Topology and its Applications 158 (2011) 1360–1368

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Versions of properties (a) and (pp)

A. Caserta [a,*], G. Di Maio [a], Lj.D.R. Kočinac [b]

[a] Dipartimento di Matematica, Seconda Università di Napoli, Via Vivaldi 43, 81100 Caserta, Italy
[b] Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia

ARTICLE INFO

MSC:
54D20
91A44

Keywords:
Selectively (a)
Selectively (pp)
Star selection principles
Alexandroff duplicate
Game theory

ABSTRACT

In this paper we define and study several versions of properties (a) and (pp) and their relationships with other covering properties. Especially, we investigate when the Alexandroff duplicate $\mathsf{AD}(X)$ of a space X has considered properties.

1. Introduction

Our notation and terminology are standard as in [11], and all spaces are assumed to be T_1. Let X be a topological space, $\mathcal{U}$ and $\mathcal{V}$ collections of subsets of X, and A a subset of X. Then we write $\mathcal{V} < \mathcal{U}$ to denote that $\mathcal{V}$ is inscribed in $\mathcal{U}$, i.e. for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$; when $\mathcal{U}$ is a cover of X, $\mathcal{V}$ may be but need not be a cover of X. The symbol $\mathrm{St}(A, \mathcal{U})$ denotes the star of A with respect to $\mathcal{U}$, i.e. the union of all elements from $\mathcal{U}$ intersecting A. For $A = \{x\}$, $x \in X$, we write $\mathrm{St}(x, \mathcal{U})$ instead of $\mathrm{St}(\{x\}, \mathcal{U})$.

In 1997, M.V. Matveev [22] introduced (a)-spaces and spaces with property (pp). On the other hand, Kočinac defined star selection principles, in particular star-Menger and strongly star-Menger spaces and their relatives in [17] (see also [6] for similar star selection properties). Also, in [20] Kočinac considered absolute versions of star selection principles. We follow these lines of investigation and define and study selective versions of (a)-spaces and (pp)-spaces following also some ideas from [12].

Selective versions of some classical covering topological properties have been already considered in the literature (see [2–4]).

Let X be a topological space and let $\mathfrak{U}$ and $\mathfrak{V}$ denote collections of some open covers of X. According to [4] the symbol $\mathsf{S}_{lf}(\mathfrak{U}, \mathfrak{V})$ denotes the following statement:

> For each sequence $(\mathcal{U}_n \colon n \in \mathbb{N})$ of elements of $\mathfrak{U}$ there is a sequence $(\mathcal{V}_n \colon n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a locally finite family of open sets with $\mathcal{V}_n < \mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n \in \mathfrak{V}$.

If in this definition we replace "$\mathcal{V}_n$ is a locally finite family" by "$\mathcal{V}_n$ is a point-finite family" (resp. "$\mathcal{V}_n$ is a disjoint family") we obtain the selection property $\mathsf{S}_{pf}(\mathfrak{U}, \mathfrak{V})$ [3] (resp. $\mathsf{S}_c(\mathfrak{U}, \mathfrak{V})$ [2], and also [1,15]).

* Corresponding author.
E-mail addresses: agata.caserta@unina2.it (A. Caserta), giuseppe.dimaio@unina2.it (G. Di Maio), lkocinac@gmail.com (Lj.D.R. Kočinac).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.05.010

In particular, if X satisfies $\mathsf{S}_{lf}(\mathcal{O},\mathcal{O})$ ($\mathsf{S}_{pf}(\mathcal{O},\mathcal{O})$, $\mathsf{S}_c(\mathcal{O},\mathcal{O})$) we say that X is *selectively paracompact* [4] (*selectively metacompact* [3], selectively screenable [2] or that X has property C [1,15]). Notice that a space X is selectively paracompact if and only if it is paracompact [4].

Here $\mathcal{O}$ denotes the collection of open covers of a space X. We use also symbols Ω, $\mathcal{K}$, Γ and Γ_k to denote the collections of ω-covers, k-covers, γ-covers and γ_k-covers of a space. Recall that an open cover $\mathcal{U}$ of a space X is said to be an *ω-cover* (a *k-cover*) if for each finite (compact) set $A \subset X$ there is an element U in $\mathcal{U}$ containing A, and $\mathcal{U}$ is called a *γ-cover* (a *γ_k-cover*) of X if it is infinite and each finite (compact) subset of X is contained in all but finitely many elements of $\mathcal{U}$.

Recall now definitions of star selection principles from [17].

Let $\mathfrak{U}$ and $\mathfrak{V}$ be collections of open covers of a space X and let $\mathcal{P}$ be a family of subsets of X. Then we say that X belongs to the class $\mathsf{SS}^*_{\mathcal{P}}(\mathfrak{U},\mathfrak{V})$ if X satisfies the following selection hypothesis: for each sequence $(\mathcal{U}_n\colon n\in\mathbb{N})$ of elements of $\mathfrak{U}$ there exists a sequence $(P_n\colon n\in\mathbb{N})$ of elements of $\mathcal{P}$ such that $\{\mathrm{St}(P_n,\mathcal{U}_n)\colon n\in\mathbb{N}\}\in\mathfrak{V}$. When $\mathcal{P}$ is the collection of all one-point [resp., finite, compact] subspaces of X we write $\mathsf{SS}^*_1(\mathfrak{U},\mathfrak{V})$ [resp., $\mathsf{SS}^*_{fin}(\mathfrak{U},\mathfrak{V})$, $\mathsf{SS}^*_{comp}(\mathfrak{U},\mathfrak{V})$] instead of $\mathsf{SS}^*_{\mathcal{P}}(\mathfrak{U},\mathfrak{V})$.

In particular, a space X is:

SSR (*strongly star-Rothberger*) if X satisfies $\mathsf{SS}^*_1(\mathcal{O},\mathcal{O})$;
SSM (*strongly star-Menger*) if it satisfies $\mathsf{SS}^*_{fin}(\mathcal{O},\mathcal{O})$;
SSH (*strongly star-Hurewicz*) if it satisfies $\mathsf{SS}^*_{fin}(\mathcal{O},\Gamma)$.

For more details regarding selection principles theory we refer the reader to the survey papers [18,19,26,30], and for star covering properties to the papers [23,10].

2. Selectively (a)-spaces

As we mentioned above, in [22] Matveev introduced the following property: a space X is said to be an *(a)-space* if for each open cover $\mathcal{U}$ of X and each dense subset D of X there is a closed discrete (in X) set $A \subset D$ such that $\mathrm{St}(A,\mathcal{U}) = X$. He also defined the class of (wa)-spaces replacing in the previous definition “closed discrete” by “discrete”. These spaces were studied in a number of papers [13,14,16,24,25].

