{A})$ satisfies condition (1.6.1). Let $\alpha : A^{n+k} \to A^n$ be an ideal vector of iterative polynomials. We are looking for solutions of

$$\alpha(\xi, e_0^k, \ldots, e_{k-1}^k) = \xi \quad \text{in } \xi : A^k \to A^n.$$

This can be rewritten as

$$\alpha(\xi(x), x) = \xi(x). \tag{2.6.1}$$

It is easy to check that there is a $n+n+k$-ary ideal n-vector $p \in T(n+n+k)^n$ such that $\alpha = (p_{\mathbf{A}})^+$.

So $\alpha(y, x)$ $(y \in A^n, x \in A^k)$ is the unique solution of the equation:

$$z = p_{\mathbf{A}}(z, y, x) \quad \text{in } z \in A^n.$$

Suppose now that $\xi(x)$ is a solution of (2.6.1), then it satisfies the equation

$$\xi(x) = p_{\mathbf{A}}(\xi(x), \xi(x), x).$$

Hence it is a solution of

$$z = p_{\mathbf{A}}(z, z, x). \tag{2.6.2}$$

Thus, there is at most one solution of (2.6.1).

Conversely, if $\varphi(x)$ is a solution of (2.6.2), then by uniqueness we get

$$\varphi(x) = \alpha(\varphi(x), x)$$

i.e. it is a solution of (2.6.1).

It means that (2.6.1) has unique solution for any vector of parameters.

Now we are going to show that a kind of converse result to that of Proposition 2.6 holds.

2.7. Proposition. *If* T *is an iterative algebraic theory, then there exists a signature* Σ *and an iterative* Σ*-algebra* $\mathbf{A}$ *with* $T \cong I(\mathbf{A})$ (*here isomorphism means an isomorphism of iterative theories*).

Proof. Take an iterative theory T. let $X = \bigcup_{n \in \omega} T(1, n)$. Define a binary relation $\equiv$ in X by

$$\alpha \equiv \beta \text{ iff } \alpha(e_0^k, \ldots, e_{n-1}^k) = \beta(e_0^k, \ldots, e_{m-1}^k),$$

where $\alpha \in T(1, n), \beta \in T(1, m)$, $k = \max(n, m)$ One easily checks that $\equiv$ is an equivalence relation in X, and

If $\alpha \in T(1, n), \beta_0, \ldots, \beta_{n-1}, \gamma_0, \ldots, \gamma_{n-1} \in x$, and $\beta_i \equiv \gamma_i$ for $i < n$, then

$$\alpha(\beta_0, \ldots, \beta_{n-1}) \equiv \alpha(\gamma_0, \ldots, \gamma_{n-1}). \tag{2.7.1}$$

This enables us to define a structure of Σ-algebra, where $\Sigma_n = T(1,n)\backslash\{e_0^n,\ldots,e_{n-1}^n\}$, on the set $A = X/\equiv$. For each $\alpha\in\Sigma_n$ we put

$$\alpha_A(|\beta_0|,\ldots,|\beta_{n-1}|) = |\alpha(\beta_0^*,\ldots,\beta_{n-1}^*)|,$$

where β_i^* is chosen from $|\beta_i|$ for each $i<n$, so that $\beta_0^*,\ldots,\beta_{n-1}^*$ have the same range.

Since T is closed under composition, each Σ-polynomial in A is either a projection or a Σ-operation.

We are going to prove that the same property holds for iterative polynomials. Let $\alpha = (\alpha_0,\ldots,\alpha_{n-1})\in\Sigma_{n+k}$ and let $a\in A^k$, i.e. $a = (|\beta_0|,\ldots,|\beta_{k-1}|)$. We may assume that $\beta_i\in T(1,m)$ for all $i<n$ and some m. Let $\beta = (\beta_0,\ldots,\beta_{k-1})$, thus $\beta\in T(k,m)$.

Observe first that for any $i<n$,

$$\begin{aligned}&\alpha_{iA}(|(\alpha^+\beta)_0|,\ldots,|(\alpha^+\beta)_{n-1}|, a)\\ &\quad = |\alpha_i(\alpha^+\beta,\beta)| = |\alpha_i(\alpha^+, e_0^k,\ldots,e_{k-1}^k)\beta| = |\alpha(\alpha^+\beta)_i|.\end{aligned}$$

Hence, the vector $(|(\alpha^+\beta)_0|,\ldots,|(\alpha^+\beta)_{n-1}|)$ is a solution of the system

$$\alpha_A(x,a) = x. \tag{2.7.2}$$

Assume now that $(|\gamma_0|,\ldots,|\gamma_{n-1}|)$ is a solution of (2.7.2). We may assume without lost of generalization that $\gamma = (\gamma_0,\ldots,\gamma_{n-1})\in T(n,p)$ and $p\geqslant m$. Let $\beta^* = (\beta_0^*,\ldots,\beta_{k-1}^*)\in T(k,p)$ be such that $\beta_i^*\equiv\beta_i$ for $i<n$. By our assumption we get for any $i<n$, $\alpha_{iA}(|\gamma_0|,\ldots,|\gamma_{n-1}|,\alpha) = |\gamma_i|$. Therefore

$$\alpha_i(\gamma,\beta^*)\equiv\gamma_i. \tag{2.7.3}$$

But since both morphisms are with the same range, condition (2.7.3) implies

$$\alpha_i(\gamma,\beta^*) = \gamma_i.$$

By the uniqueness condition for iterative theories $\gamma_i = e_i^n\alpha^+\beta^*$. Hence, $\gamma_i\equiv(\alpha^+\beta)_i$ (because $\beta_i\equiv\beta_i^*$). In this way we have proved that

$$(\alpha_A)^+ = (\alpha^+)_A. \tag{2.7.4}$$

Because α^+ is an ideal morphism for any ideal morphism so α (2.7.4) implies property (2.1.2) for the algebra A. Thus A is an iterative Σ-algebra and, as we have proved,

$$k\mathrm{IP}(\mathbf{A}) = \{\alpha_A : \alpha\in T(1,k)\backslash\{e_0^k,\ldots,e_{k-1}^k\}\}\cup\{e_0^k,\ldots,e_{k-1}^k\}.$$

This obviously implies that T and $I(A)$ are isomorphic.

Propositions 2.6 and 2.7 indicate that iterative algebraic theories are categorical counterparts of 'iterative clones' of iterative algebras. One may also prove that iterative algebraic theories play the same role for 'varieties' of iterative algebras (i.e. for classes of iterative algebras definable by certain equations between iterative polynomials) as ordinary algebraic theories (cf. [6]) play for varieties of universal algebras.

In the sequel, for a given iterative theory T, we will denote by A_T the algebra that is provided by Proposition 2.7.

3. Free iterative algebras

3.1. Let Σ be a signature. Denote by $\bar{R}_\Sigma$ the set of all Σ-trees with a finite index (cf. Section 1.10). $\bar{R}_\Sigma$ as a set of Σ-trees can be made easily, in a natural way, into a Σ-algebra. The algebra $\bar{R}_\Sigma$ has been defined for the first time, in [2]. A convenient, equivalent definition of aforementioned trees (which we are using here has been found in [3].

For any set X denote by $\bar{R}_\Sigma(X)$ the Σ-algebra of $\Sigma(X)$-trees of a finite index. Denote by $\Sigma\mathrm{Tr}$ an algebraic theory with $\Sigma\mathrm{Tr}(m, n) = \bar{R}_\Sigma(n)^m$, $m, n \in \omega$. Composition of morphisms in $\Sigma\mathrm{Tr}$ is tree-substitution. Basic morphisms in $\Sigma\mathrm{Tr}(1, n)$ are variable trees (i.e. one-vertex trees labelled by elements of n).

3.2. Theorem ([3, 5]). $\Sigma\mathrm{Tr}$ *is an iterative algebraic theory. This theory is freely generated by* Σ *in the category of iterative theories and theory mosphisms between iterative theories. More exactly, if* T *is an iterative theory and* f *is a function* $f:\Sigma \to T$ s.t. $f(\Sigma_n) \subseteq T(1, n)\backslash\{e_0^n, \ldots, e_{n-1}^n\}$ *for* $n \in \omega$, *then there is exactly one extension of* f *to a theory morphism* $F:\Sigma\mathrm{Tr} \to T$.

