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Comparing cohomology obstructions

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ABSTRACT

We show that three different kinds of cohomologies – Baues–Wirsching cohomology, the $(\mathcal{S}_*, \mathcal{O})$-cohomology of Dwyer and Kan, and the André–Quillen cohomology of a Π-algebra – are isomorphic, under certain assumptions. This is then used to identify the cohomological obstructions in three general approaches to realizability problems: the track category version of Baues and Wirsching, the diagram rectifications of Dwyer, Kan, and Smith, and the Π-algebra realization of Dwyer, Kan, and Stover. Our main tool in this identification is the notion of a *mapping algebra*: a simplicially enriched version of an algebra over a theory.

0. Introduction

A number of questions arising in topology can be framed in terms of realizing an algebraic or homotopic structure in a topological setting: for example, realizing an unstable algebra over the Steenrod algebra as the cohomology of a space, realizing a Π-algebra, or lifting a group action up to homotopy to a strict action. In these examples, the answer appears in the form of an obstruction theory, in which elements in appropriate cohomology groups serve both as the obstructions to realization, and as difference obstructions which classify the various possible realizations.

Three general approaches to dealing with such questions have been described in [6,34,35,16,33]. Our goal in this paper is to prove that these three approaches essentially coincide, in the cases where they all apply. In order to do so, we introduce the notion of a *mapping algebra* – a simplicially enriched version of an algebra over a theory, in the sense of Lawvere and Ehresmann (see Section 8) – and describe a fourth approach to the realization problem using this concept.

An important example of these methods is contained in the work of Goerss, Hopkins, and Miller on realizing ring spectra as structured spectra (cf. [37]).

To show that the four approaches coincide, we first exhibit natural isomorphisms between the various kinds of cohomologies, after identifying both the objects to which they apply, and the coefficient systems:

(a) The Baues–Wirsching cohomology $H^*_{\mathrm{BW}}(\mathcal{K}; D)$ of a small category $\mathcal{K}$ with coefficients in a natural system D (see Section 2.6),
(b) The $(\mathcal{S}_*, \mathcal{O})$-cohomology $H^*_{\mathrm{SO}}(\mathcal{Z}; M)$ of a simplicially enriched category $\mathcal{Z}$, with coefficients in a module M over the track category $\hat{\pi}_1\mathcal{Z}$ (see Section 3.3).
(c) The André–Quillen cohomology $H^*_{\mathrm{AQ}}(\Lambda; M)$ of a Π-algebra Λ, with coefficients in a Λ-module M (see Section 4.1).

The identification of (a) and (b), under suitable circumstances, is given in Theorem 3.10; that of (b) and (c) is given in Theorem 4.5. After identifying the cohomology groups, we also identify the obstructions, for which we need:

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0.1. The basic setting

Let $\mathcal{C}$ be a pointed model category. A collection of *spherical objects* for $\mathcal{C}$ is a set $\mathcal{A}$ of cofibrant homotopy cogroup objects in $\mathcal{C}$, closed under the suspension. The motivating example is the collection of spheres $\mathcal{A} = \{\mathbf{S}^n\}_{n=1}^{\infty}$ in the category of topological spaces, but there are many others.

Let $\Pi_{\mathcal{A}}$ denote the full subcategory of the homotopy category ho $\mathcal{C}$ whose objects are finite coproducts of objects from $\mathcal{A}$. A $\Pi_{\mathcal{A}}$-*algebra* is a contravariant functor $\Lambda : \Pi_{\mathcal{A}} \to \mathcal{S}et_*$ which takes coproducts to products. The category of all $\Pi_{\mathcal{A}}$-algebras is denoted by $\Pi_{\mathcal{A}}$-$\mathcal{A}lg$.

Such a $\Pi_{\mathcal{A}}$-algebra Λ is determined by its value $\Lambda\{A\} \in \mathcal{S}et_*$ on each $A \in \mathcal{A}$, together with a map $\xi^* : \prod_{i\in I} \Lambda\{A_i\} \to \Lambda\{A\}$ for every $\xi : A \to \coprod_{i\in I} A_i$ in $\Pi_{\mathcal{A}} \subseteq$ ho $\mathcal{C}$. Because each $A \in \mathcal{A}$ is a homotopy cogroup object, each $\Lambda\{A\}$ has an underlying group structure (although the operations ξ^* need not be group homomorphisms).

Thus when $\mathcal{A} = \{\mathbf{S}^n\}_{n=1}^{\infty}$, as above, a $\Pi_{\mathcal{A}}$-algebra (called simply a Π-algebra) is a graded group $(G_i)_{i=1}^{\infty}$ with Whitehead products, composition operations, and a G_1-action on each G_n, as for the homotopy groups π_*X of a space X.

For simplicity we assume that for any collection $\{A_i\}_{i\in I}$ of objects from $\mathcal{A}$ and any $B \in \mathcal{A}$, the natural map

$$\operatorname{colim}_J \left[B, \coprod_{j\in J} A_j\right]_{\mathrm{ho}\,\mathcal{C}} \longrightarrow \left[B, \coprod_{i\in I} A_i\right]_{\mathrm{ho}\,\mathcal{C}} \tag{0.2}$$

is an isomorphism, where the colimit on the left is taken over the lattice of all finite subsets $J \subseteq I$.