We give now the following general selective version of the notion of (a)-spaces.

Definition 2.1. Let X be a space. Denote by $\mathfrak{U}$ and $\mathfrak{V}$ collections of some open covers of X, and by $\mathcal{P}$ a collection of subsets of X. Then X is said to be a *selectively* $(\mathfrak{U},\mathfrak{V})$-$(a)_{\mathcal{P}}$*-space*, denoted by $X\in\mathsf{Sel}(\mathfrak{U},\mathfrak{V})$-$(a)_{\mathcal{P}}$, if for each sequence $(\mathcal{U}_n\colon n\in\mathbb{N})$ of elements of $\mathfrak{U}$ and each dense subset D of X there is a sequence $(A_n\colon n\in\mathbb{N})$ of elements of $\mathcal{P}$ such that each A_n is a subset of D and $\{\mathrm{St}(A_n,\mathcal{U}_n)\colon n\in\mathbb{N}\}\in\mathfrak{V}$.

To avoid trivial situations we always assume throughout this section that $\mathfrak{U}$ is the collection $\mathcal{O}$ of open covers of the space.

Remark 2.2. If $\mathfrak{V}$ and $\mathcal{P}$ are as in the above definition it would be natural to define and investigate $(\mathcal{O},\mathfrak{V})$-$(a)_{\mathcal{P}}$-spaces for $\mathfrak{V}\in\{\mathcal{O},\Omega,\mathcal{K},\Gamma,\Gamma_k\}$: a space X has property $(\mathcal{O},\mathfrak{V})$-$(a)_{\mathcal{P}}$ if for each open cover $\mathcal{U}$ of X and each dense $D\subset X$ there is a set $A\subset D$ such that $A\in\mathcal{P}$ and $\{\mathrm{St}(a,\mathcal{U})\colon a\in A\}\in\mathfrak{V}$. Spaces of this kind have been already studied for $\mathfrak{V}=\mathcal{O}$ and several classes $\mathcal{P}$: when $\mathcal{P}$ is the family of finite (resp. countable, Lindelöf) subspaces of X, the corresponding classes of spaces are called *absolutely countably compact* [21] (resp. *absolutely star-Lindelöf* [23,28], *absolutely $\mathcal{L}$-starcompact* [27]).

Note that certain selectively $(\mathfrak{U},\mathfrak{V})$-$(a)_{\mathcal{P}}$ spaces were investigated in the literature: selectively $(\mathfrak{U},\mathfrak{V})$-$(a)_{\mathrm{finite}}$-spaces are exactly *absolutely* $\mathsf{SS}^*_{fin}(\mathfrak{U},\mathfrak{V})$ *spaces* [20]. In particular, selectively $(\mathcal{O},\mathcal{O})$-$(a)_{\mathrm{finite}}$ spaces coincide with aSSM-spaces (absolutely strongly star-Menger spaces) which form a subclass of SSM-spaces. Similarly, $\mathsf{Sel}(\mathcal{O},\Gamma)$-$(a)_{\mathrm{finite}} = \mathsf{aSSH}\subset\mathsf{SSH}$, and $\mathsf{Sel}(\mathcal{O},\mathcal{O})$-$(a)_{\mathrm{singleton}} = \mathsf{aSSR}\subset\mathsf{SSR}$.

For a space X satisfying $\mathsf{Sel}(\mathcal{O},\mathcal{O})$-$(a)_{\mathrm{closed\ discrete}}$ we say that X is a *selectively* (a)*-space*. This is a direct generalization of the notion of (a)-spaces.

Evidently, every (a)-space is selectively (a).

So, every monotonically normal space, in particular every GO-space, is selectively (a), being an (a)-space (see [24, Theorem 1]). For the same reason every selectively paracompact space is selectively (a).

Also, every countably compact selectively $(\mathcal{O},\mathcal{O})$-$(a)_{\mathrm{closed\ discrete}}$ space (resp. selectively $(\mathcal{O},\Gamma)$-$(a)_{\mathrm{closed\ discrete}}$ space) is SSM (resp. SSH).

By small changes in the proof of Lemma 1 and its corollary in [22] one can prove:

Proposition 2.3. *Let X be a separable space. Then*:

(1) *If X is selectively* (*a*)*, then every closed discrete subset of X has cardinality* $< 2^\omega$;
(2) *If X contains a discrete subspace having cardinality* $\geqslant 2^\omega$*, then* X^2 *is not hereditarily selectively* (*a*).

Example 2.4. If a space X is a countable union of open (a)-spaces X_n, $n \in \mathbb{N}$, then X is selectively (a).

Let D be a dense subset of X and let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of X. For each $n \in \mathbb{N}$ the intersection $D_n = D \cap X_n$ is a dense subset of X_n, and $\mathcal{U}_n$ is an open cover of X_n. As X_n is an (a)-space, there is a closed discrete set $A_n \subset D_n \subset D$ such that $\mathrm{St}(A_n, \mathcal{U}_n) \supset X_n$. Thus $\bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U}_n) = X$.

It is shown in [25, Theorem 3] that there are Ψ-spaces (see [5,11,25] for more information about these spaces) which are (a)-spaces, hence selectively (a), and those which are not (a)-spaces. The next example shows that there are also Ψ-spaces which are not selectively (a). We use below the well-known small cardinals $\mathfrak{b}$ and $\mathfrak{d}$.

Example 2.5. There are Ψ-spaces which are not selectively (a), as well as which are not selectively $(\mathcal{O}, \Gamma)$-(a)-spaces.

Take an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ such that $|\mathcal{A}| \geqslant \mathfrak{d}$. Then $\Psi(\mathcal{A})$ cannot be selectively (a). Indeed, let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of $\Psi(\mathcal{A})$ and let $\mathbb{N}$ be a dense subset of $\Psi(\mathcal{A})$. Each closed discrete subset of $\mathbb{N} \subset \Psi(\mathcal{A})$ is finite. Thus by the result of Matveev stating that $\Psi(\mathcal{A})$ is SSM if and only if $|\mathcal{A}| < \mathfrak{d}$ (see [17,5]), for any choice of closed discrete subsets A_n in $\mathbb{N}$, $\bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U}_n) \neq X$.

Similarly, if we take $\mathcal{A}$ with cardinality $\geqslant \mathfrak{b}$, then by [5, Proposition 3], $\Psi(\mathcal{A})$ is not SSH and so it cannot be selectively $(\mathcal{O}, \Gamma)$-(a).

The following results show the behaviour of selectively (a)-type spaces under mappings and operations with spaces.

Theorem 2.6. *A closed-and-open image* $Y = f(X)$ *of a selectively* (*a*)*-space* X *is also selectively* (*a*).

