Radial Displacements of an Infinite Liquid Saturated Porous Medium with Cylindrical Cavity

R. Kumar and A. Miglani
Department of Mathematics
Kurukshetra University
Kurukshetra - 132 119, Haryana, India

L. Debnath
Department of Mathematics
University of Central Florida
Orlando, FL 32816, U.S.A.

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Abstract—This paper deals with radial displacement fields in solid and liquid parts of a liquid-saturated porous medium with cylindrical cavity subjected to an arbitrary time dependent force. The Laplace transform technique is used to solve the problem. A particular case of impulsive force is discussed and closed form solutions are obtained. As a special case, results of classical elasticity are derived. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The problem of the disturbance in an elastic medium containing a cavity due to arbitrary stresses on the cavity is of great importance, particularly as a model of an earthquake source. On the other hand, the propagation of elastic waves in a liquid-saturated porous medium has been a subject of continued interest due to its importance in seismology and geophysics. Chakraborty [1] studied the problem of the disturbance in an isotropic elastic infinite slab of finite thickness due to forces applied on the inner surface of a cylindrical cavity. Vodicka [2] discussed the problem of radial vibrations of an infinite medium with a cylindrical cavity. Thiruvenkatachar and Viswanathan [3] investigated the dynamic response of an elastic half-space with cylindrical cavity to time dependent surface tractions over the boundary of the cavity. In this paper, we consider the problem of radial displacement of an unbounded liquid-saturated porous medium due to a cylindrical cavity whose boundary is subjected to an arbitrary time dependent force. A particular case of impulsive force is discussed with the closed form solution. Results of classical elasticity are derived as a special case.
2. BASIC EQUATIONS

In the absence of dissipation, the field equations for the liquid-saturated porous solid, are given by Biot [4,5],

\[ N\nabla^2 u + \text{grad}\{(D + N)e + Q\varepsilon\} = \frac{\partial^2}{\partial t^2}(\rho_{11}u + \rho_{12}U), \]  
(2.1)

\[ \text{grad}\{Qe + Re\} = \frac{\partial^2}{\partial t^2}(\rho_{12}u + \rho_{22}U), \]  
(2.2)

where \(D, N, Q,\) and \(R\) are the elastic constants for the solid-liquid aggregate; \(\rho_{11}, \rho_{12}, \rho_{22}\) are dynamical coefficients. \(u\) and \(U\) are the displacements in the solid and liquid parts, respectively, and the corresponding dilatations are given by

\[ e = \text{div} \ U, \quad \varepsilon = \text{div} \ U. \]  
(2.3a,b)

The stresses in the solid \(\sigma_{ij}\) and in the liquid \(\sigma\) are given by

\[ \sigma_{ij} = (De + Qe)\delta_{ij} + 2N\varepsilon_{ij}, \]  
(2.4)

\[ \sigma = Qe + Re, \]  
(2.5)

where \(\delta_{ij}\) is the Kronecker delta, and

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]  
(2.6)

3. FORMULATION AND SOLUTION OF THE PROBLEM

We consider an isotropic, homogeneous, liquid-saturated porous medium of infinite extent with a cavity of the form of circular cylinder of radius \(a\). The surface of the cylindrical cavity is assumed to be acted upon by time-dependent pressure \(f(t)\). We take the cylindrical polar coordinates \((r, \theta, z)\), with origin on the axis of cylinder and \(z\)-axis coinciding with it. We consider the case of radial symmetry, and assume that all quantities depend upon the radial coordinate \(r\) and \(t\) only. Therefore, the displacements in the solid and liquid parts can be written as

\[ u = u(r, t)\hat{e}_r, \]  
(3.1)

\[ U = U(r, t)\hat{e}_r. \]  
(3.2)

With the help of equations (3.1) and (3.2), equations (2.1) and (2.2), reduce to

\[ \rho\nabla^2 u + Q\nabla^2 U = \frac{\partial^2}{\partial t^2}(\rho_{11}u + \rho_{12}U), \]  
(3.3)

\[ Q\nabla^2 u + R\nabla^2 U = \frac{\partial^2}{\partial t^2}(\rho_{12}u + \rho_{22}U), \]  
(3.4)

where

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}. \]  
(3.5)

