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Note

Perfect matchings in pruned grid graphs

David R. Guichard

Whitman College, Walla Walla, WA 99362, United States

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Abstract

We determine conditions sufficient to guarantee the existence of a perfect matching when vertices are removed from finite and infinite grid graphs. The conditions impose a minimum distance between the vertices that are removed. While the distances are likely not best possible, they are best possible with respect to asymptotic growth rate.
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A (finite) grid graph $G_{n,m}$, sometimes called a complete grid graph, has vertex set $\{0, 1, 2, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$ and (x_1, y_1) adjacent to (x_2, y_2) if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. The infinite grid graph G_∞ has vertex set $\mathbb{Z} \times \mathbb{Z}$, and edges defined as for the finite grid graphs. The grid graphs are bipartite. The finite grid graphs correspond to "rectangular chess boards," and the familiar black and white coloring illustrates a bipartition; we will say that a grid graph G has bipartition $V(G) = W \cup B$ and assume that $(0, 0) \in W$.

A pruned grid graph, sometimes called a grid graph, is a grid graph with some vertex deletions. A 1-factor or perfect matching in a graph is a collection of edges that partition the vertex set, that is, a collection of edges such that every vertex is incident with exactly one of the edges. Perfect matchings in bipartite graphs are characterized by Hall's Theorem, which may be found in most elementary texts on graph theory. Perfect matchings of grid graphs correspond to tilings of chess boards by "dominoes", that is, tiles that cover exactly two adjacent squares on the board.

We are concerned here with finding perfect matchings after some vertices of the grid graph have been removed. We present some sufficient conditions that specify a minimum separation between the removed vertices. Our starting point is the following result; $d(x, y)$ is the usual graph theoretic distance between vertices.

Theorem 1 ([1]). *Let m be even and let $G_{m,m}$ be the $m \times m$ grid graph with bipartition $V(G_{m,m}) = W \cup B$. Let $W' \subseteq W$, $B' \subseteq B$, and $|W'| = |B'|$. Let $d = 4\lceil\sqrt{m}\rceil + 8$. Then the graph $G_{m,m} \setminus (W' \cup B')$ has a perfect matching if*

(1) *For $x, y \in W'$, $d(x, y) > d$,*
(2) *For $x, y \in B'$, $d(x, y) > d$.*

As the authors point out, the constants 4 and 8 in this result are probably not best possible, but the asymptotic behavior of $\sqrt{m}$ is best possible.

E-mail address: guichard@whitman.edu.

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We investigate some questions left open by this result: What about rectangular grid graphs $G_{n,m}$ with $n \neq m$? What about the removal of "small" numbers of vertices—may they be closer to each other than specified by the theorem? And finally, what about infinite grid graphs? We begin with rectangular grid graphs.

Theorem 2. *Suppose that $n \geq m \geq 12$ and let $G_{n,m}$ be the $n \times m$ grid graph with bipartition $V(G_{n,m}) = W \cup B$. Let $W' \subseteq W$, $B' \subseteq B$, and $|W \backslash W'| = |B \backslash B'|$. Let $d = 4\lceil n/\sqrt{m} \rceil + 8$. Then the graph $G_{n,m} \backslash (W' \cup B')$ has a perfect matching if*

(1) *For $x, y \in W'$, $d(x, y) > d$,*
(2) *For $x, y \in B'$, $d(x, y) > d$.*

Note that while Theorem 1 is true for $m \geq 2$, it is not particularly interesting when m is small, because the minimum value of $4\lceil \sqrt{m} \rceil + 8$ is 16. The sets W' and B' cannot have more than one element until $m \geq 14$. Likewise, Theorem 2 is not interesting when m is small, so although we need to assume $n \geq m \geq 12$ for the proof, this is not a real loss. As with Theorem 1, it is unlikely that the constants are best possible, and improving the constants would presumably allow us to say more about small values of m. We could improve the restriction on m by using a larger constant in place of the 4, but this would increase the value of m at which the theorem becomes interesting. Moreover, by using the constants from Theorem 1, we get a direct generalization of the theorem: when $n = m$, Theorem 2 becomes Theorem 1, except for the restriction on size.

Before proceeding with the proof, we need to introduce some concepts from [1].