We are going to show that $\bar{R}_\Sigma(X)$ is an iterative algebra freely generated by X.

3.3. Theorem. *Let* Σ *be any signature*:

The reduct of $A_{\Sigma\mathrm{Tr}}$ *to a* Σ*-algebra is isomorphic to* $\bar{R}_\Sigma$. *The isomorphism* $\varphi:\bar{R}_\Sigma \to A_{\Sigma\mathrm{Tr}}$ *is given by* $\varphi(t) = |t|$; (3.3.1)

For any set X, $\bar{R}_\Sigma(X)$ *is an iterative* Σ*-algebra*; (3.3.2)

For any iterative Σ*-algebra* A *there is exactly one homomorphism* $h_{\mathbf{A}}:\bar{R}_\Sigma \to A$ (3.3.3)

For any set X, *any iterative* Σ*-algebra* $\mathbf{A}$, *and any function* $f:X \to A$ *there is exactly one homorphism* $\bar{f}:\bar{R}(X) \to A$ *that extends* f. (3.3.4)

Proof. (3.3.1): it is an immediate consequence of Definition 2.1 that reduct of iterative Σ-algebra to a Σ'-algebra ($\Sigma' \subseteq \Sigma$) is again iterative. Thus $A_{\Sigma\mathrm{Tr}}$ treated as Σ-algebra is iterative.

It is a routine checking to prove that φ is a Σ-isomorphism. Since $\bar{R}_\Sigma = \Sigma\mathrm{Tr}(1, 0)$ one easily checks, using Theorem 3.2, that $\bar{R}_\Sigma$ is an iterative Σ-algebra

(3.3.2): By (3.3.1) $\bar{R}_{\Sigma(X)}$ is an iterative $\Sigma(X)$-algebra, therefore $\bar{R}_\Sigma(X)$, as a reduct, is iterative as well.

(3.3.3): By results concerning the algebra of regular trees in [2] follows that $\bar{R}_\Sigma$ has no proper iterative Σ-subalgebras. This result can be expressed in other way as the equality $\bar{R}_\Sigma = O\mathrm{IP}(\bar{R}_\Sigma)$ (i.e. every element of $\bar{R}_\Sigma$ is a nullary iterative polynomial). Hence by proposition 2.4, there is at most one homomorphism from $\bar{R}_\Sigma$ into $\boldsymbol{A}$. Denote by $F_{\boldsymbol{A}}:\Sigma\mathrm{Tr}\to I(\boldsymbol{A})$ the unique theory morphism that extends the map: $\Sigma\ni\sigma\mapsto\sigma_{\boldsymbol{A}}$. Let $h_{\boldsymbol{A}} = F_{\boldsymbol{A}/\Sigma\mathrm{Tr}(1,0)}$. Since $\Sigma\mathrm{Tr}(1,0)=\bar{R}_\Sigma$, $h_{\boldsymbol{A}}$ maps $\bar{R}_\Sigma$ into $\boldsymbol{A}$. One easily checks that $h_{\boldsymbol{A}}$ is a homomorphism from $\bar{R}_\Sigma$ to $\boldsymbol{A}$.

(3.3.4): make $\boldsymbol{A}$ into a $\Sigma(X)$-algebra interpreting each $x\in X$ as $x_{\boldsymbol{A}} = f(x)$. Since equations considered in the definition of iterative algebras (Definition 2.1) involve parameters, any extension of signature by constants leads to iterative algebras. Thus $\boldsymbol{A}$ becomes an iterative $\Sigma(X)$-algebra. By (3.3.3) there is exactly one $\Sigma(X)$-homomorphism $\bar{f}:\bar{R}_{\Sigma(X)}\to\boldsymbol{A}$. Hence $\bar{f}$ is the unique extension of f to a Σ-homomorphism from $\bar{R}_\Sigma(X)$ into $\boldsymbol{A}$.

In the above proof we have used several times the equality $\bar{R}_\Sigma = \Sigma\mathrm{Tr}(1,0)$. The following property that relates $\bar{R}_\Sigma$ and $\Sigma\mathrm{Tr}$ can be proved easily, as well.

3.4. Proposition. *For any signature Σ, $I(\bar{R}_\Sigma)\cong\Sigma\mathrm{Tr}$.*

The above proposition says that if we introduce, using Theorem 3.3, derived operations in iterative algebras, then we will get exactly the notion of iterative polynomial.

The procedure of defining derived operations, having free algebras, is standard. Suppose X is a set and $\boldsymbol{A}$ is an iterative Σ-algebra. Each $t\in\bar{R}_\Sigma(X)$ determines in $\boldsymbol{A}$ a function (called *derived operation*) $t_{\boldsymbol{A}}:A^X\to A$ defined by $t_{\boldsymbol{A}}(a)=\bar{a}(t)$, where $a\in A^X$, and $\bar{a}$ is the unique extension of a to a homomorphism. Because each $t\in\bar{R}_\Sigma(X)$ depends on a finite number of variables from X, so we may define only finitary derived operations in iterative algebras. Proposition 3.4 states that in each iterative algebra notions of iterative polynomial and derived operation coincide.

3.5. Proposition. *For arbitrary $n, k\in\omega$, $p\in T_\Sigma(n+k)^n$, and an iterative Σ-algebra $\boldsymbol{A}$ the following holds:*

$$(p_{\boldsymbol{A}})^+ = (p^+)_{\boldsymbol{A}},$$

where $p^+ = (p_{\bar{R}_\Sigma(k)})^+(0,\ldots,k-1)$, i.e. *$p^+$ is the unique solution of $p(x,0,\ldots,k-1)=x$ in $\bar{R}_\Sigma(k)$.*

Proof. Let $a\in A^k$, and let $\bar{a}:\bar{R}_\Sigma(k)\to\boldsymbol{A}$ be the unique extension of a to a homomorphism. Then

$$(p^+)_{\boldsymbol{A}}(a)=\bar{a}(p^+)=\bar{a}((p_{\bar{R}_\Sigma(k)})^+(0,\ldots,k-1))=(p_{\boldsymbol{A}})^+(a).$$

The above result may be called a Mezei–Wright like result. It says that one may first interpret and then solve a given system of fixed-point equations, or first solve it in free algebra and then interpret result, getting the same. Observe that the above proposition still holds when we interpret p in $\bar{R}_\Sigma(X)$, where $k \subseteq X$, rather than in $\bar{R}_\Sigma(k)$. We shall use this observation later.

4. Basic results on regular algebras

In this section we recall basic definitions and results concerning regular algebras. Full exposition of these results the reader may find in [10, 11].

4.1. In this section we assume that all algebras have ordered carrier with least element denoted by $\perp$.

Let $\boldsymbol{A}$ be a Σ-algebra. Let $n, k \in \omega$, $p \in T_\Sigma(n+k)^n$, $a \in A^k$. Denote by $f: A^n \to A^n$ the map $f(x) = p_{\mathbf{A}}(x, a)$. Let

$$L_f = \{f^i(\perp, \ldots, \perp): i \in \omega\}.$$

Usually f is called an *algebraic* (*vector*) *map* in $\boldsymbol{A}$.

A subset $X \subseteq A$ is called an *iteration* if there is $n \in \omega$ and an algebraic n-vector map f in $\boldsymbol{A}$ such that $X = e_0^n(L_f)$. Subsets of iterations play essentially the same role for regular algebras as directed sets for Δ-continuous algebras (cf. [2]).

4.2. A Σ-algebra $\boldsymbol{A}$ is said to be a *regular algebra* if for any $n \in \omega$ and for any algebraic map $f: A^n \to A^n$ the following conditions hold:

$$f(\perp, \ldots, \perp) \leq f(x) \quad \text{for all } x \in A^n, \tag{4.2.1}$$

$$\text{The set } L_f \text{ has least upper bound in } A^n, \tag{4.2.2}$$

$$f(\sup L_f) = \sup L_f. \tag{4.2.3}$$

In other words, regular algebras are all those algebras with ordered carrier, where every finite system of fixed-point equations with parameters can be solved by taking least upper bound of a ω-chain of approximants (i.e. of iterations). Then by (4.2.1) the got result is the least solution.