We assume that the initial displacements and their corresponding velocities are zero throughout the medium, that is,

\[ u(r, 0) = U(r, 0) = 0 \quad \text{for } r > a. \]  
(3.6)

The radiation condition imply that

\[ \lim_{r \to \infty} u(r, t) = \lim_{r \to \infty} U(r, t) = 0, \quad \text{for all } t > 0. \]  
(3.7)
We define the potentials $\phi(r, t)$ and $\psi(r, t)$ by

$$u = \frac{\partial \phi}{\partial r} \quad \text{and} \quad U = \frac{\partial \psi}{\partial r}. \quad (3.8a, b)$$

Substituting equations (3.8a, b), in equations (3.3) and (3.4), we obtain the coupled equations

$$P \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right\} \phi + Q \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right\} \psi = \rho_{11} \frac{\partial^2 \phi}{\partial t^2} + \rho_{12} \frac{\partial^2 \psi}{\partial t^2}, \quad (3.9)$$

$$Q \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right\} \phi + R \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right\} \psi = \rho_{12} \frac{\partial^2 \phi}{\partial t^2} + \rho_{22} \frac{\partial^2 \psi}{\partial t^2}. \quad (3.10)$$

If $\phi$ or $\psi$ is eliminated from these equations, both $\phi$ and $\psi$ satisfy the same equation

$$\left\{ A \nabla^4 - B \frac{\partial^2}{\partial t^2} \nabla^2 + C \frac{\partial^4}{\partial t^4} \right\} (\phi, \psi) = 0, \quad (3.11a, b)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}, \quad A = PR - Q^2, \quad B = P \rho_{22} + R \rho_{11} - 2Q \rho_{12},$$

$$C = \rho_{11} \rho_{22} - \rho_{12}^2, \quad \text{and} \quad P = D + 2N. \quad (3.12)$$

Application of the Laplace transform to equations (3.11a, b) with respect to $t$ gives the solutions of the transformed equations satisfying the radiation condition

$$\overline{\phi} = A_1 K_0(p_s1 r) + A_2 K_0(p_s2 r), \quad (3.13)$$

$$\overline{\psi} = E_1 K_0(p_s1 r) + E_2 K_0(p_s2 r), \quad (3.14)$$

where $K_0(z)$ are the modified Bessel functions and $p$ is the Laplace transform variable, and

$$s_1^2 = \frac{1}{\alpha_1^2} = \frac{B - \sqrt{B^2 - 4AC}}{2A}, \quad s_2^2 = \frac{1}{\alpha_2^2} = \frac{1}{\alpha_1^2} = \frac{B + \sqrt{B^2 - 4AC}}{2A}, \quad (3.15a, b)$$

and $\alpha_1, \alpha_2$ are the velocities of fast $P$ (or $P_f$) wave and slow $P$ (or $P_s$) wave respectively; $A_1, A_2, E_1,$ and $E_2$ are arbitrary constants.

Application of the Laplace transform to equation (3.9) and (3.10), and the use of equations (3.13) and (3.14) yields

$$E_j = m_j A_j, \quad (j = 1, 2), \quad (3.16)$$

where

$$m_j = \frac{P s_j^2 - \rho_{11}}{\rho_{12} - Q s_j^2} = \frac{Q s_j^2 - \rho_{12}}{\rho_{22} - R s_j^2}, \quad (j = 1, 2). \quad (3.17)$$

With the help of equations (3.8a, b), (3.13), (3.14) and (3.16), we obtain the Laplace transformed solutions

$$\overline{u}(r, p) = -[A_1 p_s1 K_1(p_s1 r) + a_2 p_s2 K_1(p_s2 r)], \quad (3.18)$$

$$\overline{U}(r, p) = -[m_1 A_1 p_s1 K_1(p_s1 r) + m_2 A_2 p_s2 K_1(p_s2 r)]. \quad (3.19)$$
4. BOUNDARY CONDITIONS

Deresiewicz and Skalak [6] formulated the boundary conditions appropriate for the boundaries of liquid-saturated porous solid.