A row R_j of $G_{n,m}$ is the vertex set $\{(i, j) \mid 0 \leq i < n\}$, and a column C_i is $\{(i, j) \mid 0 \leq j < m\}$. Given a set $A \subseteq W$, we say that row R_j is complete if $R_j \cap A = R_j \cap W$, and similarly for a column. The neighborhood $N(A)$ of A is, as usual, $\{v \mid \exists w \in A$such that v is adjacent to $w\}$. We define the height of set A to be

$$h(A) = 1 + \max\{j \mid \exists i\ (i, j) \in A \cup N(A)\} - \min\{j \mid \exists i\ (i, j) \in A \cup N(A)\}$$

and the width of A to be

$$w(A) = 1 + \max\{i \mid \exists j\ (i, j) \in A \cup N(A)\} - \min\{i \mid \exists j\ (i, j) \in A \cup N(A)\}.$$

A row R_j is left justified if $R_j \cap A = \{(0, j), (2, j), \ldots, (2k, j)\}$ for some k, and right justified if $R_j \cap A = \{(n - 2k - 1, j), (n - 2k + 1, j), \ldots, (n - 1, j)\}$ for some k. When n is odd a row R_j may be both left and right justified, precisely when the row is complete and j is even (recall that $(0, 0) \in W$). A row R_j is bi-justified if $R_j \cap A = \{(0, j), (2, j), \ldots, (2k, j), (n - 2l - 1, j), (n - 2l + 1, j), \ldots, (n - 1, j)\}$ for some k, l; this is possible only if n is odd. We adopt similar definitions for columns.

The surplus S^r_j of row R_j is $|R_j \cap N(A)| - |R_j \cap A|$, and similarly for the surplus S^c_i of column C_i. The surplus of a row or column is at least -1. The surplus of a row R_j is -1 if and only if n is odd, the row is complete, and j is even, and similarly for columns. Otherwise, $S^r_j = 0$ only if R_j is left justified, right justified, or bi-justified, and similarly for columns.

If $S^r_j = 0$ and R_j is not complete, then $S^r_{j+1} > 0$ and $S^r_{j-1} > 0$, provided of course that $j + 1 < m$ and $j - 1 \geq 0$, respectively. If $S^r_j = 0$ and R_j is not complete, $(0, j \pm 1) \notin A$ and $(n - 1, j \pm 1) \notin A$. Similarly, if $S^c_i = 0$ and C_i is not complete, then $S^c_{i+1} > 0$ and $S^c_{i-1} > 0$, provided that $i + 1 < n$ and $i - 1 \geq 0$, respectively.

Finally, we need the following lemma from [1]; note that the lemma is true whether or not $(0, 0) \in W$.

Lemma 3. *Suppose $G_{n,m}$ is a grid graph with bipartition $V(G_{n,m}) = W \cup B$. Suppose d is a positive integer and $B' \subseteq B$ has the property that for any $x, y \in B'$, $d(x, y) > d$. Then*

$$|B'| \leq \frac{2nm}{d^2} + \frac{2(n + m)}{d} + 1,$$

and similarly for $W' \subseteq W$.

Proof of Theorem 2. By Hall's Theorem, it suffices to show that for all $A \subseteq W \backslash W'$, $|N(A) \backslash B'| = |N(A)| - |N(A) \cap B'| \geq |A|$, and for this it suffices to prove $|N(C) \backslash B'| = |N(C)| - |N(C) \cap B'| \geq |C|$ for each $C \subseteq A$ such that $C \cup N(C)$ is a component of the subgraph induced by $A \cup N(A)$. We will in fact show that $|N(C)| - |C| \geq |N(C) \cap B'|$. Let $h = h(C)$, $w = w(C)$. We consider the $h \times w$ rectangle containing $C \cup N(C)$ one row or column at a time. The

quantity $|N(C)| - |C|$ is the sum of the surpluses of the rows that intersect this rectangle, or alternately, the sum of the surpluses of the columns that intersect the rectangle.

Case 1. Suppose first that there is a complete column, so $h = m$, but no complete row. We show that $|N(C)| - |C| \geq (h-1)/2 \geq |N(C) \cap B'|$.

If n is odd, the odd numbered rows R_j have $S_j^r > 0$, and $|N(C)| - |C| \geq (h-1)/2$ follows immediately. Suppose now that n is even, and let Z be the number of rows with zero surplus. Since there is no complete row, $S_j^r = 0$ only if R_j is left or right justified, and in both cases we have $S_{j-1}^r > 0$ and $S_{j+1}^r > 0$, provided of course that $j - 1 \geq 0$ and $j + 1 < m$, respectively. Now define $\phi(R_j) = R_{j+1}$ if R_j is left justified, $S_j^r = 0$, and $j < m-1$, and $\phi(R_j) = R_{j-1}$ if R_j is right justified and $S_j^r = 0$. ϕ is an injection from the set of rows with zero surplus (except possibly row $m-1$, when m is odd) to the set of rows with positive surplus. It follows easily that $h \geq Z + Z - 1$, so $Z \leq (h+1)/2$ and $|N(C)| - |C| \geq (h-1)/2$. Note that we have not used the fact that $h = m$ in arriving at this conclusion. Now the second inequality:

$$\begin{aligned}
|N(C) \cap B'| \leq |B'| &\leq \frac{2mn}{d^2} + \frac{2m}{d} + \frac{2n}{d} + 1 \\
&\leq \frac{2mn}{16n^2}m + \frac{1}{2n/\sqrt{m} + 4}m + \frac{1}{2\sqrt{m} + 4m/n}m + \frac{3}{2} - \frac{1}{2} \\
&\leq \frac{1}{8}m + \frac{1}{2\sqrt{n} + 4}m + \frac{1}{2\sqrt{m}}m + \frac{3}{2m}m - \frac{1}{2} \\
&\leq \frac{m-1}{2} = \frac{h-1}{2},
\end{aligned}$$

since $n \geq m \geq 12$.

Case 2. Next, suppose there is no complete column and that $w \geq h$. We will show that $|N(C) \cap B'| \leq \lceil (w-1)/2 \rceil \leq |N(C)| - |C|$.

By an argument almost identical to that in case 1, $|N(C)| - |C| \geq \lceil (w-1)/2 \rceil$, and

$$\begin{aligned}
|N(C) \cap B'| &\leq \frac{2wh}{d^2} + \frac{4w}{d} + 1 \\
&\leq \frac{hm}{8n^2}w + \frac{1}{n/\sqrt{m} + 2}w + \frac{3}{2} - \frac{1}{2} \\
&\leq \frac{1}{8}w + \frac{1}{\sqrt{n} + 2}w + \frac{3}{2w}w - \frac{1}{2} \\
&\leq \frac{1}{8}w + \frac{1}{2\sqrt{12} + 2}w + \frac{3}{2w}w - \frac{1}{2} \\
&\leq \frac{w-1}{2},
\end{aligned}$$

if $w \geq 8$. If $w < 8$, then since $h \leq w$ no two vertices can be at distance more than 12 in the $h \times w$ rectangle, but $12 < 22 \leq d = 4\lceil n/\sqrt{m} \rceil + 8$, since $n \geq m \geq 12$. Hence $|N(C) \cap B'| \leq 1 \leq \lceil (w-1)/2 \rceil$ since $w \geq 2$.

Case 3. Next, suppose there is no complete column and that $w < h$. Note that $w \geq 2$ and $h \geq 3$. Since $w < h \leq m \leq n$, there is no complete row. As in Case 1, it is therefore true that $|N(C)| - |C| \geq (h-1)/2$, and by an argument nearly identical to that for case 2, $|N(C) \cap B'| \leq (h-1)/2$.

Case 4. Now suppose there is both a complete row and a complete column in C. Let $A' = B \backslash N(C)$; note that $N(A') \subseteq W \backslash C$. Interchanging the roles of W and B, A' contains no complete row or column because C contains both a complete row and complete column. As in the first part of the proof, we may consider $A' \cup N(A')$ one component at a time. If $D \cup N(D)$ is such a component, slight modifications of the arguments in cases 2 and 3 show that $|W' \cap N(D)| \leq |N(D)| - |D|$ and so $|W' \cap N(A')| \leq |N(A')| - |A'|$. (The modifications are needed because $(0, 0) \notin B$ and indeed there may be no corner vertex in B.) Let $W'' = W' \backslash N(A')$, so $|W'| = |W''| + |W' \cap N(A')|$.

$|W| = |B| + \delta$, where δ is 0 or 1, depending on whether mn is even or odd. Hence, $|W'| = |B'| + \delta$. Now

$$|N(C) \cap B'| \leq |B'| = |W'| - \delta$$

$$= |W''| + |W' \cap N(A')| - \delta$$
$$\leq |W''| + |N(A')| - |A'| - \delta.$$

Note that $C \cup N(A') \subseteq W \backslash W''$: If $w \in C$, then $w \in W$ and $w \notin W'$, so $w \notin W''$. If $w \in N(A')$, $w \in W$. If $w \notin W'$ then $w \notin W''$. If $w \in W'$, then since also $w \in N(A')$, $w \notin W''$. Since $C \cap N(A') = \emptyset$, $|C| + |N(A')| \leq |W| - |W''|$. Now