$\mathbf{A}$ is said to be an *ordered regular algebra* if instead of (4.2.1) stronger condition holds:

$$\text{all } \Sigma\text{-operations are monotonic.} \tag{4.2.4}$$

4.3. Let $\boldsymbol{A}, \boldsymbol{B}$ be regular Σ-algebras. A map $f: A \to B$ is said to be *algebraically continuous* if for any iteration E in A there is an iteration E' in B such that

$$f(E) \subseteq E', \tag{4.3.1}$$

$$\sup f(E) = f(\sup E). \tag{4.3.2}$$

Remark. From (4.3.1) it follows that $f(E)$ has least upper bound in B. A map f is said to be *regular homorphism* if

it is algebraically continuous, (4.3.3)

it is strict, i.e. $f(\perp) = \perp$, (4.3.4)

it is a Σ-homomorphism, i.e. it preserves all Σ-operations. (4.3.5)

4.4. Now we are going to give an example of a regular algebra. Let $R_\Sigma = \bar{R}_\Sigma(\perp)$. Define order relation on R_Σ:

> $t \leq t'$ iff one obtains t' replacing some occurences of $\perp$ in t by some elements of R_Σ.

R_Σ is in obvious way a Σ-algebra (isomorphic, as algebra, to $R_\Sigma(\perp)$). it is also a poset under above-defined relation, with least element $\perp$. This poset is not Δ-complete (i.e. there are directed sets in R_Σ that have no least upper bounds).

The same construction can be made for $R_\Sigma(X)$, for any set X, i.e. $R_\Sigma(X) = \bar{R}_\Sigma(X \cup \{\perp\})$ and order relation is defined as above.

4.5. Proposition ([2]). *$R_\Sigma(X)$ is a regular algebra, and it is not Δ-continuous if $\Sigma_n \neq \emptyset$ for some $n > 1$, or if card$(\Sigma_1) \geq 2$.*

4.6. Theorem ([10]). *$R_\Sigma(X)$ is freely generated by X in the class of regular homomorphisms. More exactly, for any regular Σ-algebra $\mathbf{A}$ and for any map $f : X \to A$, there is exactly one extension of f to a regular homomorphism $\bar{f} : R_\Sigma(X) \to \mathbf{A}$.*

4.7. Having free algebras one may define derived operations in regular algebras. Let $t \in R_\Sigma(X)$. and $\mathbf{A}$ let be a regular Σ-algebra. The tree t induces in $\mathbf{A}$ a map (called a *derived operation*) $t_\mathbf{A} : A^X \to A$ defined by: $t_\mathbf{A}(a) = \bar{a}(t)$, where $a \in A^X$, and $\bar{a}$ is the unique extension of a to a regular homomorphism.

Similarly, as in iterative algebra case, t depends only on a finite number of variables from X. Thus we may define only finitary derived operations in regular algebras.

The next result indicated that the notion of regular homomorphism was chosen properly.

4.8. Theorem ([10]). *Let $\mathbf{A}$, $\mathbf{B}$ be regular Σ-algebras, and $f : A \to B$ let be a map. The following conditions are equivalent:*

f is a regular homorphism, (4.8.1)

for any $n \in \omega$, and for any $t \in R_\Sigma(n)$, $ft_\mathbf{A} = t_\mathbf{B}\tilde{f}$, where $\tilde{f}$ is the extension of f to vectors (cf. Section 1.2). (4.8.2)

4.9. Suppose A is a regular Σ-algebra. let $p \in T_{\Sigma(\perp)}(n+k)^n$, $n, k \in \omega$, $a \in A^k$. Denote by f (as in Section 4.1) the map $f(x) = p_A(x, a)$. It can be easily proved (cf. [10]) that $\sup L_f$ is the least solution of the equation $f(x) = x$. Denote this solution by $(p_A)^\nabla(a)$. This gives rise to a map $(p_A)^\nabla : A^k \to A^n$. Call each map f of the form $f = (p_A)^\nabla_0$ for some $p \in T_{\Sigma(\perp)}(n+k)^n$, $n, k \in \omega$, a *regular polynomial* in A.

It follows from Normal Form Theorem in [11] that notions of derived operations and regular polynomials coincide in regular algebras.

It turns out that similar result to that presented in Section 2 holds for ordered regular algebras and rational algebraic theories (the latter notion is due to ADJ (1976)). Details of this correspondence the reader may find in [12] Now we formulate a Mezei–Wright like result for regular algebras.

4.10. Proposition. *For any $n, k \in \omega$, $p \in T_\Sigma(n+k)^n$, and for arbitrary regular Σ-algebra $\mathbf{A}$,*

$$(p_{\mathbf{A}})^\nabla = (p^\nabla)_{\mathbf{A}},$$

where $p^\nabla \in R_\Sigma(k)^n$, and $p^\nabla = (p_{R_\Sigma(k)})^\nabla(0, \ldots, k-1)$.

Proof. Is essentially the same as that of Proposition 3.5.

4.11. Notice that since $R_\Sigma(k)$ is both iterative and regular, so for any $n, k \in \omega$, $p \in T_\Sigma(n+k)^n$,

$$p^+ = p^\nabla.$$

(cf. for notations Proposition 4.10 and 3.5).

The next results follow from algebraic continuity of regular polynomials in regular algebras (cf. [11]).

4.12. Proposition. *If $f: A^n \to A$ is a regular polynomial in a regular algebra $\mathbf{A}$, and $E_0, \ldots, E_{n-1}$ are iterations in $\mathbf{A}$, then there is an iteration E in $\mathbf{A}$ with the properties:*

$$f(E_0, \ldots, E_{n-1}) \subseteq E, \tag{4.12.1}$$

$$E \leqslant f(E_0, \ldots, E_{n-1}), \tag{4.12.2}$$

$$\sup E = f(\sup E_0, \ldots, \sup E_{n-1}). \tag{4.12.3}$$

5. Formulation of problem

We are interested in a relationship between unique fixed points and least fixed points. Iterative algebras and/or iterative algebraic theories deal with unique fixed points, while regular algebras and/or rational algebraic theories deal with least fixed points. The following problem arises naturally. Given an iterative Σ-algebra $\mathbf{A}$, is it possible to extend the carrier A to A_{R} and to define on A_{R} a partial order and a structure of Σ-algebra so that $\mathbf{A}_{\mathrm{R}}$ becomes a regular algebra, and any solution of ideal system of equations in $\mathbf{A}$ is also the least solution of the same system in $\mathbf{A}_{\mathrm{R}}$? Moreover, it is natural to require the construction $\mathbf{A} \mapsto \mathbf{A}_{\mathrm{R}}$ to be universal (cf. [7]). This leads to the following formulation.

5.1. An iterative Σ-algebra $\mathbf{A}$ is said to *admit* a *regular extension* if there is a regular algebra $\mathbf{A}_{\mathrm{R}}$ and a map $\varphi_{\mathbf{A}}: A \to A_{\mathrm{R}}$ such that

for any $n, k \in \omega$, $p \in T_\Sigma(n+k)^n$, if p is ideal, then

$$\varphi_{\mathbf{A}}(p_{\mathbf{A}})^+ = (p_{\mathbf{A}_{\mathrm{R}}})^\nabla \varphi_{\mathbf{A}}, \tag{5.1.1}$$

for any regular Σ-algebra $\mathbf{B}$ and for any $f: A \to B$ satisfying

$$f(p_{\mathbf{A}})^+ = (p_{\mathbf{B}})^\nabla f \tag{5.1.2}$$

for any $n, k \in \omega$, and ideal $p \in T_\Sigma(n+k)^n$, there is exactly one regular homomorphism $f^*: \mathbf{A}_{\mathrm{R}} \to \mathbf{B}$ with $f^* \varphi_{\mathbf{A}} = f$.

The pair $(\varphi_{\mathbf{A}}, \mathbf{A}_R)$ is called a *regular extension of* $\mathbf{A}$.

5.2. A is said to admit a *faithful regular extension* if there is a regular extension $(\varphi_{\mathbf{A}}, \mathbf{A}_{\mathrm{R}})$ with $\varphi_{\mathbf{A}}$ being injective.

5.3. We shall consider in that paper the following questions:

Does every iterative algebra admit a faithful regular extension? (5.3.1)

Does every iterative algebra admit a regular extension? (5.3.2)

We will see in Section 6 that the anser to (5.3.1) is negative, while the main result of this paper (Theorem 6.12) solves (5.3.2) in affirmative. In addition the regular algebra $\mathbf{A}_{\mathrm{R}}$ obtained by this result turns out to be ordered. This links iterative algebraic theories with rational algebraic theories. The following standard result shows that every iterative algebra admist at most one (up to a regular isomorphism) regular extension.