The boundary conditions for the present problem are given by

\[(i) \quad \sigma_r = -f(t), \quad r = a, \quad t > 0, \quad (4.1)\]
\[(ii) \quad \sigma = 0, \quad r = a, \quad t > 0, \quad (4.2)\]

where
\[
\sigma_{rr} = P \frac{\partial u}{\partial r} + D \frac{u}{r} + Q \left( \frac{\partial U}{\partial r} + \frac{U}{r} \right), \quad (4.3)
\]
\[
\sigma = Q \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) + R \frac{\partial U}{\partial r} + \frac{U}{r}. \quad (4.4)
\]

Applying Laplace transform to the boundary conditions (4.1) and (4.4), and making use of equations (3.18) and (3.19), we get the following equations:

\[
\{(P + Qm_1)p^2 s_a^2 K_0(ps_1 a) + 2Nps_1 K_1(ps_1 a)\} A_1 + \{(P + Qm_2)p^2 s_2^2 a K_0(ps_2 a) + 2Nps_2 K_1(ps_2 a)\} A_2 = -a \tilde{f}(p), \quad (4.5)
\]
\[
\{(Q + Rm_1)p^2 s_1^2 a K_0(ps_1 a)\} A_1 + \{(Q + Rm_2)p^2 s_2^2 a K_0(ps_2 a)\} A_2 = 0. \quad (4.6)
\]

Solving equations (4.5) and (4.6) gives

\[
A_1 = -\frac{1}{\Delta} \left\{ a \tilde{f}(p)(Q + m_2 R)p^2 s_2^2 \right\} K_0(ps_2 a), \quad (4.7)
\]
\[
A_2 = \frac{1}{\Delta} \left\{ a \tilde{f}(p)(Q + m_1 R)p^2 s_1^2 \right\} K_0(ps_1 a), \quad (4.8)
\]

where
\[
\Delta = s_1 s_2 (B_1 p^4 + B_2 p^3),
\]
\[
B_1 = a s_1 s_2 K_0(ps_1 a) K_0(ps_2 a) A(m_2 - m_1),
\]
\[
B_2 = 2N\{Q\{s_2 K_0(ps_2 a) K_1(ps_1 a) - s_1 K_0(ps_1 a) K_1(ps_2 a)\} + R\{m_2 s_2 K_0(ps_2 a) K_1(ps_1 a) - m_1 s_1 K_0(ps_1 a) K_1(ps_2 a)\}\}. \quad (4.9)
\]

Substituting the values of $A_1, A_2$ from equations (4.7) and (4.8) in equations (3.18) and (3.19) gives

\[
\bar{u} = a \left[ \tilde{f}(p) \bar{w}_1(r,p) - \tilde{f}(p) \bar{w}_2(r,p) \right], \quad (4.10)
\]
\[
\bar{U} = a \left[ m_1 \tilde{f}(p) \bar{w}_1(r,p) - m_2 \tilde{f}(p) \bar{w}_2(r,p) \right], \quad (4.11)
\]

where
\[
\bar{w}_1(r,p) = \frac{(Q + m_2 R)s_2 K_0(ps_2 a)}{B_1 p + B_2} K_1(ps_1 r), \quad (4.12)
\]
\[
\bar{w}_2(r,p) = \frac{(Q + m_1 R)s_1 K_0(ps_1 a)}{B_1 p + B_2} K_1(ps_2 r). \quad (4.13)
\]

Making use of the convolution theorem of the Laplace transform, we obtain from (4.10) and (4.11)

\[
u(r,t) = a[f(t) * w_1(r,t) - f(t) * w_2(r,t)], \quad (4.14)
\]
\[
U(r,t) = a[m_1 f(t) * w_1(r,t) - m_2 f(t) * w_2(r,t)], \quad (4.15)
\]

where $*$ denotes the convolution operation and $w_j(r,t), (j = 1, 2)$ are the functions whose transforms are $\bar{w}_j(r,p), (j = 1, 2)$, respectively.

