Ordinary and Modular Characters of $SU(3, p)$

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This article deals with the relationship between ordinary and $p$-modular characters of the group $SU(3, p)$, parallel to the treatment of $SL(3, p)$ in J. E. Humphreys, Ordinary and modular characters of $SL(3, p)$, J. Algebra 72 (1981), 8–16. Here we denote by $SU(3, q)$ the special unitary subgroup of $SL(3, q^2)$, where $q$ is a power of the prime $p$; the notation $SU(3, q^2)$ is perhaps more standard, but we want to emphasize the comparison with $SL(3, q)$. Our main conclusion is that the decomposition behavior is essentially the “same” for both groups, in a sense to be made more precise below.

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1. CHARACTER DATA

First we assemble the relevant data about ordinary and modular characters of $SU(3, q)$. This is closely analogous to the data for $SL(3, q)$ summarized in the first section of [10], so we just indicate briefly the differences between the two groups.

The character table of $SU(3, q)$ is given in [17] (modulo some small errors in the description of parameters). In case $q \equiv 2 \pmod{3}$, the group has a center of order 3, and the parameter $d$ in the table takes the value 3. Otherwise $SU(3, q) = PSU(3, q)$ and $d = 1$. The characters may be divided into three families $A, B, C$ having respective generic degrees,

$q^3 - 2q^2 + 2q - 1, \quad q^3 + 1, \quad q^3 + q^2 - q - 1$.

For example, we denote the character $\chi_{st}^{(u, v, w)}$ of [17] by $A(u, v, w)$. For characters not of the generic degrees, we write

$\chi_1 = 1$;

$\chi_{uv} = q^2 - q$;

$\chi_{at} = St$;

$\chi_{st}^{(w)} = B'(u)$;

$\chi_{t}^{(w)} = B''(u)$.

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Note that the $q$ characters $B'(u)$ (resp. $B''(u)$) have degree $q^3 - q^2 + q$ (resp. $q^2 - q + 1$). When $d = 3$ there are also several triples of characters having degrees equal to a third of the generic degrees associated with $A$, $C$, $C$.

From now on we set $q = p$. The irreducible $p$-modular representations of $SU(3, p)$ are the same as those of $SL(3, p)$, coming from irreducible rational representations of the algebraic group $G = SL(3, K)$ (where $K$ is algebraically closed of characteristic $p$). They are parametrized by restricted highest weights which we abbreviate as in [10] by pairs $(a, b)$, $0 < a, b < p$. Write $L(a, b)$ for the $G$-module in question. When $a = b = p - 1$ we get the Steinberg module of dimension $p^3$, which is the reduction modulo $p$ of the representation with character $St$. This determines the unique $p$-block of defect 0.

Contrary to the notation in [10], we use $U(a, b)$ to denote the principal indecomposable module (PIM) for $SU(3, p)$ corresponding to the irreducible module $L(a, b)$. (We use the same notation for the character when convenient.) This module occurs once as a direct summand in the restriction to $SU(3, p)$ of the $G$-module $Q(a, b)$ which lifts the corresponding PIM for the restricted enveloping algebra of the Lie algebra of $G$. In fact, $Q(a, b) = U(a, b)$ except in the following cases:

\[
Q(0, 0) = U(0, 0) + U(p - 1, 1) + U(1, p - 1) + St;
\]
\[
Q(p - 2, 0) = U(p - 2, 0) + 2 St;
\]
\[
Q(0, p - 2) = U(0, p - 2) + 2 St;
\]
\[
Q(r, 0) = U(r, 0) + U(r + 1, p - 1) \quad \text{for} \quad 0 < r < p - 2;
\]
\[
Q(0, s) = U(0, s) + U(p - 1, s + 1) \quad \text{for} \quad 0 < s < p - 2.
\]

This can be worked out straightforwardly using the general formula found independently by Chastkofsky [3] and by Jantzen [12]. Unfortunately, there are errors in the previously published results for $SU(3, p)$: In [9, p. 981, the author suggested how the $Q(a, b)$ might decompose, based on incomplete evidence; in a letter to the author (January 1978) Chastkofsky pointed out some errors and supplied the correct versions. Different errors appear in the recent paper of Andersen [1, p. 399].