5.4. Proposition. *If* $(\varphi_{\mathbf{A}}, \mathbf{A}_{\mathrm{R}})$ *and* $(\psi_{\mathbf{A}}, \mathbf{A}'_{\mathrm{R}})$ *are regular extensions of iterative algebra* $\mathbf{A}$, *then there is a regular isomorphism*

$$\xi: \mathbf{A}_{\mathrm{R}} \to \mathbf{A}'_{\mathrm{R}} \quad \textit{with } \xi\varphi_{\mathbf{A}} = \psi_{\mathbf{A}}.$$

Moreover, if $(\varphi_{\mathbf{A}}, \mathbf{A}_{\mathrm{R}})$ *is a regular extension of* $\mathbf{A}$ *and if* $\mathbf{B}$ *is a regular algebra isomorphic to* $\mathbf{A}_{\mathrm{R}}$ *with a regular isomorphism* $\xi: \mathbf{A}_{\mathrm{R}} \to \mathbf{B}$, *then* $(\xi\varphi_{\mathbf{A}}, \mathbf{B})$ is *a regular extension of* $\mathbf{A}$ *as well.*

Proof.

$$\begin{tikzcd}
& \boldsymbol{A}_{\mathbf{R}} \arrow[dd, dashed, shift left, "\tau"] \\
\boldsymbol{A} \arrow[ur, "\varphi_{\boldsymbol{A}}"] \arrow[dr, "\Psi_{\boldsymbol{A}}"'] & \\
& \boldsymbol{A}_{\mathbf{R}} \arrow[uu, dashed, shift left, "\xi"]
\end{tikzcd}$$

Let $\xi = \psi^*_{\mathbf{A}}$ and $\tau = \varphi^*_{\mathbf{A}}$, then by the uniqueness condition (5.1.2) $\xi\tau = \mathrm{id}_{A_{\mathbf{R}}}$ and $\tau\xi = \mathrm{id}_{A_{\mathbf{R}}}$. Thus ξ is a required isomorphism.

Proof of the second part being trivial is omitted.

6. Regular extensions of iterative algebras

Let $\boldsymbol{A}$ be an iterative Σ-algebra. This algebra will be kept fixed in this section.

6.1. For any $n \in \omega$, and any $p \in \bar{R}_\Sigma(A)^n$ let $p^A \in A^n$ be a vector defined by

$$p_i^A = f(p_i) \quad \text{for } i < n.$$

where $f : \bar{\boldsymbol{R}}_\Sigma(A) \to \boldsymbol{A}$ is the unique extension of $\mathrm{id}_A : A \to A$ to a homomorphism.

6.2. Define on $R_\Sigma(A)$ the following binary relation $\equiv_{\boldsymbol{A}}$. For $t, t' \in R_\Sigma(A)$ $t \equiv_{\boldsymbol{A}} t'$ iff there are $n \in \omega$, $q \in R_\Sigma(n)$, $p, p' \in \bar{R}_\Sigma(A)^n$ such that

$$t = q[p], \qquad t' = q[p']; \tag{6.2.1}$$

$$p^A = p'^A. \tag{6.2.2}$$

6.3. Proposition. *$\equiv_{\boldsymbol{A}}$ is an equivalence relation on $R_\Sigma(A)$.*

Proof. It is a reflexive relation since any $t \in R_\Sigma(A)$ depends only on a finite part of A, i.e. there are $n \in \omega$, $q \in R_\Sigma(n)$, and $a \in A^n$ such that $t = q[a]$.

Obviously this relation is also symmetric.

The non-trivial part of the proof is transitivity of $\equiv_{\boldsymbol{A}}$. For suppose $t \equiv_{\boldsymbol{A}} t'$ and $t' \equiv_{\boldsymbol{A}} t''$ for some $t, t', t'' \in R_\Sigma(A)$.

There exist $n, m \in \omega$, $q \in R_\Sigma(n)$, $q' \in R_\Sigma(m)$, $p, p' \in \bar{R}_\Sigma(A)^n$, $p^*, p'' \in \bar{R}_\Sigma(A)^m$ such that

$$t = q[p], \tag{6.3.1}$$

$$t' = q[p'] = q'[p^*], \tag{6.3.2}$$

$$t'' = q'[p'']. \tag{6.3.3}$$

$$p^A = p''^A, \tag{6.3.4}$$

$$p^{*A} = p''^A. \tag{6.3.5}$$

Let $\{q_i : i < k_1\}$ be all distinct subtrees of q, and let $\{q'_i : i < k_2\}$ be all distinct subtrees of q'. For $i < n, j < k_2$ define a set

$$S^{(0)}_{i,j} = \{w \in \omega^* : q(w) = i, w \in \mathrm{Dom}(q'), q' \upharpoonright w = q'_j\}$$

Analogously, for $i < m, j < k_1$ define

$$S^{(1)}_{i,j} = \{w \in \omega^* : q'(w) = i, w \in \mathrm{Dom}(q), q(w) \in \Sigma, q \upharpoonright w = q_j\}.$$

Let

$$S = \bigcup_{\substack{i<n\\ j<k_2}} S^{(0)}_{i,j} \cup \bigcup_{\substack{i<m\\ j<k_1}} S^{(1)}_{i,j},$$

and let

$$\bar{S} = \{w \in \omega^* : \text{there is an } w' \in S \text{ with } w' \leqslant w\}.$$

Let $k = (n \times k_2) \oplus (m \times k_1)$ (cf. Section 1.14). Thus, in fact $k = n \cdot k_2 + m \cdot k_1$. Now we are going to define $r \in R_\Sigma(k)$ and two vectors $u, u'' \in \bar{R}_\Sigma(A)^k$, that provide equivalence $t \equiv_A t''$.

$$\mathrm{Dom}(r) = S \cup (\mathrm{Dom}(q) \setminus \bar{S}). \tag{6.3.6}$$

For $w \in \mathrm{Dom}(r)$ we define

$$r(w) = \begin{cases} q(w), & \text{if } w \in \mathrm{Dom}(q)\setminus\bar{S},\\ (0, i, j), & \text{if } w \in S^{(0)}_{i,j},\\ (1, i, j) & \text{if } w \in S^{(1)}_{i,j}. \end{cases} \tag{6.3.7}$$

Observe that the above definition is correct since sets $S^{(0)}_{i,j}$, $S^{(1)}_{1,1}$, are pairwise disjoint for $i < n, j < k_2, 1 < m, 1' < k_1$. Moreover, it is easy to see that r is of finite index, i.e. $r \in R_\Sigma(k)$.

$$u_{(0,i,j)} = p_i \quad \text{for } i < n, j < k_2, \tag{6.3.8}$$

$$u_{(1,i,j)} = \begin{cases} q_j[p], & \text{if } S^{(1)}_{i,j} \neq \emptyset\\ p''_i, & \text{otherwise} \end{cases} \quad \text{for } i < m, j < k_1 \tag{6.3.9}$$

$$u''_{(0,i,j)} = \begin{cases} q'_j[p''], & \text{if } S^{(0)}_{i,j} \neq \emptyset\\ p_i, & \text{otherwise} \end{cases} \quad \text{for } i < n, j < k_2. \tag{6.3.10}$$

$$u''_{(1,i,j)} = p''_i \quad \text{for } i < m, j < k_1. \tag{6.3.11}$$

Observe that vectors u and u'' can be filled out arbitrary in those positions $(1, i, j)$ for which $S^{(1)}_{i,j} = \emptyset$, since variable $(1, i, j)$ does not occur in r. We prove the following facts:

$$t = r[u], \tag{6.3.12}$$

$$t'' = r[u''], \tag{6.3.13}$$

$$u, u'' \in \bar{R}_\Sigma(A)^k, \tag{6.3.14}$$

$$u^A = u''^A. \tag{6.3.15}$$

Proof of (6.3.12): Let $w \in \mathrm{Dom}(t)$. let $\bar{w}$ be a maximal path with $\bar{w} \leqslant w$, $\bar{w} \in \mathrm{Dom}(q)$, $\bar{w} \in \mathrm{Dom}(q')$. Consider the following three cases:

$$q(\bar{w}) \in \Sigma, \quad q'(\bar{w}) \in \Sigma, \tag{6.3.12a}$$

$$q(\bar{w}) \in \Sigma, \quad q'(\bar{w}) = i \qquad \text{for some } i < m, \tag{6.3.12b}$$

$$q(\bar{w}) = i \quad \text{for some } i < n. \tag{6.3.12c}$$

(6.3.12a): If $\bar{w} = w$, then $w \in \mathrm{Dom}(q) \setminus \bar{S}$ and

$$\begin{aligned} t(w) &= q(w) \quad (\text{by } (6.3.1)) \\ &= r[u](w) \quad (\text{by } (6.3.7)) \end{aligned}$$

Suppose $\bar{w} < w$, this means $w \notin \mathrm{Dom}(q)$ or $w \notin \mathrm{Dom}(q')$.