$$\begin{aligned} |N(C) \cap B'| &\leq |W''| + |N(A')| - |A'| - \delta \\ &\leq |W''| + |W| - |W''| - |C| - |A'| - \delta \\ &\leq |B| + \delta - |C| - |A'| - \delta \\ &\leq |A'| + |N(C)| - |C| - |A'| \\ &\leq |N(C)| - |C| \end{aligned}$$

as desired. □

Suppose that W' and B' are "small". Then they might be contained in a small rectangle of the full grid graph $G_{n,m}$, in which case we might expect that the elements of W' and B' could be closer together than n and m would indicate, since we could first tile the small rectangle and then the rest of the big grid graph. On the other hand, if W' and B' are "small" but spread out over the whole grid graph, that should make it easier to tile the pruned grid graph. We might therefore expect that there is a minimum separation based on the size of W' and B' that guarantees a tiling; we can indeed prove such a result in the case that $n = m$.

Theorem 4. *Let $G_{m,m}$ be the $m \times m$ grid graph with bipartition $V(G_{m,m}) = W \cup B$. Let $W' \subseteq W$, $B' \subseteq B$, $|W \backslash W'| = |B \backslash B'|$, and $M = \max(|W'|, |B'|)$. Let $d = 8\left\lceil \sqrt{M} \right\rceil + 8$. Then the graph $G_{m,m} \backslash (W' \cup B')$ has a perfect matching if*

(1) *For $x, y \in W'$, $d(x, y) > d$,*
(2) *For $x, y \in B'$, $d(x, y) > d$.*

Proof. It suffices to show that for all $A \subseteq W \backslash W'$, $|N(A) \backslash B'| = |N(A)| - |N(A) \cap B'| \geq |A|$, and for this it suffices to prove $|N(C) \backslash B'| = |N(C)| - |N(C) \cap B'| \geq |C|$ for each $C \subseteq A$ such that $C \cup N(C)$ is a component of the subgraph induced by $A \cup N(A)$. We will in fact show that $|N(C)| - |C| \geq |N(C) \cap B'|$.

Let $h = h(C)$ and $w = w(C)$; without loss of generality we may assume that $h \geq w$. Once again, we consider the $h \times w$ rectangle containing $C \cup N(C)$ one row or column at a time. The quantity $|N(C)| - |C|$ is the sum of the surpluses of the rows that intersect this rectangle, or alternately, the sum of the surpluses of the columns that intersect the rectangle.

Case 1. Suppose first that there is no complete row. By an argument nearly identical to case 1 from Theorem 2, $|N(C)| - |C| \geq (h - 1)/2$. Now

$$\begin{aligned} |N(C) \cap B'| &\leq \frac{2hw}{d^2} + \frac{2(h + w)}{d} + 1 \\ &\leq \frac{h}{8} \frac{16w}{(8\sqrt{M} + 8)^2} + \frac{h}{2\sqrt{M} + 2} + \frac{3}{2} - \frac{1}{2}. \end{aligned}$$

If $16w \leq (8\sqrt{M} + 8)^2$, $M > 1$, and $h \geq 9$, then the last expression is at most

$$\frac{h}{8} + \frac{h}{2\sqrt{2} + 2} + \frac{3}{18}h - \frac{1}{2} \leq \frac{h - 1}{2}.$$

If $w \leq h \leq 8$, two points within the $h \times w$ rectangle are at distance at most 14; since $d = 8\left\lceil \sqrt{M} \right\rceil + 8 \geq 16$, $|N(C) \cap B'| \leq 1 \leq h/2$. If $M = 1$, $|N(C) \cap B'| \leq M = 1 \leq h/2$.

If $16w > (8\sqrt{M} + 8)^2$ then

$$h \geq w > \frac{64M + 128\sqrt{M} + 64}{16} \geq 4M \geq 2M,$$

so $|N(C) \cap B'| \leq M \leq h/4 \leq (h - 1)/2$.

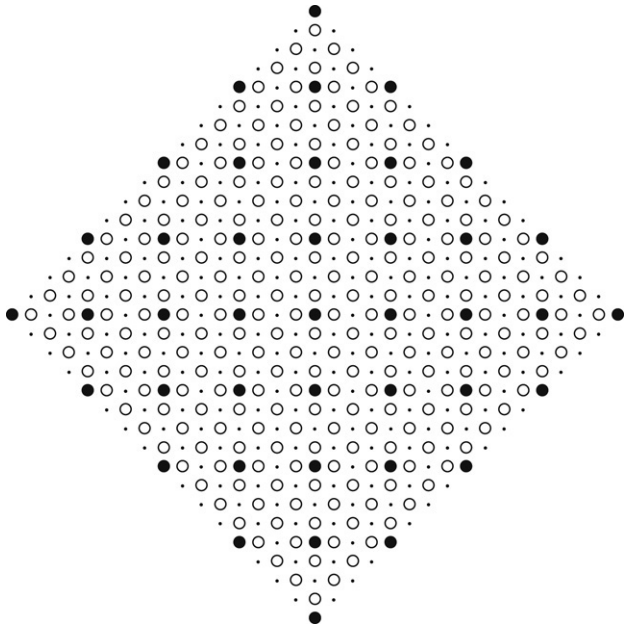

Fig. 1. A pruned graph that cannot be tiled.