As in [10] one can go on to express the principal indecomposable characters of $SU(3, p)$ as explicit $Z$-linear combinations of class functions $s(a, b)St$. For example,

\[
U(0, 0) = (s(p - 1, p - 1) + s(1, 1) - s(0, p - 2) - s(p - 2, 0) - s(0, 0)) St.
\]
2. Projective Cover of the Trivial Module

Theorem. With the above notation, the principal indecomposable character $U(0, 0)$ of $SU(3, p)$ ($p \geq 7$) decomposes as follows into ordinary irreducible characters:

$$1 + B''(3) + B''(-3) + A(1, p, p + 1) + A(2, p - 1, p + 1) + B(p + 1) + C(p - 2) + C(2 - p).$$

As in [10], the proof involves a routine comparison of the character of $U(0, 0)$ with the indicated sum of characters as given in [17]. The theorem shows that (for $p \geq 7$) the first Cartan invariant is equal to 8, just as in the case of $SL(3, p)$; this calculation, done by the author around 1984, was mentioned in [11]. The coincidence of first Cartan invariants has been shown by Ye [18] to hold more generally when $p$ is replaced by an arbitrary power $q = p^n$ ($p \geq 7$). For the prime 2, cf. the work of Chastkofsky and Feit [4], together with that of Cheng [5].

Fig. 1. $U(0, 0)$ as part of the Brauer complex of $SU(3, p)$, $p \geq 7$. 
It is difficult at first sight to see any resemblance between the theorem and the corresponding result in [10]. However, the configuration of alcoves representing $U(0,0)$ in the Brauer complex of $SU(3, p)$ is identical with the configuration for $SL(3, p)$ given in [10, Figure 1], cf. Fig. 1.

3. Decomposition Patterns for Projectives

As we did in [10, Table 1], we summarize in Table 1 the various ways in which principal indecomposable characters in the principal block decompose into ordinary irreducible characters, when $p \equiv 1 \pmod{3}$. The asterisks indicate the "generic" cases, for weights inside the two restricted alcoves.

We actually discovered these patterns by carrying over to the Brauer complex of $SU(3, p)$ the configurations of alcoves representing various PIM's in the Brauer complex of $SL(3, p)$. The point is that the same configurations work in both cases, as in our treatment above of $U(0,0)$. However, to verify the table rigorously requires detailed comparison of the character values involved, as in the above theorem. In view of recent work

<table>
<thead>
<tr>
<th>Conditions on $(a, b)$</th>
<th>$\dim U(a, b)/p^3$</th>
<th>Ordinary characters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = b = 0$</td>
<td>5</td>
<td>$1 + 2A + 2B'' + B + 2C$</td>
</tr>
<tr>
<td>$0 &lt; a + b &lt; p - 3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ab \neq 0$</td>
<td>+ 12</td>
<td>$2A + 6B + 4C$</td>
</tr>
<tr>
<td>$ab = 0$</td>
<td>9</td>
<td>$2A + B' + 4B + 3C$</td>
</tr>
<tr>
<td>$a + b = p - 3$</td>
<td>12</td>
<td>$A + B'(\text{twice}) + 5B + 4C$</td>
</tr>
<tr>
<td>$ab \neq 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ab = 0$</td>
<td>9</td>
<td>$A + B'(\text{twice}) + B'' + 3B + 3C$</td>
</tr>
<tr>
<td>$a + b = p - 2$</td>
<td>6</td>
<td>$A + 3B + 2C$</td>
</tr>
<tr>
<td>$ab \neq 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ab = 0$</td>
<td>4</td>
<td>$A + 2B + C + (p^2 - p)$</td>
</tr>
<tr>
<td>$a + b = p - 1$</td>
<td>6</td>
<td>$A + B' + B'' + 2B + 2C$</td>
</tr>
<tr>
<td>$ab \neq 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ab = 0$</td>
<td>3</td>
<td>$2B' + B'' + C$</td>
</tr>
<tr>
<td>$p - 1 &lt; a + b &lt; 2p - 2$</td>
<td>* 6</td>
<td>$A + 3B + 2C$</td>
</tr>
<tr>
<td>$a \neq p - 1, b \neq p - 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a = p - 1$ or $b = p - 1$</td>
<td>3</td>
<td>$B' + B + C$</td>
</tr>
</tbody>
</table>
ORDINARY AND MODULAR CHARACTERS

of Chastkofsky on the Brauer complex and its interpretation (not yet published), it is reasonable to hope for a more systematic approach to PIM's along geometric lines.