Assume $w \notin \mathrm{Dom}(q)$. Let $\tilde{w}$ be a maximal path with $\tilde{w} \leqslant w$, $\tilde{w} \in \mathrm{Dom}(q)$. By (6.3.1) we get $q(\tilde{w}) = i$ for some $i < n$, and $\tilde{w} \notin \mathrm{Dom}(q')$. obviously $\bar{w}$ is maximal with $\bar{w} \leqslant \tilde{w}$, $w \in \mathrm{Dom}(q')$. By (6.3.2) we deduce that $q'(\bar{w}) = j$ for some $j < m$. Obtained contradiction shows that $w \in \mathrm{Dom}(q)$ holds. Thus $w \notin \mathrm{Dom}(q')$. Since $\bar{w}$ is maximal with $\bar{w} \leqslant w$, $\bar{w} \in \mathrm{Dom}(q')$, by (6.3.2) and the assumption $\bar{w} < w$ we get $q'(\bar{w}) = j$ for some $j < m$. This shows that the case $\bar{w} < w$ is impossible.

(6.3.12b): In this case $\bar{w} \in S^{(1)}_{i,j}$, where $q_j = q \restriction \bar{w}$. Then

$$t \restriction \bar{w} = q \restriction \bar{w}[p] \quad (\text{by } (6.3.1))$$

$$q_j[p] = r[u] \restriction w \quad (\text{by } (6.3.7), (6.3.8)).$$

Now we use the argument that will be used very often in the sequel: for any trees t_1, t_2 if $t_1 \restriction \bar{w} = t_2 \restriction \bar{w}$ for some path $\bar{w}$, then $t_1(w) = t_2(w)$ for any path w with $\bar{w} \leqslant w$.

(6.3.12c): In this case $\bar{w} \in S^{(0)}_{i,j}$, where $q'_j = q' \restriction w$. Then

$$\begin{aligned} t \restriction \bar{w} &= p_i \quad (\text{by } (6.3.1)) \\ &= r[u] \restriction \bar{w} \quad (\text{by } (6.3.7), (6.3.8)). \end{aligned}$$

In this way we have proved $t \subseteq r[u]$.

Let $w \in \mathrm{Dom}(r[u])$. Consider the following three cases:

$$w \in \mathrm{Dom}(q) \setminus \bar{S} \tag{6.3.12d}$$

$$w = \bar{w}v, \quad \text{where } \bar{w} \in S^{(1)}_{i,j},\ v \in \mathrm{Dom}(u_{(1,i,j)}), \tag{6.3.12e}$$

$$w = \bar{w}v. \quad \text{where } \bar{w} \in S^{(0)}_{i,j},\ v \in \mathrm{Dom}(u_{0,i,j)}). \tag{6.3.12f}$$

In the first case $q(w) \in \Sigma$, and applying the same argument as in the first part of (6.3.12a) one shows $r[u](w) = t(w)$.

In the second case, likewise in (6.3.12b) we prove $r[u] | \bar{w} = t | \bar{w}$. While in the third case we proceed as in (6.3.12c) to show $r[u] | \bar{w} = t | \bar{w}$.

Proof of (6.3.13): Similar to that of (6.3.12) and we omit it.

Proof of (6.3.14): it is enough to prove that $q_j \in \bar{R}_\Sigma(n)$ for all those $j < k_1$ for which $\bigcup_{i<m} S^{(1)}_{i,j} \neq \emptyset$ and that $q'_j \in \bar{R}_\Sigma(m)$ for all those $j < k_2$ for which $\bigcup_{i<n} S^{(0)}_{i,j} \neq \emptyset$.

For suppose $\bar{w} \in S^{(1)}_{i,j}$ for some $i < m$, $j < k_1$, and $q_j(w) = \perp$ for some $w \in \mathrm{Dom}(q_j)$. hence $q \restriction \bar{w} = q_j$, and $q(\bar{w}w) = \perp$. Thus $q[p'](\bar{w}w) = \perp$ and $q'[p^*](\bar{w}w) = \perp$ by (6.3.2). But $q'(\bar{w}) = i$ and we get $p^*_i(w) = \perp$ – contradiction with $p^*_i \in \bar{R}_\Sigma(A)$.

In the same way one shows that if $S^{(0)}_{i,j} \neq \emptyset$ for some $i < n$, $j < k_2$, and $q'_j(w) = \perp$ for some $w \in \mathrm{Dom}(q'_j)$, then $p'_i(w) = \perp$.

Proof of (6.3.15): First we prove $u^A_{(0,i,j)} = u''^A_{(0,i,j)}$ for all $i < n$, $j < k_2$. If $S^{(0)}_{i,j} = \emptyset$, then it follows immediately from (6.3.8) and (6.3.10). If $w \in S^{(0)}_{i,j}$, then, by (6.3.2), $p'_i = q[p'](w) = q'[p^*](w) = q'_j[p^*]$. Using this we get

$$\begin{aligned} u^A_{(0,i,j)} &= p^A_i = p'^A_i \quad \text{(by (6.3.4)} \\ &= (q'_j[p^*])^A = q'_{jA}[p^{*A}] \quad \text{(cf. Section 6.1)} \\ &= q'_{jA}[p^{*A}] \quad \text{(by (6.3.5))} \\ &= (q'_{jA}[p''])^A \quad \text{(cf. Section 6.1)} \\ &= u''^A_{(0,i,j)}. \end{aligned}$$

In the same way one proves $u^A_{(1,i,j)} = U''^A_{(1,i,j)}$ for all $i < m$, $j < k_1$. This completes the proof of Proposition 6.3.

6.4. Let $A^* = R_\Sigma(A)/\equiv_A$. On A^* we define a sequence of binary relations $\leqslant_\alpha$, $\alpha \in \mathbf{Ord}$. Remind that elements of A^*, as equivalence classes, are subsets of $R_\Sigma(A)$. By $|t|$, where $t \in R_\Sigma(A)$ we denote the class determined by t.

We extend our notations to subsets according to Section 1.12 and 1.13 $\leqslant_0 = \leqslant$ (i.e. the extension of $\leqslant$ in $R_\Sigma(A)$ (cf. Section 4.4) to subsets).

For $t, t' \in R_\Sigma(A)$: $|t| \leqslant_{\alpha+1} |t'|$ iff there exist $n \in \omega$ and iterations $E_0, \ldots, E_n$ in $R_\Sigma(A)$ such that:

$$\sup E_0 \equiv_A t, \tag{6.4.1}$$

$$|E_i| \leqslant_\alpha |\sup E_{i+1}| \quad \text{for } i < n. \tag{6.4.2}$$

$$|E_n| \leqslant_\alpha |t'|. \tag{6.4.3}$$

If α is a limit ordinal, then

$$\leqslant_\alpha = \bigcup_{\beta < \alpha} \leqslant_\beta$$

6.5. Proposition.

For any $p, q \in R_\Sigma(A)$, $|p| \leqslant_0 |q|$ iff there exist $p' \in |p|$, $q' \in |q|$ such that $p' \leqslant q'$, (6.5.1)

$\leqslant_0$ is a partial order in A^*. (6.5.2)

Proof. (6.5.1): '$\to$' is obvious. To prove '$\leftarrow$' implication assume $p' \leqslant q'$ for some $p' \in |p|$, $q' \in |q|$. Take arbitrary $p^* \in |p|$. Since $p' \equiv_{\mathbf{A}} p^*$, there is $n \in \omega$, $t \in R_\Sigma(n)$, $t', t^* \in \bar{R}_\Sigma(A)^n$ such that $p' = t[t']$, $p^* = t[t^*]$, and $t'^A = t^{*A}$.