The solutions $u(r,t), U(r,t)$ given by (4.14) and (4.15) are known if $w_1(r,t)$ and $w_2(r,t)$ are known. Thus, the problem reduces to the determination of $w_1(r,t)$ and $w_2(r,t)$.
5. EVALUATION OF $w_1(r, t)$ AND $w_2(r, t)$

Applying the inverse Laplace transform to equation (4.12) gives

$$w_1(r, t) = \frac{1}{2\pi i} \int_{\gamma - \infty}^{\gamma + \infty} e^{st} \overline{w}_1(r, p) \, dp,$$

(5.1)

and with the help of asymptotic relation, we get

$$e^{st} \overline{w}_1(r, p) \sim \frac{a}{2N} \frac{(Q + m_2 R)s_2}{((Q + m_2 R)s_2 - (Q + m_1 R)s_1)} \frac{(a/r)^{1/2}}{(a p + 1)} \exp \left\{ \frac{p}{\alpha_1} \left( t - \frac{r - a}{\alpha_1} \right) \right\},$$

where

$$\alpha_1 = \frac{a}{2N} \frac{s_1 s_2 A(m_2 - m_1)}{(Q + m_2 R)s_2 - (Q + m_1 R)s_1}.$$

Therefore, we have two different expressions for $w_1(r, t)$ corresponding to the cases, $t < (r - a)/\alpha_1$ and $t > (r - a)/\alpha_1$.

If $t < (r - a)/\alpha_1$, we have by Cauchy’s theorem (see Figure 1)

$$w_1(r, t) = 0, \quad t < \frac{r - a}{\alpha_1},$$

(5.2)

since the integral over $BC'A$ vanishes as $R \to \infty$.

For the case of $t > (r - a)/\alpha_1$, we take the contour $ABCDEFGA$ of Figure 1. The integrals over the arcs $BC$, $GA$ and $DEF$ tend to zero as $R \to \infty$, $\varepsilon \to 0$ and there are no poles of the integrand within the contour. Consequently,

$$\int_{\gamma - \infty}^{\gamma + \infty} e^{st} \overline{w}_1(r, p) \, dp = \lim \left[ \int_{DC} e^{st} \overline{w}_1(r, p) \, dp - \int_{FG} e^{st} \overline{w}_1(r, p) \, dp \right],$$

(5.3)

the limit being taken for $R \to \infty$ and $\varepsilon \to 0$. 
On DC and FG, we put $p = \zeta_1 e^{i\tau}$ and $p = \zeta_1 e^{-i\tau}$, respectively, and equation (5.3) becomes

$$
\int_{-\infty}^{\infty} e^{p\tau} \mathcal{W}_1(r,p) \, dp = \int_{0}^{\infty} e^{-\zeta_1 t} \{ \mathcal{W}_1(r,\zeta_1 e^{-i\tau}) - \mathcal{W}_1(r,\zeta_1 e^{i\tau}) \} \, d\zeta_1,
$$

$$
= 2i \int_{0}^{\infty} e^{-\zeta_1 t} \text{Im} [\mathcal{W}_1(r,\zeta_1 e^{-i\tau})] \, d\zeta_1,
$$

where $\text{Im}(z)$ denotes the imaginary part.