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of Y and D a dense subset of Y. First note that $f^{\leftarrow}(D)$ is a dense subset of X because f is open. The sequence $(f^{\leftarrow}(\mathcal{U}_n)\colon n \in \mathbb{N})$ is a sequence of open covers of X. The space X is selectively (a); let $(A_n\colon n \in \mathbb{N})$ be a sequence of closed discrete (in X) subsets of D witnessing for $(f^{\leftarrow}(\mathcal{U}_n)\colon n \in \mathbb{N})$ this fact. For each n let $B_n = f(A_n)$. We claim that the sequence $(B_n\colon n \in \mathbb{N})$ witnesses that Y is selectively (a).

1. For each $n \in \mathbb{N}$, $B_n \subset D$, and B_n is a closed discrete set in Y because f is closed and open.

2. $\bigcup_{n\in\mathbb{N}} \mathrm{St}(B_n, \mathcal{U}_n) = Y$.

Let $y \in Y$ and let x be a point of X such that $f(x) = y$. There are $m \in \mathbb{N}$ and an element $f^{\leftarrow}(U) \in f^{\leftarrow}(\mathcal{U}_m)$ such that $A_m \cap f^{\leftarrow}(U) \neq \emptyset$ and $x \in f^{\leftarrow}(U)$. It follows $y \in U \in \mathcal{U}_m$ and $B_m \cap U \neq \emptyset$, i.e. $y \in \mathrm{St}(B_m, \mathcal{U}_m)$. □

The product of two selectively (a)-spaces need not be selectively (a); the Sorgenfrey line S and its square S^2 can serve as an example (by Proposition 2.3 S^2 is not selectively (a)). It would be interesting to answer the following question (compare with [16, Theorem 16]):

Question 2.7. Is the product of a selectively (a)-space X and a metrizable compact space Y selectively (a)?

We have the following

Theorem 2.8. *If the product* $X \times Y$ *of spaces* X *and* Y *is selectively* (*a*) *and the projection* $p_X : X \times Y \to X$ *is closed, then* X *is selectively* $(\mathcal{O}, \mathcal{O})$-$(a)_{\text{closed}}$.

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of X and D a dense subset of X. For each $n \in \mathbb{N}$, $\mathcal{W}_n := \{U \times Y\colon U \in \mathcal{U}_n\}$ is an open cover of $X \times Y$. Since $X \times Y$ is a selectively (a)-space, and $D \times Y$ a dense subset of $X \times Y$, there is a sequence $(F_n\colon n \in \mathbb{N})$ of closed discrete subsets of $X \times Y$ such that for each n, $F_n \subset D$ and $\{\mathrm{St}(F_n, \mathcal{W}_n)\colon n \in \mathbb{N}\}$ is an open cover of $X \times Y$. The sets $p_X(F_n)$, $n \in \mathbb{N}$, are closed in X and each is contained in D. We claim that $\{\mathrm{St}(p_X(F_n), \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is an open cover of X. Let $x \in X$ and let (x, y) be a point in $X \times Y$. There are $m \in \mathbb{N}$, $F_m \subset D$ and $U \in \mathcal{U}_m$ such that $(x, y) \in U \times Y$ and $F_m \cap (U \times Y) \neq \emptyset$; pick a point $(p, q) \in F_m \cap (U \times Y)$. Then $p \in U \cap p_X(F_m)$ and thus $x \in \mathrm{St}(p_X(F_m), \mathcal{U}_m)$. □

Clearly, the previous theorem is true for a space X and a compact space Y.

In what follows we need a well-known construction. Let (X, τ) be a topological space. The Alexandroff duplicate of X (see [11,8]) is the set $\mathrm{AD}(X) := X \times \{0, 1\}$ equipped with the following topology. For each $U \in \tau$ let $\widehat{U} = U \times \{0, 1\}$. Define a base for a topology on AD(X) by $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$, where $\mathcal{B}_0$ is the family of all sets $\widehat{U} \setminus (F \times \{1\}) \subset \mathrm{AD}(X)$, with $U \in \tau$ and F a finite subset of X, and $\mathcal{B}_1 = \{\{(x, 1)\}\colon x \in X\}$. For every $x \in X$ put $\tau_x = \{U \in \tau\colon x \in U\}$ and $\mathcal{B}_{(x,0)} = \{\widehat{U} \setminus \{(x, 1)\}\colon U \in \tau_x\}$, and $\mathcal{B}_{(x,1)} = \{\{(x, 1)\}\}$. Then, if X is a T_1-space, $\mathcal{B}_{(x,0)}$ is a local base at each $(x, 0) \in \mathrm{AD}(X)$, and $\mathcal{B}' = \bigcup_{x\in X}(\mathcal{B}_{(x,0)} \cup B_{(x,1)})$

is a base in $\mathsf{AD}(X)$ such that $\mathcal{B}' \subset \mathcal{B}$. If $\mathcal{U}$ is a family of open sets in X, then we say that the family $\mathcal{U}^* := \{\widehat{U} \setminus (F \times \{1\})\colon U \in \mathcal{U}, F$ a finite subset of $X\}$ of open subsets of $\mathsf{AD}(X)$ is associated to $\mathcal{U}$ and vice versa.

For many topological properties $\mathcal{P}$ the space $\mathsf{AD}(X)$ has $\mathcal{P}$ if X has $\mathcal{P}$ (see, for example, [8]). Such properties are, for instance, complete regularity, normality, compactness, Lindelöfness, (hereditary) paracompactness. We investigate similar questions in connection with properties studied in this paper.

In what follows denote by I_X the set of isolated points of a space X, and by $e(X)$ the extent of X, the supremum of cardinalities of closed discrete subsets of X.

Theorem 2.9. *If $X \in \mathsf{Sel}(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{discrete}}$ and $e(\mathsf{AD}(X)) < \omega_1$, then $\mathsf{AD}(X)$ is also in $\mathsf{Sel}(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{discrete}}$.*

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of $\mathsf{AD}(X)$ and D a dense subset of $\mathsf{AD}(X)$. Note that each dense subset of $\mathsf{AD}(X)$ contains the set $(I_X \times \{0\}) \cup (X \times \{1\})$, so we suppose that $D = (D^* \times \{0\}) \cup (X \times \{1\})$ for some dense subset $D^* \subset X$ containing I_X. For each $x \in X$ and each $n \in \mathbb{N}$ there is $U_n \in \mathcal{U}_n$ with $(x, 0) \in U_n$ and hence there is a set $W_x^{(n)} = \widehat{V}_x^{(n)} \setminus \{(x, 1)\} \in \mathcal{B}_{(x,0)}$, with $V_x^{(n)}$ a neighbourhood of x in X, such that $W_x^{(n)} \subset U_n$. For each $n \in \mathbb{N}$ the set $\mathcal{V}_n = \{V_x^{(n)}\colon x \in X\}$ is an open cover of X. Since X belongs to $\mathsf{Sel}(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{discrete}}$ there is a sequence $(A_n\colon n \in \mathbb{N})$ of subsets of D^* such that each A_n is discrete in X and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{V}_n) = X$. For every $x \in A_n \setminus I_X$ pick a point $y_x \in V_x^{(n)} \setminus \{x\}$. By definition of $W_x^{(n)}$ we have $(x, 0) \in W_x^{(n)}$ and $(y_x, 1) \in W_x^{(n)}$. For each n set

$$B_n = \big(A_n \times \{1\}\big) \cup \big\{(y_x, 1)\colon x \in A_n \setminus I_X\big\} \cup \big((A_n \cap I_X) \times \{0\}\big).$$