Because $p' \leqslant q'$, $p' = t[t']$, and $t' \in \bar{R}_\Sigma(A)^n$, so it is easily seen that there exists (unique) $\bar{t} \in R_\Sigma(n)$ with the properties:

$$t \leqslant \bar{t}, \tag{6.5.3}$$

$$q' = \bar{t}[t']. \tag{6.5.4}$$

From (6.5.3) and (6.5.4) it follows that if we put $q^* = \bar{t}[t^*]$, then $q' \equiv_{\mathbf{A}} q^*$ and $p^* \leqslant q^*$.

(6.5.2): $\leqslant_0$ is obviously reflexive. From (6.5.1) it follows that it is transitive. It remains to prove antisymmetry. For suppose $|p| \leqslant_0 |q|$ and $|q| \leqslant_0 |p|$. Take $p' \in |p|$, there is $q' \in |q|$ such that $p' \leqslant q'$. There is $p^* \in |p|$, such that $q' \leqslant p^*$. Since $p' \equiv_{\mathbf{A}} p^*$ so $p' = t[t']$, $p^* = t[t^*]$ for some $t \in R_\Sigma(n)$, $n \in \omega$, $t', t^* \in \bar{R}_\Sigma(A)$. Since $p' \leqslant p^*$ so it must be $p' = p^*$. Therefore $p' = q'$ and $|p| = |q|$.

In the last part of the above proof we have proved, by the way, that

$$\text{If } p \equiv_{\mathbf{A}} q \text{ and } p \leqslant q, \text{ then } p = q. \tag{6.5.5}$$

6.6. Proposition.

$$\textit{If } \alpha < \beta, \textit{ then } \leqslant_\alpha \subseteq \leqslant_\beta; \tag{6.6.1}$$

$$\textit{For any } \alpha \in \mathbf{Ord}, \leqslant_\alpha \textit{ is reflexive and transitive.} \tag{6.6.2}$$

Proof. (6.6.1): It is enough to prove that $\leqslant_\alpha \subseteq \leqslant_{\alpha+1}$ for all $\alpha \in \mathbf{Ord}$. For $\alpha = 0$. Suppose $|p| \leqslant |q|$ for some $p, q \in R_\Sigma(A)$. Take an arbitrary iteration E in $R_\Sigma(A)$ with $\sup E = p$. From (6.5.1) it follows that $|E| \leqslant |p|$. From (6.5.2) we get $|E| \leqslant |q|$. Hence, $|p| \leqslant_1 |q|$.

Suppose $\leqslant_\alpha \subseteq \leqslant_{\alpha+1}$ for all $\alpha < \beta$. Let $|p| \leqslant_\beta |q|$. If $\beta = \gamma + 1$ then there are $n \in \omega$, and iterations E_i, $i < n+1$ with $\sup E_0 \equiv_{\mathbf{A}} p$,

$$|E_i| \leqslant_\gamma |\sup E_{i+1}| \quad \text{for } i < n, \qquad |E_n| \leqslant |q|. \tag{6.6.3}$$

By inductive hypothesis we may write $\gamma + 1$ in places of occurence of γ in (6.6.3), and get $|p| \leqslant_{\beta+1} |q|$.

If, however, β is a limit ordinal, then there is an ordinal $\gamma + 1 < \beta$ with $|p| \leqslant_{\gamma+1} |q|$. There are iterations E_i, $i < n$ such that (6.6.3) holds. Since $\leqslant_\gamma \subseteq \leqslant_\beta$ we may write β in places of occurence of γ in (6.6.3), and get $|p| \leqslant_{\beta+1} |q|$.

(6.6.2): By (6.5.2) and (6.6.1) each $\leqslant_\alpha$ is reflexive. By (6.5.2) $\leqslant_0$ is transitive. Assume $\leqslant_\alpha$ is transitive for all $\alpha < \beta$.

Let $|p| \leq_\beta |q| \leq_\beta |t|$. If $\beta = \gamma + 1$, then there are iterations $E_0, \ldots, E_n, B_0, \ldots, B_m$ with $\sup E_0 \equiv_A p$,

$$|E_i| \leq_\gamma |\sup E_{i+1}| \quad (i<n), \qquad |E_n| \leq_\gamma |q| = |\sup B_0|,$$

$$|B_i| \leq_\gamma |\sup B_{i+1}| \quad (i<m), \qquad |B_m| \leq_\gamma |t|.$$

Hence, $|p| \leq_\beta |t|$.

If β is a limit ordinal then $\leq_\beta$ is transitive as a union of a chain of transitive relations.

6.7. For any $n \in \omega$ and any $t \in R_\Sigma(n)$ we define a function $t_{A^*} : A^{*n} \to A^*$ by

$$t_{A^*}(|t_0|, \ldots, |t_{n-1}|) = |t[t_0, \ldots, t_{n-1}]|.$$

It follows immediately from Definition 6.2 that the above definition is correct.

6.8. Proposition. *For any $n \in \omega$, and any $t \in R_\Sigma(n)$, t_{A^*} is monotone in each quasi-order $\leq_\alpha$ ($\alpha \in$ **Ord**).*

Proof. For $\alpha = 0$ this is obvious (it follows from monotonicity of derived operations in $R_\Sigma(A)$).

Assume t_{A^*} is monotone under each $\leq_\alpha$ for $\alpha < \beta$, $\beta = \gamma + 1$, and $|t_0| \leq_\beta |p_0|, \ldots, |t_{n-1}| \leq_\beta |p_{n-1}|$. Let E_j^i ($i<n, j<k_i+1$) be iterations such that for $i<n$,

$$\sup E_0^i \equiv_A t_i$$

$$|E_j^i| \leq_\gamma |\sup E_{j+1}^i| \quad \text{for } j<k_i, \qquad |E_{k_i}^i| \leq_\gamma |p_i|.$$

Of course we may assume that $k_i = k$ for all $i<n$.

Let $t_i' = \sup E_0^i$, $i<n$. For every $j<k+1$ denote by E_j the iteration in $R_\Sigma(A)$ obtained by proposition 4.12 applied to $t(E_j^0, \ldots, E_j^{n-1})$. Thus by Proposition 4.12 $\sup E_0 = t[t_0', \ldots, t_{n-1}']$. Since $t_i' \equiv_A t_i$ for $i<n$, so $\sup E_0 \equiv_A t[t_0, \ldots, t_{n-1}]$. The iterations E_j for $j<k+1$ have the following properties:

$$\sup E_j = t(\sup E_j^0, \ldots, \sup E_j^{n-1}); \tag{6.8.1}$$

$$E_j \leq t(E_j^0, \ldots, E_j^{n-1}). \tag{6.8.2}$$

By the inductive hypothesis, (6.8.1) and (6.8.2) it follows that:

$$|E_j| \leq_\gamma |\sup E_{j+1}| \quad \text{for } j<k$$

and

$$|E_k| \leq_\gamma |t[p_0, \ldots, p_{n-1}]|.$$

Thus we have proved that $t_{A^*}(|t_0|, \ldots, |t_{n-1}|) \leq_\beta t_{A^*}(|p_0|, \ldots, |p_{n-1}|)$. The case β is limit ordinal is obvious.

6.9. Since $\equiv_A$ is a Σ-congruence in $R_\Sigma(A)$, so on A^* there is naturally defined a structure of Σ-algebra. In particular for $p \in T_\Sigma(n)$, the meaning of the polynomial p in A^* coincides with p_{A^*} defined in Definition 6.7. It is easily seen $|\perp| \leqslant_\alpha |t|$ for any $\alpha \in \mathbf{Ord}$ and any $t \in R_\Sigma(A)$.

6.10. Proposition. *Let $\tilde{E}$ be an arbitrary iteration in $\mathbf{A}^*$. There is an iteration E in $R_\Sigma(A)$ such that*

$$|E| = \tilde{E}, \tag{6.10.1}$$

$$\textit{For any } t \in R_\Sigma(A), \textit{ if } |E| \leqslant_{\omega_1} |t|, \textit{ then } |\sup E| \leqslant_{\omega_1} |t|, \tag{6.10.2}$$

$$|E| \leqslant_{\omega_1} |\sup E|. \tag{6.10.3}$$

Proof. (6.10.1): Follows from the fact that the natural surjection $|\ |$ is a Σ-homomorphism preserving bottom element.