In equations (4.12) and (4.13), we make use of the following results:

$$
K_0(se^{-i\tau}) = K_0(s) + i\pi I_0(s),
$$

$$
K_1(se^{-i\tau}) = -K_1(s) + i\pi I_1(s).
$$

Then, we have

$$
\mathcal{W}_1(r,\zeta_1 e^{-i\tau}) = \left( \frac{c + id}{e + if} \right),
$$

where

$$
c = -a(Q + m_2 R)s_2 \{ K_0(\zeta_1 s_2 a)K_1(\zeta_1 s_1 r) + \pi^2 I_0(\zeta_1 s_2 a)I_1(\zeta_1 s_1 r) \},
$$

$$
d = a\pi(Q + m_2 R)s_2 \{ K_0(\zeta_1 s_2 a)I_1(\zeta_1 s_1 r) - I_0(\zeta_1 s_2 a)K_1(\zeta_1 s_1 r) \},
$$

$$
e = -as_1 s_2 A(m_2 - m_1) \zeta_1 \{ K_0(\zeta_1 s_1 a)K_0(s_1 s_2 a) + \pi^2 I_0(\zeta_1 s_1 a)I_0(\zeta_1 s_2 a) \}
$$

$$
- 2N(Q + m_2 R)s_2 \{ K_0(\zeta_1 s_2 a)K_1(\zeta_1 s_1 a) + \pi^2 I_0(\zeta_1 s_2 a)I_1(\zeta_1 s_1 a) \}
$$

$$
+ 2N(Q + m_1 R)s_1 \{ K_0(\zeta_1 s_1 a)K_1(\zeta_1 s_2 a) + \pi^2 I_0(\zeta_1 s_1 a)I_1(\zeta_1 s_2 a) \},
$$

$$
f = -a\pi s_1 s_2 A(m_2 - m_1) \zeta_1 \{ K_0(\zeta_1 s_1 a)I_0(\zeta_1 s_2 a) + K_0(\zeta_1 s_2 a)I_0(\zeta_1 s_1 a) \}
$$

$$
+ 2N\pi(Q + m_2 R)s_2 \{ K_0(\zeta_1 s_2 a)I_1(\zeta_1 s_1 a) - I_0(\zeta_1 s_2 a)K_1(\zeta_1 s_1 a) \}
$$

$$
- 2N\pi(Q + m_1 R)s_1 \{ K_0(\zeta_1 s_1 a)I_1(\zeta_1 s_2 a) - I_0(\zeta_1 s_1 a)K_1(\zeta_1 s_2 a) \}.
$$

Therefore,

$$
\text{Im} [\mathcal{W}_1(r,\zeta_1 e^{-i\tau})] = \frac{Z_1(r,\zeta_1)}{N_1(\zeta_1)},
$$

where

$$
Z_1(r,\zeta_1) = de - cf,
$$

$$
N_1(\zeta_1) = e^2 + f^2.
$$

Substituting (5.4) in (5.1) and making use of equation (5.6) gives

$$
w_1(r,t) = \frac{1}{\pi} \int_{0}^{\infty} \exp(-t\zeta_1) \frac{Z_1(r,\zeta_1)}{N_1(\zeta_1)} \, d\zeta_1, \quad t > \frac{r - a}{\alpha_1}.
$$

With the help of equations (5.2) and (5.8), we obtain the expression for $w_1(r,t)$ as

$$
w_1(r,t) = \frac{1}{\pi} \cdot H \left( t - \frac{r - a}{\alpha_1} \right) \int_{0}^{\infty} \exp(-t\zeta_1) \frac{Z_1(r,\zeta_1)}{N_1(\zeta_1)} \, d\zeta_1,
$$

and $H(x)$ is the Heaviside unit step function.

Similarly, we obtain the expression for $w_2(r,t)$ as

$$
w_2(r,t) = \frac{1}{\pi} \cdot H \left( t - \frac{r - a}{\alpha_2} \right) \int_{0}^{\infty} e^{-\zeta_1 t} \frac{Z_2(r,\zeta_2)}{N_2(\zeta_2)} \, d\zeta_2.
$$
Substituting equations (5.9) and (5.10), in equations (4.14) and (4.15), and using the convolution theorem, we obtain the displacement fields in the integral form