Case 2. Suppose now that there is a complete row but no complete column. Then $w = m$, so $h = m$. If we interchange rows and columns (take the "transpose" of $G_{m,m}$) we have no complete rows, and we may apply case 1.

Case 3. The final case is identical to case 4 of Theorem 2; this finishes the proof. □

We might also hope for a rectangular version of this result, but at least any direct generalization can't work. To see why, suppose that $k = |W'| = |B'| = m/2 + 1$, where m is even. Suppose that d is a function of $|W'|$ alone, not of m or n, and let d' be $d + 1$ or $d + 2$, whichever is even. Let $B' = \{(0, 1), (d', 1), (2d', 1), \ldots, ((k-1)d', 1)\}$; the elements of B' are more than d apart. Pick k elements of W to be W', with the first coordinate of each greater than kd'. This is possible so long as n is large enough.

Now let A be $W \cap \{0, 1, 2, \ldots, (k-1)d' - 1\} \times \{0, 1, \ldots, m-1\}$; $|A| = m(k-1)d'/2$. The number of vertices in B available to match the elements of A is $m((k-1)d' + 1)/2 - k = m(k-1)d'/2 + m/2 - k = m(k-1)d'/2 - 1 < |A|$. Hence, no tiling of $G_{n,m} \backslash (W' \cup B')$ is possible.

Finally, we consider the infinite grid graph.

Theorem 5. *Let G_∞ be the infinite grid graph with bipartition $V(G_\infty) = W \cup B$. Suppose that $W' \subseteq W$ and $B' \subseteq B$ are finite and $M = \max(|W'|, |B'|)$. Let $d = 8\left\lceil \sqrt{M} \right\rceil + 8$. Then the graph $G_\infty \backslash (W' \cup B')$ has a perfect matching if*

(1) *For $x, y \in W'$, $d(x, y) > d$,*
(2) *For $x, y \in B'$, $d(x, y) > d$.*

Proof. Without loss of generality, suppose $|W'| \leq |B'|$. Choose m even and large enough so that there is a subgraph H of G_∞ isomorphic to $G_{m,m}$ that contains W' and B', and moreover is large enough so that $|B'| - |W'|$ vertices in W may be removed from the perimeter of H while maintaining the separation d. That is, we can choose $U \subseteq W$ from the perimeter of H and form $W'' = W' \cup U$ so that $|W''| = |B'|$ and for $x, y \in W''$, $d(x, y) > 8\left\lceil \sqrt{M} \right\rceil + 8$. By Theorem 4, $H \backslash (W'' \cup B')$ has a perfect matching. This can easily be extended to a perfect matching of $G_\infty \backslash (W' \cup B')$ because $(G_\infty \backslash H) \cup U$ is easily seen to have a perfect matching. □

As with the earlier theorems, the constants here are certainly not best possible, but the form of d is asymptotically correct, as this example shows: In the infinite grid graph, pick the vertices of B' in a diamond pattern as shown in Fig. 1, with the separation between nearest vertices in B' equal to d; of course, d is even. In the figure, $d = 4$, the

elements of B' are shown as large black dots, other elements of B are small black dots, and elements of W are open circles. The longest row and column of the diamond contain $2d + 1$ elements of B', so $|B'| = 2d^2 + 2d + 1$.

The total number of vertices in the diamond is $2d^4 + 2d^2 + 1$, consisting of d^4 white vertices and $d^4 + 2d^2 + 1$ black vertices. The black vertices completely enclose the white vertices, so if the whole grid graph can be tiled, the white vertices inside the diamond must be matched with black vertices in the diamond. But the number of black vertices in the diamond after B' is removed is $d^4 + 2d^2 + 1 - 2d^2 - 2d - 1 = d^4 - 2d$, less than the number of white vertices. The separation of elements of B' is approximately $\sqrt{|B'|/2}$, as desired.

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References

[1] R.E.L. Aldred, R.P. Anstee, S.C. Locke, Perfect matchings after vertex deletions, Discrete Mathematics 307 (23) (2007) 3048–3054.