Now we prove (6.10.2). Let $E = \{t_n : n \in \omega\}$. By assumption for every $n \in \omega$ there is $\alpha_n \leqslant \omega_1$ with $|t_n| \leqslant_{\alpha_n} |t|$. Hence there is $\alpha < \omega_1$ with $|E| \leqslant_\alpha |t|$. Thus, by Definition 6.4 $|\sup E| \leqslant_{\alpha+1} |t|$, and by (6.6.1) $|\sup E| \leqslant_{\omega_1} |t|$.

(6.10.3) follows from (6.5.1) and (6.6.1).

It should be remarked that, in general, (6.10.2) is not true for a quasi-order $\leqslant_\alpha$ with $\alpha < \omega_1$. For example, for $\mathbf{A} = \langle R, f \rangle$, where R stands for reals and $f(x) = \frac{1}{2}x$, one easily proves that $\leqslant_0$ does not satisfy (6.10.2).

6.11. Define in A^* a binary relation $\sim$ by $|p| \sim |q|$ iff $|p| \leqslant_{\omega_1} |q|$ and $|q| \leqslant_{\omega_1} |p|$. It is a standard fact that $\sim$ is an equivalence in A^*. By $\|p\|$ we denote the equivalence class determined by $|p| \in A^*$. By proposition 6.8 it follows that $\sim$ is a Σ-congruence in $\mathbf{A}^*$. Denote by $\mathbf{A}_\mathrm{R}$ the quotient Σ-algebra $\mathbf{A}^*/\sim$. The carrier is ordered by relation $\sqsubseteq$ defined as follows:

$$\|p\| \sqsubseteq \|q\| \quad \text{iff} \quad |p| \leqslant_{\omega_1} |q|.$$

Now we are in a position to formulate the main results of this section.

6.12. Theorem. *$(\varphi_\mathbf{A}, \mathbf{A}_\mathrm{R})$ is a regular extension of $\mathbf{A}$, where $\varphi_\mathbf{A}(a) = \|a\|$ for $a \in A$. Moreover, $\mathbf{A}_\mathrm{R}$ is an ordered regular algebra.*

Proof. First we prove that $\mathbf{A}_\mathrm{R}$ is a regular Σ-algebra. By Proposition 6.10 it follows that every iteration in $\mathbf{A}_\mathrm{R}$ has a least upper bound in the poset $(A_\mathrm{R}, \sqsubseteq)$. Consider the map $\|\ \| : R_\Sigma(A) \to A_\mathrm{R}$. Since $\equiv_A$ and $\sim_A$ are Σ-congruences, $\|\ \|$ is a surjective Σ-homomorphism. Since $\leqslant_0 \subseteq \leqslant_{\omega_1}$, $\|\ \|$ is a monotone map. If E is an iteration in $R_\Sigma(A)$, then $\|E\|$ is an iteration in $\mathbf{A}_\mathrm{R}$, and (by Proposition 6.10) $\|\sup E\| = \sup\|E\|$.

It follows from proposition 6.8 that $\mathbf{A}_\mathrm{R}$ satisfies condition (4.2.1). If $f : A_\mathrm{R}^n \to A_\mathrm{R}^n$ is an algebraic map, then, by the above-made remarks, there is an algebraic map $\hat{f} : R_\Sigma(A)^n \to R_\Sigma(A)^n$ such that $\|\hat{f}(x)\| = f(\|x\|)$ for all $x \in R_\Sigma(A)^n$. Then $\|\sup L_{\hat{f}}\| =$

$\sup\|L_{\hat f}\| = \sup L_f$. And

$$f(\sup L_f) = f(\|\sup L_{\hat f}\|) = \|\hat f(\sup L_{\hat f})\| = \|\sup L_{\hat f}\| = \sup L_f.$$

By proposition 6.8 all Σ-operations in $\boldsymbol{A}_{\mathbf{R}}$ are monotonic. Thus

$\boldsymbol{A}_{\mathbf{R}}$ is an ordered regular Σ-algebra and $\|\ \|$ is a regular monotone surjective homomorphism. (6.12.1)

Observe that if $p \in T_\Sigma(n+k)^n$, $n, k \in \omega$ and a $\in A^k$, then

$$(p^+)_{\mathbf{A}}(a) \equiv_{\mathbf{A}} p^+[a]. \tag{6.12.2}$$

Notice that on the left-hand side in (6.12.2) stands an element of A, while on the right-hand side a tree in $R_\Sigma(A)$. Obviously, (6.12.2) follows immediately from definition of $\equiv_{\mathbf{A}}$.

Now we may prove that $(\varphi_{\mathbf{A}}, \boldsymbol{A}_{\mathbf{R}})$ satisfies (5.1.1):

$$\begin{aligned}\|(p_{\mathbf{A}})^+(a)\| &= \|(p^+)_{\mathbf{A}}(a)\| \quad \text{(by 3.5)}\\ &= \|p^+[a]\| \quad \text{(by (6.12.2))}\\ &= \|p^\nabla[a]\| \quad \text{(by (4.11))}\\ &= (p^\nabla)_{\mathbf{A}_{\mathbf{R}}}(\|a\|) \quad \text{(by (6.12.1))}\\ &= (p_{\mathbf{A}_{\mathbf{R}}})^\nabla(\|a\|) \quad \text{by (4.10).}\end{aligned}$$

Observe that, by the way, we have proved that (5.1.1) is equivalent to

$$\text{for any } k \in \omega,\ t \in \bar R_\Sigma(k),\ \varphi_{\mathbf{A}} t_{\mathbf{A}} = t_{\mathbf{A}_{\mathbf{R}}}\varphi_{\mathbf{A}}. \tag{6.12.3}$$

Let $\boldsymbol{B}$ be a regular Σ-algebra and let $f : A \to B$ be a map satisfying:

$$\text{for any } k \in \omega,\ t \in \bar R_\Sigma(k),\ f t_{\mathbf{A}} = t_{\boldsymbol{B}} f. \tag{6.12.4}$$

Denote by $\bar f : R_\Sigma(A) \to \boldsymbol{B}$ the extension of f to a regular homomorphism We have to define a regular homomorphism $f : \boldsymbol{A}_{\mathbf{R}} \to \boldsymbol{B}$ satisfying

$$f^* \varphi_{\mathbf{A}} = f. \tag{6.12.5}$$

Since $\{\|a\|: a \in A\}$ generates $\boldsymbol{A}_{\mathbf{R}}$ (i.e. for any $x \in \boldsymbol{A}_{\mathbf{R}}$ there is $n \in \omega$, $t \in R_\Sigma(n)$, $a \in A^n$ with $x = t_{\mathbf{A}_{\mathbf{R}}}(\|a\|)$), there is at most one f^* satisfying (6.12.5). Define f^* by the formula

$$f^*(\|t\|) = \bar f(t) \quad \text{for } t \in R_\Sigma(a). \tag{6.12.6}$$

Observe that to prove correctness of (6.12.4) it is enough to prove the following two statements:

$$\text{For } p, q \in R_\Sigma(A), \text{ if } p \equiv_{\mathbf{A}} q, \text{ then } \bar f(p) = \bar f(q); \tag{6.12.7}$$

$$\text{For } p, q \in R_\Sigma(A), \text{ and for any } \alpha \in \mathbf{Ord}, \text{ if } |p| \leqslant_\alpha |q|, \text{ then } \bar f(p) \leqslant \bar f(q). \tag{6.12.8}$$

Proof of (6.12.7); Suppose $p \equiv_A q$ for some $p, q \in R_\Sigma(A)$. There exist $n \in \omega$, $t \in R_\Sigma(n)$, $p', q' \in \bar{R}_\Sigma(A)^n$ with $p = t[p']$, $q = t[q']$, $p'^A = q'^A$. The n-vector tree p' can be presented as $r[a]$, where $r \in \bar{R}_\Sigma(m)^n$, $a \in A^m$. Then $p'^A = r_A(a)$. The same for q', there exist $k \in \omega$, $s \in \bar{R}_\Sigma(k)^n$, $b \in A^k$ with $q' = s[b]$, $q'^A = s_A(b)$. Then

$$f(p') = \bar{f}(r[a]) = r_B\bar{f}(a) = r_B f(a) = f r_A(a) = f(p'^A) = f(q'^A) = f(q'^A) = \bar{f}(q').$$

By the above equality we get

$$\bar{f}(p) = \bar{f}(q).$$

Proof of (6.12.8): Let $p, q \in R_\Sigma(A)$. The proof is by induction on α. For $\alpha = 0$ it follows from monotonicity of regular homomorphisms defined on free regular algebras (cf. [10]).