Hence for each $n \in \mathbb{N}$, $B_n \subset D$ and B_n is discrete in $\mathsf{AD}(X)$. We claim that

$$X \times \{0\} \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(B_n, \mathcal{U}_n).$$

To show this take $(x, 0) \in X \times \{0\}$. Clearly, $x \in \bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{V}_n)$ and so there are $m \in \mathbb{N}$ and $V_m \in \mathcal{V}_m$ such that $x \in V_m$ and $V_m \cap A_m \neq \emptyset$. Consider three possible cases:

Case 1: $x \in A_m \cap I_X$.

Then $V_m \cap A_m = \{x\}$ and thus $(x, 0) \in W_x^{(m)} \cap B_m \neq \emptyset$, and since $W_x^{(m)}$ is contained in some $U_m \in \mathcal{U}_m$ we obtain $U_m \cap B_m \neq \emptyset$, hence $(x, 0) \in \mathrm{St}(B_m, \mathcal{U}_m)$.

Case 2: $x \in A_m \setminus I_X$.

There exists $y_x \neq x$ such that $(y_x, 1) \in W_x^{(m)} \cap B_m \neq \emptyset$. Therefore, $(x, 0) \in \mathrm{St}(B_m, \mathcal{U}_m)$.

Case 3: $x \notin A_m$.

Let $z \in V_m \cap A_m$. Then $(z, 1) \in W_x^{(m)} \cap B_m$, so that $(x, 0) \in \mathrm{St}(B_m, \mathcal{U}_m)$.

So, we have proved $X \times \{0\} \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(B_n, \mathcal{U}_n)$.

Let $E = \mathsf{AD}(X) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{St}(B_n, \mathcal{U}_n)$. Since $E \subset X \times \{1\}$ is closed and discrete in $\mathsf{AD}(X)$ and $e(\mathsf{AD}(X)) < \omega_1$, E is countable. Enumerate E bijectively as $E = \{p_n\colon n \in \mathbb{N}\}$ and let for each $n \in \mathbb{N}$

$$C_n = B_n \cup \{p_n\} \subset D.$$

For each n, C_n is discrete in $\mathsf{AD}(X)$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(C_n, \mathcal{U}_n) = \mathsf{AD}(X)$. □

Remark 2.10. Notice that the previous theorem remains true if "selectively $(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{discrete}}$" is replaced by "$(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{discrete}}$".

Question 2.11. Is a space X selectively (a) provided the space $\mathsf{AD}(X)$ is selectively (a)? What about the selectively $(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{discrete}}$ property?

The following result is related to Theorem 2.9. The property selectively $(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{countable}}$ in the next theorem is a selective version of absolute star-Lindelöfness (see [23,28]).

Theorem 2.12. *If the Alexandroff duplicate $\mathsf{AD}(X)$ of a space X is selectively $(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{countable}}$, then $e(X) < \omega_1$.*

Proof. Suppose to the contrary that there is a closed discrete subset B of X having cardinality $\geqslant \omega_1$. The set $B \times \{1\}$ is closed and open in $\mathsf{AD}(X)$. For each $n \in \mathbb{N}$ let $A_n = (B \times \{1\}) \setminus (C_n \times \{1\})$, where each C_n is a countable subset of B. Every A_n is a closed (discrete) subset of $\mathsf{AD}(X)$. For each n define $\mathcal{U}_n = \{\mathsf{AD}(X) \setminus A_n\} \cup \{\{(x, 1)\}\colon (x, 1) \in A_n\}$. We claim that the sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of $\mathsf{AD}(X)$ and the dense set $D = (I_X \times \{0\}) \cup (X \times \{1\}) \subset \mathsf{AD}(X)$ witness that $\mathsf{AD}(X)$ is not selectively $(\mathcal{O}, \mathcal{O})\text{-}(a)_{\mathrm{countable}}$. Indeed, if $(F_n\colon n \in \mathbb{N})$ is a sequence of countable subsets of D, then there is a point $b \in B$ such that $(b, 1) \notin \bigcup_{n\in\mathbb{N}} F_n$. Since $(b, 1)$ is an isolated point in $\mathsf{AD}(X)$, the set $\{(b, 1)\}$ is the only element of every $\mathcal{U}_n$ that contains $(b, 1)$, and $(b, 1) \notin \mathrm{St}(F_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$. This contradicts the assumption on $\mathsf{AD}(X)$. □

In [28] it was proved: the Alexandroff duplicate of a (T_1) space X is absolutely star-Lindelöf if and only if $e(X) < \omega_1$. So, we have the following corollary.

Corollary 2.13. *The Alexandroff duplicate of a space X is absolutely star-Lindelöf if and only if it is selectively* $(\mathcal{O},\mathcal{O})$*-*$(a)_{countable}$.

A natural question is to know when subspaces of the Alexandroff duplicate $\mathsf{AD}(X)$ of a space X have properties of selectively (a)-type.

Recall the definition of such subspaces (called lines). Let X be a space, A and B disjoint subspaces of X. The subspace $Z=(A\times\{1\})\cup(B\times\{0\})$ of $\mathsf{AD}(X)$ is called a *Michael-type line* (see [7, Definition 2.6.16], [8, Definition 3.14]).

Theorem 2.14. *Let A and B be subspaces of a space X such that* $\overline{A}\cap B=\emptyset$ *and* $Z=(A\times\{1\})\cup(B\times\{0\})$*. If* $e(Z)<\omega_1$ *and B is selectively* $(\mathcal{O},\mathcal{O})$*-*$(a)_{\text{discrete}}$*, then Z is selectively* $(\mathcal{O},\mathcal{O})$*-*$(a)_{\text{discrete}}$.