The table alone does not allow one to compute the decomposition matrix for a given \( p \); for that one has to deal very explicitly with parameters for weights and for ordinary characters. However, it is quite easy to work out such a small case as \( p = 7 \) by hand, using the Brauer complex as a guide (cf. [10, example \( p = 5 \))). After we did this, a computer calculation was used to verify the determinant of the resulting \( 48 \times 48 \) matrix of Cartan invariants for the principal block (namely, \( 7^{10} \)). For \( SU(3, 7) \) the decomposition and Cartan matrices appear in [20], but without any identifying parameters for the representations; our results appear to be consistent with these. (For \( p = 2, 3, 5 \) the ordinary and modular character tables, together with decomposition and Cartan matrices for the principal block, also appear in [20].)

When \( p \equiv 2 \pmod{3} \), there are 3 blocks of highest defect. In each block 3 extra characters occur, of degrees equal to one third the degrees of the type \( A \) or \( C \) characters; such a triple of characters will share the same modular constituents. The table above requires some modifications, analogous to those described in [10] for \( \Gamma(3, p) \) when \( p \equiv 1 \pmod{3} \). In particular, each block has an occurrence of 3 as a decomposition number.

4. DECOMPOSITION BEHAVIOR OF FAMILIES OF CHARACTERS

We have just been discussing, in effect, the transpose of the decomposition matrix. One may also ask for more direct descriptions of the way in which various ordinary characters decompose modulo \( p \), e.g., the character of degree \( p^2 - p \) has modular constituents of highest weights \( (p - 2, 0) \) and \( (0, p - 2) \).

In a recent preprint [19], A. E. Zalesskii has described in detail the composition factors modulo \( p \) of a family of \( q + 1 \) ordinary irreducible representations of \( SU(n, q) \), \( n > 2 \), where \( q \) is a power of \( p \). These representations were described by G. Seitz [15] as restrictions of representations of symplectic groups constructed earlier by H. N. Ward. In case \( n = 3 \) and \( q = p \), these \( p + 1 \) ordinary characters are the ones we have denoted by \( B'' \), along with the single character of degree \( p^2 - p \). Zalesskii gives an explicit algorithm (his Corollary 0.5) for writing down the highest weights of the composition factors in question, which in our case translates into a simple recipe: for \( 0 \leq a \leq p - 1 \), we get the triple of highest weights \( (a, p - 1 - a), (0, a - 2), (p - 3 - a, 0) \), omitting nondominant weights in the four cases when they appear.

For the family of \( B' \) characters there is a similar pattern involving
usually five composition factors: \((a, p - 1 - a), (p - 1, a - 1), (p - 2 - a, p - 1), (a - 1, p - 2 - a)\) (repeated).

For the \(A, B, C\) characters, one gets patterns of a generic sort, usually involving nine distinct composition factors, cf. [12]. The author has used the rubric "deformations of linkage classes" to describe what is going on here, by comparison with induced modules for the restricted enveloping algebra.

5. Twisted and Untwisted Groups

Underlying our summary of the decomposition behavior for \(SU(3, p)\) is the very close analogy with \(SL(3, p)\). There is a well known transition (going back to a conjecture of V. Ennola [7], cf. [14, 8, 13]) from the character table of one group to that of the other, making possible the combined approach of [17]. The two groups share the same modular irreducible representations (a fact exploited in [19]) and have "almost" the same PIM's. Even so, the relationship between decomposition and Cartan matrices is not straightforward (e.g., because of the different block structures depending on congruences modulo 3). We expect the Brauer complexes for the two groups (cf. [2, 3.8]) to be the best tool for working out the connections conceptually. The goal, not yet fully realized, is to predict formally all decomposition behavior of one group from the known behavior of the other group. (This depends in turn on the comparison of projective modules, cf. [4, 18, 6].)

More generally, we expect a close resemblance between decomposition behavior for \(SL(3, q)\) and for \(SU(3, q)\); but we cannot formulate a precise conjecture at this time. Similar questions will of course have to be asked about other Chevalley groups and their twisted analogues, where the ordinary (resp. modular) characters are again known to be closely related (cf. [16, Theorem 4.18]).

References

4. L. Chastkofsky and W. Feit, On the projective characters in characteristic 2 of the groups \(SL_3(2^n)\) and \(SU_3(2^n)\), \(J.\ Algebra 63\) (1980), 124–142.
5. Y. Cheng, On the first Cartan invariants in characteristic 2 of the groups \(SL_3(2^n)\) and \(SU_3(2^n)\), \(J.\ Algebra 82\) (1983), 194–244.