Suppose (6.12.8) holds for all $\alpha < \beta$. Let $|p| \leq_\beta |q|$, and let $\beta = \gamma + 1$. There exist iterations $E_0, \ldots, E_n$ with $\sup E_0 \equiv_A p$, $|E_i| \leq_\gamma |\sup E_{i+1}|$ for $i < n$, $|E_n| \leq_\gamma |q|$.

Then, by inductive hypothesis and by (6.12.7)

$$\begin{aligned}\bar{f}(p) &= \bar{f}(\sup E_0) = \sup \bar{f}(E_0) \leq \bar{f}(\sup E_1)\\ &= \sup \bar{f}(E_1) \leq \cdots = \sup \bar{f}(E_n) \leq \bar{f}(q).\end{aligned}$$

Now we are going to prove that f^* is a regular homomorphism. For, take arbitrary $n \in \omega$, $r \in R_\Sigma(n)$, $t \in R_\Sigma(A)^n$. Then

$$\begin{aligned}f^*(r_{A_R}(\|t\|)) &= f^*(\|r[t]\|) \quad \text{(by (6.12.1))}\\ &= f(r[t]) = r_B\bar{f}(t) \quad \text{(by (6.12.6))}\\ &= r_B f^*(\|t\|) \quad \text{(by (6.12.6))}.\end{aligned}$$

Thus, by Theorem 4.8, f^* is a regular homomorphism.

6.13. Corollary. *An iterative algebra A admits a faithful regular extension if and only if the quasi-order $\leq_{\omega_1}$ restricted to the set $\{|a|: a \in A\}$ is a partial order.*

Proof. By uniqueness (Proposition 5.3) and by existence (Theorem 6.12) of regular extensions an algebra A admits a faithful regular extension iff the extension (φ_A, A_R) provided by Theorem 6.12 is faithful. It is obvious that for $a, b \in A$, $a \equiv_A b$ iff $a = b$, thus φ_A is an injection iff $\leq_{\omega_1}$ restricted to $\{|a|: a \in A\}$ is symmetric.

One may prove, by rather long technical reasoning that $\leq_{\omega_1}$ restricted to $\{|a|: a \in A\}$ is a partial order iff $\leq_{\omega_1}$ is a partial order as whole on A^*. Thus, if A admits a faithful regular extension then A^* and A_R are isomorphic.

6.14. We are going to give an example of an iterative algebra that does not admit a faithful regular extension. Let $A = (A, f, g)$, where $A = R$ is the set of reals, $f(x) = \frac{1}{2}x$, $g(x) = \frac{1}{2}x + \frac{1}{2}$. One easily proves that A is an iterative algebra.

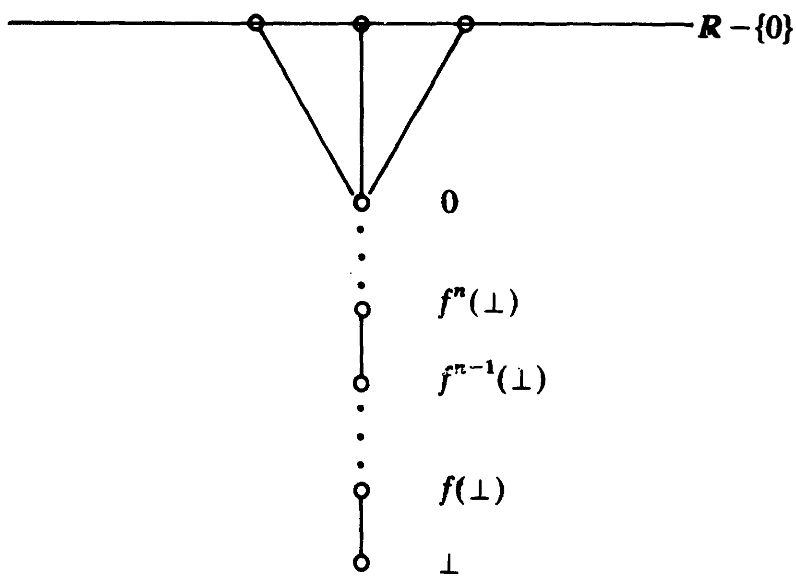
6.15. Theorem. *The algebra* **A** *defined in Section* 6.14 *does not admit a faithful regular extension. More exactly, in each regular extension the elements* 0 *and* 1 *have to be identified.*

Proof. Observe that $E=\{f^n(\perp): n\in\omega\}$ is an iteration and $\sup E=f^{\nabla}=f^{+}\equiv_{\mathbf{A}} 0$. Moreover, $|E|\leqslant|1|$ since $f^n(2^n)\in|1|$ for $n\in\omega$. Thus we have proved $|0|\leqslant_1|1|$.

Similarly for $B=\{g^n(\perp): n\in\omega\}$ we get $\sup B=g^{\nabla}=g^{+}\equiv_{\mathbf{A}} 1$, and $|B|\leqslant|0|$. The last follows from $g^n(1-2^n)\in|0|$, for all $n\in\omega$. Thus $|1|\leqslant_1|0|$, by Corollary 6.13 **A** does not admit a faithful regular extension. Because $\varphi_{\mathbf{A}}(0)=\varphi_{\mathbf{A}}(1)$, by Theorem 6.12 and by Proposition 5.3 0 and 1 have to be identified in each regular extension.

6.16. One can easily check, that the regular extension of $\bar{\boldsymbol{R}}_{\Sigma}(X)$ is isomorphic to $\boldsymbol{R}_{\Sigma}(X)$.

6.17. Let $\boldsymbol{A}=\langle R, f\rangle$, where R stands for reals and $f(x)=\frac{1}{2}x$. it can be shown that $\boldsymbol{A}$ admits a faithful regular extension, and $\boldsymbol{A}_{\mathrm{R}}$ can be presented pictorially as follows:



Operation f is defined in $\boldsymbol{A}_R$ naturally.

7. Concluding remarks

The following few problems seem to be worth of solving, as answers to them should throw a bit more light on relationships between unique and least fixed-points.

7.1. We conjecture that for any iterative algebra **A** the regular extension $(\varphi_{\mathbf{A}}, \boldsymbol{A}_{\mathrm{R}})$ is such that $\boldsymbol{A}_{\mathrm{R}}$ is still an iterative algebra. If this is true, then it would mean that the category of iterative regular Σ-algebras with regular homomorphisms is a reflexive subcategory of the category of iterative Σ-algebras with Σ-homomorphisms

7.2. Is the algebra $\boldsymbol{A}^*$ (cf. Section 6.9) iterative? This question is connected with Section 7.1 since for an iterative algebra $\boldsymbol{A}$ admiting faithful regular extensions, $\boldsymbol{A}^* \cong \boldsymbol{A}_{\mathbf{R}}$.

7.3. Observe that algebras $\boldsymbol{A}^*$ and $\boldsymbol{A}_{\mathbf{R}}$ are quotient algebras of $R_\Sigma(A)$. Thus the following problem seems to be natural. Characterize those congruences $\sim$ on $R_\Sigma(X)$, where X is a set, for which $R_\Sigma(X)/\sim$ becomes an iterative algebra.

7.4. Necessary and sufficient conditions on an iterative algebra to admit a faithful regular extension, found in Corollary 6.13 are not satisfactory. Find simpler conditions.

One possible way to achieve this would be as follows.

7.5. Define on A^* a sequence of equivalence relations ($\sim_\alpha$, $\alpha \in \mathbf{Ord}$) by

$$|t| \sim_\alpha |t'| \quad \text{iff} \quad |t| \leqslant_\alpha |t'| \text{ and } |t'| \leqslant_\alpha |t|.$$

One may prove that $\boldsymbol{A}$ admits a faithful regular extension iff $\sim\omega_1$ is the equality relation. Is it always the case that $\sim_1 = \sim_{\omega_1}$? If it is, then this would provide a simpler characterization mentioned in Section 7.4.

7.6. Find concrete representations of regular extensions of various known iterative theories like: theories of sequacious functions and various matricial theories (cf. [4]). Do they admit faithful extensions?

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