Proof. Let $(\mathcal{U}_n\colon n\in\mathbb{N})$ be a sequence of open covers of Z and D a dense subset of Z. One may suppose that $D=(E\times\{0\})\cup(A\times\{1\})$, where E is a dense subset of $B\setminus A'$ because every dense subset of Z contains the set D (see [7, Lemma 2.2.9], [8, Lemma 2.17]). So, E is dense in B. It remains to repeat the proof of Theorem 2.12 with appropriate changes. □

3. Selectively (pp)-spaces

In this section we introduce and investigate selective versions of (pp)-spaces introduced first in [22] (and in a more general form in [12]): a space X is said to be a *(pp)-space* if for each open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that if $x_V\in V$ for each $V\in\mathcal{V}$, then the set $\{x_V\colon V\in\mathcal{V}\}$ is closed and discrete in X. For investigation of these spaces see [9,12–14].

The next definition is exactly a general selective version of the $(pp)^{\mathfrak{Q}}_{\mathcal{P}}$ property [12, Definition 1].

Definition 3.1. Let $\mathfrak{U}$ and $\mathfrak{V}$ be collections of some open covers of a space X, $\mathcal{P}$ a collection of subsets of X, and $\mathfrak{Q}$ a collection of families of subsets of X. Then X is said to be *selectively* $(\mathfrak{U},\mathfrak{V})$-$(pp)^{\mathfrak{Q}}_{\mathcal{P}}$, denoted by $X\in\mathsf{Sel}(\mathfrak{U},\mathfrak{V})\text{-}(pp)^{\mathfrak{Q}}_{\mathcal{P}}$, if for each sequence $(\mathcal{U}_n\colon n\in\mathbb{N})$ of elements of $\mathfrak{U}$ there is a sequence $(\mathcal{V}_n\colon n\in\mathbb{N})$ such that:

(a) for each n, $\mathcal{V}_n<\mathcal{U}_n$;
(b) $\mathcal{V}=\bigcup_{n\in\mathbb{N}}\mathcal{V}_n\in\mathfrak{V}$;
(c) if $W_V\subset V$ and $W_V\in\mathcal{P}$ for each $V\in\mathcal{V}_n$, $n\in\mathbb{N}$, then the set $\{W_V\colon V\in\mathcal{V}_n\}$ belongs to $\mathfrak{Q}$.

Remark 3.2. There is an infinitely long two-person game played on X associated to this selection principle. Players ONE and TWO play a round for each $n\in\mathbb{N}$. In the n-th round ONE chooses a cover $\mathcal{U}_n\in\mathfrak{U}$, and then TWO responds by choosing a partial open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$. TWO wins a play $\mathcal{U}_1,\mathcal{V}_1;\mathcal{U}_2,\mathcal{V}_2;\ldots$ if $\bigcup_{n\in\mathbb{N}}\mathcal{V}_n\in\mathfrak{V}$, and for each $n\in\mathbb{N}$ and any choice $W_V\subset V$, $W_V\in\mathcal{P}$ for each $V\in\mathcal{V}_n$, the set $\{W_V\colon V\in\mathcal{V}_n\}$ is in $\mathfrak{Q}$.

If TWO has a winning strategy in this game, then the selection principle holds. Call a space X *strongly selectively* $(\mathfrak{U},\mathfrak{V})$-$(pp)^{\mathfrak{Q}}_{\mathcal{P}}$ if TWO has a winning strategy in the game described above.

Conjecture 3.3. *The class of strongly selectively* $(\mathfrak{U},\mathfrak{V})$-$(pp)^{\mathfrak{Q}}_{\mathcal{P}}$ *spaces is a proper subclass of the class of selectively* $(\mathfrak{U},\mathfrak{V})$-$(pp)^{\mathfrak{Q}}_{\mathcal{P}}$ *spaces.*

$\mathsf{Sel}(\mathcal{O},\mathcal{O})\text{-}(pp)^{\text{closed discrete}}_{\text{singleton}}$ spaces are called simply *selectively (pp)-spaces*. They are a generalization of (pp)-spaces as defined in [22].

Notice that σ-(pp)-spaces are selectively (pp), and that selectively (pp)-spaces are selectively (a). Selectively paracompact (= paracompact) spaces are selectively (pp).

For some $\mathfrak{U}$, $\mathfrak{V}$, $\mathcal{P}$ and $\mathfrak{Q}$ from Definition 3.1 the class $\mathsf{Sel}(\mathfrak{U},\mathfrak{V})\text{-}(pp)^{\mathfrak{Q}}_{\mathcal{P}}$ is trivially equivalent to a known class of spaces. For instance, $\mathsf{Sel}(\mathcal{O},\mathcal{O})\text{-}(pp)^{\text{locally finite}}_{\text{open}}=\mathsf{Sel}(\Omega,\mathcal{O})\text{-}(pp)^{\text{locally finite}}_{\text{open}}=\mathsf{S}_{pf}(\mathcal{O},\mathcal{O})$, while $\mathsf{Sel}(\Omega,\Omega)\text{-}(pp)^{\text{locally finite}}_{\text{open}}=\mathsf{Sel}(\Omega,\Omega)\text{-}(pp)^{\text{finite}}_{\text{open}}=$ the class of spaces having the Menger covering property in all finite powers.

Example 3.4. Let $\mathbb{R}$ be endowed with the countable complement extension topology τ, also known as open-minus-countable topology [29]: a set U is open in τ if and only if $U=O\setminus C$, O open in the usual metric topology on $\mathbb{R}$ and $C\subset\mathbb{R}$ countable. For each n, the space $[-n,n]$ is Lindelöf. If $(\mathcal{U}_n\colon n\in\mathbb{N})$ is a sequence of open covers, then countably many elements from $\mathcal{U}_n$ cover $[-n,n]$, so for every choice of elements from U, $U\in\mathcal{U}_n$, we get a countable set. But countable sets in this space are closed and discrete. Thus the space is selectively (pp).

Theorem 3.5. *For a space X the following are equivalent*:

(1) *X belongs to the class* $\mathsf{Sel}(\mathfrak{U},\mathfrak{V})\text{-}(pp)^{\text{closed discrete}}_{\text{singleton}}$;
(2) *X belongs to the class* $\mathsf{Sel}(\mathfrak{U},\mathfrak{V})\text{-}(pp)^{\text{closed}}_{\text{singleton}}$.