\[ u(r,t) = \frac{a}{\pi} \left[ H \left( t - \frac{r-a}{\alpha_1} \right) \int_{(r-a)/\alpha_1}^t f(t-\tau) \, d\tau \int_0^\infty e^{-\zeta_1 \tau} \frac{Z_1(r, \zeta_1)}{N_1(\zeta_1)} \, d\zeta_1 \right. \]
\[ - H \left( t - \frac{r-a}{\alpha_2} \right) \int_{(r-a)/\alpha_2}^t f(t-\tau) \, d\tau \int_0^\infty e^{-\zeta_2 \tau} \frac{Z_2(r, \zeta_2)}{N_2(\zeta_2)} \, d\zeta_2 \right] , \tag{5.11} \]

\[ \frac{\partial u}{\partial \alpha} \bigg|_{\alpha=\infty} \left[ H(t-r \rho_2^2 a) \frac{a}{\alpha_2} \int_{(r-a)/\alpha_2}^t f(t-\tau) \, d\tau \int_0^\infty e^{-\zeta_2 \tau} \frac{Z_2(r, \zeta_2)}{N_2(\zeta_2)} \, d\zeta_2 \right] . \tag{5.12} \]

\[ U(r,t) = \frac{a}{\pi} \left[ m_1 H \left( t - \frac{r-a}{\alpha_1} \right) \int_{(r-a)/\alpha_1}^t f(t-\tau) \, d\tau \int_0^\infty e^{-\zeta_1 \tau} \frac{Z_1(r, \zeta_1)}{N_1(\zeta_1)} \, d\zeta_1 \right. \]
\[ - m_2 H \left( t - \frac{r-a}{\alpha_2} \right) \int_{(r-a)/\alpha_2}^t f(t-\tau) \, d\tau \int_0^\infty e^{-\zeta_2 \tau} \frac{Z_2(r, \zeta_2)}{N_2(\zeta_2)} \, d\zeta_2 \right] . \]

6. SPECIAL CASE

We consider the disturbance produced by an impulsive force at the boundary \( r = a \) as

\[ f(t) = F \delta(t), \tag{6.1} \]

where \( \delta(t) \) is the Dirac delta function and \( F \) is the constant magnitude of the force and the Laplace transform of \( f(t) \) is \( \tilde{f}(p) = F \).

Therefore, equations (4.10) and (4.11) becomes

\[ \bar{u}(r, p) = aF \left[ \bar{w}_1(r, p) - \bar{w}_2(r, p) \right], \tag{6.2} \]
\[ \bar{U}(r, p) = aF \left[ m_1 \bar{w}_1(r, p) - m_2 \bar{w}_2(r, p) \right]. \tag{6.3} \]

Thus, using the inverse \( w_j(r,t), \ (j = 1, 2), \) of \( \bar{w}_j(r, p), \ (j = 1, 2), \) as given by equations (5.9) and (5.10), we obtain the displacement fields as

\[ u(r,t) = \frac{aF}{\pi} \left[ H \left( t - \frac{r-a}{\alpha_1} \right) \int_0^\infty e^{-\zeta_1 \tau} \frac{Z_1(r, \zeta_1)}{N_1(\zeta_1)} \, d\zeta_1 \right. \]
\[ - H \left( t - \frac{r-a}{\alpha_2} \right) \int_0^\infty e^{-\zeta_2 \tau} \frac{Z_2(r, \zeta_2)}{N_2(\zeta_2)} \, d\zeta_2 \right] , \tag{6.4} \]

\[ \frac{\partial u}{\partial \alpha} \bigg|_{\alpha=\infty} \left[ H(t-r \rho_2^2 a) \frac{a}{\alpha_2} \int_0^\infty e^{-\zeta_2 \tau} \frac{Z_2(r, \zeta_2)}{N_2(\zeta_2)} \, d\zeta_2 \right] . \tag{6.5} \]

In the limit as \( Q/R \to 0, \rho_{12}/\rho_{22} \to 0, \) we obtain the expressions for displacement in the solid part, \( u(r,t) \) due to [6], whereas displacement in liquid part, \( U(r,t) \) vanishes.

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