Proof. We have to prove (2) implies (1). Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of elements of $\mathfrak{U}$ and let $(\mathcal{V}_n\colon n \in \mathbb{N})$ be a sequence witnessing that X belongs to $\mathsf{Sel}(\mathfrak{U}, \mathfrak{V})\text{-}(pp)^{\text{closed}}_{\text{singleton}}$. We prove that for each n the set $\{x_V\colon V \in \mathcal{V}_n\}$ is discrete. Suppose not. There are $n \in \mathbb{N}$ and $V^* \in \mathcal{V}_n$ such that each neighbourhood of x_{V^*} intersects $\{x_V\colon V \in \mathcal{V}_n\}$ in a point different from x_{V^*}. Then take for each $V \in \mathcal{V}_n$ a point y_V in the following way:

$$\begin{cases} y_V = x_V, & \text{if } x_V \neq x_{V^*}; \\ y_V \in V^* \setminus \{x_{V^*}\} \text{ arbitrary}, & \text{if } x_V = x_{V^*}. \end{cases}$$

Evidently, the set $\{y_V\colon V \in \mathcal{V}_n\}$ is not closed because $x_{V^*} \in \overline{\{y_V\colon V \in \mathcal{V}_n\}} \setminus \{y_V\colon V \in \mathcal{V}_n\}$, a contradiction. □

A natural question for selectively (pp)-spaces is when they are (selectively) paracompact. Also, because paracompact spaces are both selectively (pp) and selectively metacompact, it is reasonable to ask what about relations between the latter two classes of spaces.

In [9] it was shown that every separable (pp)-space is Lindelöf, and so every regular separable (pp)-space is paracompact. With slight modifications in the proof from [9] it is not difficult to prove

Theorem 3.6. *Every regular separable selectively (pp)-space is selectively paracompact, and so paracompact.*

The following three theorems establish relations between the selective (pp) property and selective metacompactness.

Theorem 3.7. *A selectively (pp)-space X with $\chi(x, X) = \omega$ for each $x \in X$ is selectively metacompact.*

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of X and let $(\mathcal{V}_n\colon n \in \mathbb{N})$ be a sequence witnessing (for $(\mathcal{U}_n\colon n \in \mathbb{N})$) that X is selectively (pp); by Theorem 3.5 one may assume that the set $\{x_V\colon V \in \mathcal{V}_n\}$ is closed for each $n \in \mathbb{N}$. We claim that each $\mathcal{V}_n$ is point-finite. Suppose not and let x be a point in X such that for some $n \in \mathbb{N}$ the set $\mathcal{V}_n^* = \{V \in \mathcal{V}_n\colon x \in V\}$ is infinite, say $\mathcal{V}_n^* = \{V_{n,j}\colon j \in \mathbb{N}\}$. Pick a countable base $\{B_1, B_2, \cdots\}$ at x. As $\chi(x, X) = \omega$, for each $i \in \mathbb{N}$ and each $V \in \mathcal{V}_n^*$, the intersection $V \cap (B_i \setminus \{x\})$ is not empty. Associate to each $V \in \mathcal{V}_n$ a point $x_V \in X$ as follows:

(i) if $V = V_{n,i}$, then $x_V \in V_{n,i} \cap (B_i \setminus \{x\})$;
(ii) if $V \in \mathcal{V}_n \setminus \mathcal{V}_n^*$, then x_V is an arbitrary point in $V \setminus \{x\}$.

It is clear that $x \in \overline{\{x_V\colon V \in \mathcal{V}_n\}} \setminus \{x_V\colon V \in \mathcal{V}_n\}$. We have a contradiction, and so X is selectively metacompact. □

It is worth observing that this result can be extended in such a way that the condition "$\chi(x, X) = \omega$" can be replaced by "the q-character of X is countable" (compare with [12]).

In [9, Lemma 2.6] it was shown that a $(pp)^{\text{locally finite}}_{\text{finite}}$ refinement is point-finite. A similar proof works for the selective version of this result, and using it we can prove the following assertion.

Theorem 3.8. *Every selectively $(\mathcal{O}, \mathcal{O})\text{-}(pp)^{\text{locally finite}}_{\text{finite}}$ space is selectively metacompact.*

Theorem 3.9. *If the product $X \times Y$ of a space X and an infinite countably compact space Y is selectively (pp), then X is selectively metacompact.*

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of X. For each n set $\mathcal{W}_n = \{U \times Y\colon U \in \mathcal{U}_n\}$. Apply the assumption on $X \times Y$ to the sequence $(\mathcal{W}_n\colon n \in \mathbb{N})$ of open covers of $X \times Y$ and choose for each $n \in \mathbb{N}$ a partial refinement $\mathcal{H}_n$ of $\mathcal{W}_n$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n$ is an open cover of $X \times Y$ and for any choice $z_H \in H$, $H \in \mathcal{H}_n$, the set $\{z_H\colon H \in \mathcal{H}_n\}$ is closed discrete in $X \times Y$. Since Y is infinite and countably compact there is a non-isolated point $y_0 \in Y$. For each $n \in \mathbb{N}$ let $\mathcal{G}_n = \{H \cap (X \times y_0)\colon H \in \mathcal{H}_n\}$ and $\mathcal{V}_n = \{p_X(G)\colon G \in \mathcal{G}_n\}$, where $p_X : X \times Y \to X$ is the projection. Then $\mathcal{V}_n$ is a partial refinement of $\mathcal{U}_n$ and $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an open cover of X.

We prove that each $\mathcal{V}_n$ is point-finite and so the sequence $(\mathcal{V}_n;\ n \in \mathbb{N})$ testifies to X being selectively metacompact. Suppose there is $m \in \mathbb{N}$ such that $\mathcal{V}_m$ is not point-finite. Let $p \in X$ be a point belonging to infinitely many (distinct) elements V_i, $i \in \mathbb{N}$, of $\mathcal{V}_m$. For each i let G_i be the element from $\mathcal{G}_m$ such that $p_X(G_i) = V_i$. Then (p, y_0) belongs to infinitely many distinct elements G_i, $i \in \mathbb{N}$. Pick points $(p, y_i) \in G_i$, $i \in \mathbb{N}$, in such a way that $y_i \neq y_j$ whenever $i \neq j$. We get an infinite closed, discrete set $\{(p, y_i)\colon i \in \mathbb{N}\} \subset \{p\} \times Y$, $(p, y_i) \in H \in \mathcal{H}_m$, contradicting $\{p\} \times Y$ being a countably compact space. □

Example 3.10. The selective (pp) property is not finitely productive.

The Sorgenfrey line S is selectively (pp) being paracompact, but its square S^2 does not have this property. Otherwise, by Theorem 3.7, S^2 would be selectively metacompact, but it is known [3] that it is not the case.

We consider now which spaces are preserving factors for the selectively (pp)-like properties.

First we need to define a game which is a modification of the game defined in Remark 3.2. Given a space X define the following infinite game $G(X)$ on X played between two players ONE and TWO; they play a round for each natural number n. In the n-th round ONE chooses an open cover $\mathcal{U}_n$ of X, and TWO must then respond by choosing a point-finite family $\mathcal{V}_n$ of open subsets of X such that $\mathcal{V}_n < \mathcal{U}_n$. TWO wins a play $\mathcal{U}_1, \mathcal{V}_1; \mathcal{U}_2, \mathcal{V}_2; \dots$ if $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ covers X and for each $n \in \mathbb{N}$ and each choice $F_V \in [V]^{<\omega}$, $V \in \mathcal{V}_n$, the set $\bigcup_{V\in\mathcal{V}_n} F_V$ is closed and discrete. A similar game $G^*(X \times Y)$ is played on the product $X \times Y$, but in the n-th round TWO chooses $\mathcal{V}_n < \mathcal{U}_n$ (not necessarily point-finite). TWO wins if $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ covers $X \times Y$ and for each $n \in \mathbb{N}$ and each choice $z_V \in V$, $V \in \mathcal{V}_n$, the set $\{z_V\colon\ V \in \mathcal{V}_n\}$ is discrete.

Theorem 3.11. *If* TWO *has a winning strategy in the game $G(X)$, then* TWO *has a winning strategy in the game $G^*(X \times Y)$ for every compact space Y.*

Proof. Let σ be a winning strategy for TWO in $G(X)$. Define a strategy Σ for TWO in the game $G^*(X \times Y)$ for a compact space Y.

Let $\mathcal{W}_1$ be an open cover of $X\times Y$ chosen by ONE in the first round of the game $G^*(X\times Y)$. Without loss of generality one may assume that all elements of all $\mathcal{W}_n$ are rectangular sets of the form $U^{(n)} \times V^{(n)}$, $U^{(n)}$ open in X, $V^{(n)}$ open in Y. TWO looks at the open cover $p_Y(\mathcal{W}_1)$ of Y, and argues as follows. For each $x \in X$ there are finitely many elements $U^{(1)}_{x_i} \times V^{(1)}_{x_i}$, $i \leqslant k^{(1)}_x$ covering $\{x\} \times Y$; let $O^{(1)}_x = \bigcap_{i\leqslant k^{(1)}_x} U^{(1)}_{x_i}$. Then $\mathcal{O}_1 := \{O^{(1)}_x\colon\ x \in X\}$ is an open cover of X, and the move $\mathcal{O}_1$ is a legal move for ONE in the game $G(X)$. Let the response of TWO in $G(X)$ be $\sigma(\mathcal{O}_1) = \mathcal{G}_1$, with $\mathcal{G}_1$ a point-finite partial open refinement of $\mathcal{O}_1$. For each $G \in \mathcal{G}_1$ there is $O^{(1)}_{x_G} = \bigcap_{i\leqslant k^{(1)}_{x_G}} U^{(1)}_{x_i} \in \mathcal{O}_1$, with $G \subset O^{(1)}_{x_G}$. Let $\mathcal{H}_1 = \{G \times V^{(1)}_{x_i}\colon\ i \leqslant k^{(1)}_{x_G}, G \in \mathcal{G}_1\}$. Then $\mathcal{H}_1$ is an open partial refinement of $\mathcal{W}_1$. TWO plays $\Sigma(\mathcal{W}_1) = \mathcal{H}_1$ (in the game $G^*(X \times Y)$).

Let $\mathcal{W}_2$ be the second move of ONE in $G^*(X \times Y)$. TWO argues as in the first round and plays $\Sigma(\mathcal{W}_1, \mathcal{W}_2) = \mathcal{H}_2$. And so on.

We use the fact that TWO wins a play

$$\mathcal{O}_1, \sigma(\mathcal{O}_1) = \mathcal{G}_1;\ \mathcal{O}_2;\ \sigma(\mathcal{O}_1, \mathcal{O}_2) = \mathcal{G}_2;\ \dots$$

in the game $G(X)$, and prove that the play

$$\mathcal{W}_1, \Sigma(\mathcal{W}_1) = \mathcal{H}_1;\ \mathcal{W}_2, \Sigma(\mathcal{W}_1, \mathcal{W}_2) = \mathcal{H}_2;\ \dots$$

is won by TWO in $G^*(X \times Y)$.

1. $\bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ is an open cover of $X \times Y$.

Let $(x, y) \in X \times Y$. For each $n \in \mathbb{N}$ there is $U^{(n)}_{x_i} \times V^{(n)}_{x_i}$, $i \leqslant k^{(n)}_x$, containing (x, y), hence $(x, y) \in O^{(n)}_x \times V^{(n)}_{x_i}$, where $O^{(n)}_x \in \mathcal{O}_n$. But $\bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ is an open cover of X, and it follows there exist $n_0 \in \mathbb{N}$ and $G \in \mathcal{G}_{n_0}$ such that $x \in G$. Thus $(x, y) \in G \times V^{(n_0)}_{x_i} \in \mathcal{H}_{n_0}$.

2. For each $G \in \mathcal{G}_n$ and each $i \leqslant k^{(n)}_{x_G}$ pick a point $(\alpha^{(n)}_i, \beta^{(n)}_i) \in G \times V^{(n)}_i \in \mathcal{H}_n$. We claim that for each n the set T_n of points chosen in this way is discrete in $X \times Y$.

For each $G \in \mathcal{G}_n$, the set $A_G = \{\alpha^{(n)}_i\colon\ i \leqslant k^{(n)}_{x_G}\}$ is a finite subset of G and consequently $A_n = \bigcup\{A_G\colon\ G \in \mathcal{G}_n\}$ is closed and discrete in X since TWO won in $G(X)$.

Let $(p, q) \in X \times Y$. Consider two possible cases.

Case 1: $p \notin A_n$. There is a neighbourhood M_p of p such that $M_p \cap A_n = \emptyset$, and thus $(p, q) \in M_p \times Y$ and $(\alpha^{(n)}_i, \beta^{(n)}_i) \notin M_p \times Y$ for each $i \leqslant k^{(n)}_{x_G}$ and each $G \in \mathcal{G}_n$.

Case 2: $p \in A_n$. There is a neighbourhood S_p of p such that $S_p \cap A_n = \{p\}$; it follows that $S_p \times Y$ contains only points $(\alpha^{(n)}_i, \beta^{(n)}_i)$ with $\alpha^{(n)}_i = p$. Since $\mathcal{G}_n$ is point-finite there are finitely many $(p, \beta^{(n)}_{i_1}), \cdots (p, \beta^{(n)}_{i_s})$ in $S_p \times Y$. The set $(S_p \times Y) \setminus \{(p, \beta^{(n)}_{i_1}), \cdots (p, \beta^{(n)}_{i_s})\}$ is also a neighbourhood of (p, q) and intersects the set T_n only in (p, q), i.e. T_n is discrete. □

In a similar way one can prove the following theorem.

Theorem 3.12. *Let X be a selectively $(\mathcal{O}, \mathcal{O})$-$(pp)^{\text{closed discrete}}_{\text{finite}}$ space such that all $\mathcal{V}_n$ from the definition of selectively (pp)-spaces are point-finite. Then the product $X \times Y$ is selectively $(pp)^{\text{discrete}}_{\text{singleton}}$ for every compact space Y.*

In the next part of this section we are interested in the question regarding the Alexandroff duplicate and selectively (pp)-spaces. First we need the following theorem.

Theorem 3.13. *A closed irreducible image $Y = f(X)$ of a selectively (pp)-space X is selectively (pp).*

Proof. Let $(\mathcal{U}_n\colon\ n \in \mathbb{N})$ be a sequence of open covers of Y. X is a selectively (pp)-space, and thus for the sequence $(f^{\leftarrow}(\mathcal{U}_n)\colon\ n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{W}_n\colon\ n \in \mathbb{N})$ such that $\mathcal{W}_n < f^{\leftarrow}(\mathcal{U}_n)$, $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is an open cover of X and for each n and each choice $x_W \in W$, $W \in \mathcal{W}_n$, the set $\{x_W\colon W \in \mathcal{W}_n\}$ is closed in X (which is enough by Theorem 3.5). Since f is closed irreducible, for each n and each $W \in \mathcal{W}_n$ the set $f^\#(W) = Y \setminus f(X \setminus W)$ is non-empty open subset of Y. Therefore if for each n we put $\mathcal{V}_n = \{f^\#(W)\colon W \in \mathcal{W}_n\}$ we have the sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ which witnesses that Y is selectively (pp). We have only to prove that for each n and each choice y_V, $V \in \mathcal{V}_n$, the set $S_n = \{y_V\colon V \in \mathcal{V}_n\}$ is closed. Fix n. For each $y_V \in V \in \mathcal{V}_n$ take an element $W \in \mathcal{W}_n$ and a point $x_W \in W$ such that $f^\#(W) = V$ and $f(x_W) = y_V$. The set $\{x_W\colon W \in \mathcal{W}_n\}$ is closed in X for each n, because X is selectively (pp). Therefore, the set $\{y_V\colon V \in \mathcal{V}_n\}$ is closed in Y because f is a closed mapping. By Theorem 3.5, Y is selectively (pp). □

From Theorem 3.13 we obtain:

Corollary 3.14. *If the Alexandroff duplicate* $\mathsf{AD}(X)$ *of a space* X *is selectively* (pp)*, then* X *is also selectively* (pp)*.*

Proof. In [8, Lemma 1.6] it was shown that the mapping $r : \mathsf{AD}(X) \to X \times \{0\} \cong X$, defined by $r \restriction (X \times \{0\}) = \mathrm{id}_{(X\times\{0\})}$, $r(x,1) = (x,0)$, is a closed retraction, while by [7, Lemma 2.3.8] the mapping $r^\#$ is open. □

Parallel to Corollary 3.14 we have the following.

Theorem 3.15. *If a space* X *is selectively* $(\mathcal{O},\mathcal{O})\text{-}(pp)^{\text{discrete}}_{\text{singleton}}$*, then* $\mathsf{AD}(X)$ *has the same property.*

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of $\mathsf{AD}(X)$. Without loss of generality we may suppose that for each n, $\mathcal{U}_n \subset \mathcal{B}$, where $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ is the base in $\mathsf{AD}(X)$ described above. For each $n \in \mathbb{N}$ let $\mathcal{U}'_n$ be an open cover of X associated with $\mathcal{U}_n$. Take a sequence $(\mathcal{V}'_n\colon n \in \mathbb{N})$ which witnesses that X is selectively $(\mathcal{O},\mathcal{O})\text{-}(pp)^{\text{discrete}}_{\text{singleton}}$: $\mathcal{V}'_n < \mathcal{U}'_n$, $\bigcup_{n\in\mathbb{N}} \mathcal{V}'_n$ covers X, and for each choice $p_V \in V$, $V \in \mathcal{V}'_n$, the set $P := \{p_V\colon V \in \mathcal{V}'_n\}$ is discrete in X. For every $V \in \mathcal{V}'_n$ pick $U(V) \in \mathcal{U}'_n$ such that $V \subset U(V)$ and take $F_{U(V)} = F_V$, where F_V is a finite subset of X associated to V. Define for each $n \in \mathbb{N}$,

$$\mathcal{V}^{(0)}_n = \bigl\{\widehat{V} \setminus \bigl(F_V \times \{1\}\bigr)\colon V \in \mathcal{V}'_n,\ F_V \in [X]^{<\omega}\bigr\}, \qquad \mathcal{V}^{(1)}_n = \bigl\{\bigl\{(x,1)\bigr\}\colon (x,1) \notin \cup\mathcal{V}^{(0)}_n\bigr\}.$$

Then $\mathcal{V}^{(0)}_n < \mathcal{U}_n \cap \mathcal{V}_0$, $n \in \mathbb{N}$, and $X \times \{0\} \subset \bigcup_{n\in\mathbb{N}} \cup\mathcal{V}^{(0)}_n$. Define $\mathcal{V}_n = \mathcal{V}^{(0)}_n \cup \mathcal{V}^{(1)}_n$, $n \in \mathbb{N}$. Then the sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ testifies to $\mathsf{AD}(X)$ being selectively $(pp)^{\text{discrete}}_{\text{singleton}}$.

Clearly, for each n, $\mathcal{V}_n < \mathcal{U}_n$ and $\mathsf{AD}(X) = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$. It remains to prove that for each n and each choice $x_V \in V$, $V \in \mathcal{V}_n$, the set $S_n := \{x_V\colon V \in \mathcal{V}^{(0)}_n \cup \mathcal{V}^{(1)}_n\}$ is a discrete subset of $\mathsf{AD}(X)$. Consider three cases:

1. $x_V = (x,0) \in \mathsf{AD}(X)$.

Then there is a neighbourhood of x_V that meets the set S_n only in x_V which follows from the fact that the set $\{p_V\colon V \in \mathcal{V}'_n\}$ is discrete.

2. $x_V = (x,1)$ and $V \in \mathcal{V}^{(1)}_n$.

Then $\{(x,1)\}$ is a neighbourhood of x_V which intersects the set S_n only in the point x_V.

3. $x_V = (x,1)$ and $V \in \mathcal{V}^{(0)}_n$.

In this case $(x,1) \in \widehat{V} \setminus (F_V \times \{1\})$ for some $V \in \mathcal{V}'_n$, and we use again the fact that the set $\{p_V\colon V \in \mathcal{V}'_n\}$ is discrete. □

Acknowledgements

The paper has its roots in visits by Kočinac to the Department of Mathematics of SUN in December 2009 and May 2010. The third author thanks the Department for the hospitality he enjoyed during these visits. The authors thank the referee for his careful reading of the paper and a number of suggestions which led to improvements on several places.

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