0.3. The basic problem

The canonical example of a $\Pi_{\mathcal{A}}$-algebra is a *realizable* one, denoted by $\pi_{\mathcal{A}}X$, for fixed $X \in \mathcal{C}$. This is defined by setting $(\pi_{\mathcal{A}}X)\{A\} := [A, X]_{\mathrm{ho}\,\mathcal{C}}$ for each $A \in \Pi_{\mathcal{A}}$.

The problem we consider in this paper is that of *realizing* an abstract $\Pi_{\mathcal{A}}$-algebra Λ: that is, finding an object $X \in \mathcal{C}$ with $\pi_{\mathcal{A}}X \cong \Lambda$. Such an X may not exist, and need not be unique. There are three main approaches to the realization problem, each describing the obstructions in terms of appropriate cohomology classes:

(a) Trying to lift Λ to a "secondary $\Pi_{\mathcal{A}}$-algebra", which has additional structure encoding the second-order homotopy operations in the model category $\mathcal{C}$ in terms of track categories. In this case, the obstruction to such a lifting lies in Baues–Wirsching cohomology (see Section 6.7).

One could try in principle to continue this process to "higher order track categories", but the appropriate setting for this is not yet clear (see [8] and [21]).

(b) Starting with a simplicial $\Pi_{\mathcal{A}}$-algebra-resolution of Λ, we obtain a "simplicial object up to homotopy" over $\mathcal{C}$. We try to rectify it in $\mathcal{C}$ to a strict simplicial object. If we succeed, we can show that its "geometric realization" realizes the given $\Pi_{\mathcal{A}}$-algebra Λ.

In this setting Λ, together with $\Pi_{\mathcal{A}}$, can be used to construct a certain category $\mathcal{K}$, as well as a simplicially enriched category, such that the Dwyer–Kan–Smith obstructions to rectifying the "simplicial object up to homotopy" lie in the $(\mathcal{S}_*, \mathcal{O})$-cohomology of $\mathcal{K}$ (see Section 5.5).

(c) Starting again with a simplicial $\Pi_{\mathcal{A}}$-algebra-resolution of Λ, and trying to lift it to a strict simplicial object over $\mathcal{C}$ through a Postnikov tower, as in [16], we find that in this case the obstructions lie in the André–Quillen cohomology of Λ (see Theorem 7.5).

The identification of the obstructions appearing in (a) and (b) is given in Theorem 6.5. In order to do this for (b) and (c), we set up yet another, fourth, version of the obstruction theory in terms of $\mathcal{A}$-mapping algebras. The identification is then given via Theorem 10.11 and Remark 10.12.

Remark 0.4. We observe that one can dualize this setting by taking a set $\mathcal{A}$ of *group* objects in ho $\mathcal{C}$ as our dual spherical objects, and define $\Pi^{\mathcal{A}}$ to be the full subcategory of ho $\mathcal{C}$ consisting of finite products of objects from $\mathcal{A}$. A $\Pi^{\mathcal{A}}$-algebra is then a covariant product-preserving functor $\Pi^{\mathcal{A}} \to \mathcal{S}et_*$. This is one reason why we work in a general categorical setting, which can readily be dualized. However, the dual of (0.2) is unlikely to hold, so more care is needed in dealing with infinite products of objects from $\mathcal{A}$.

An important example is provided by letting $\mathcal{A} = \{K(\mathbb{F}_p, n)\}_{n=1}^{\infty}$ consist of the mod p Eilenberg–Mac Lane spaces. In this case a $\Pi^{\mathcal{A}}$-algebra is just an unstable algebra over the mod p Steenrod algebra (cf. [45, Section 1.4]). See [7,14] for more details.

0.5. Organization

Section 1 describes the respective abstract model category settings for the cohomology theories and the general realization problem. Section 2 provides some background on track categories and the Baues–Wirsching cohomology of small categories. In Section 3 we define $(\mathcal{S}_*, \mathcal{O})$-categories, and show how Baues–Wirsching cohomology can be identified with $(\mathcal{S}_*, \mathcal{O})$-cohomology (Theorem 3.10). In Section 4 we similarly show how the André–Quillen cohomology of a $\Pi_{\mathcal{A}}$-algebra can be identified with relative $(\mathcal{S}_*, \mathcal{O})$-cohomology (Theorem 4.5).

In the second half of the paper, we describe the various obstruction theories and show how they correspond: the Dwyer–Kan–Smith $(\mathcal{S}_*, \mathcal{O})$-obstructions to rectifying homotopy commutative diagrams are defined in Section 5, and in

Section 6 the Baues–Wirsching class for classifying linear track extensions is identified with the first $(\mathcal{S}_*, \mathcal{O})$-obstruction (Theorem 6.5). The Dwyer–Kan–Stover approach to realizing Π-algebras via André–Quillen cohomology obstructions is described in Section 7. In Section 8 we introduce the concept of an $\mathcal{A}$-*mapping algebra*, and describe the main example, the Stover mapping algebras, in Section 9. Finally, Section 10 reinterprets the obstruction theory of [16] in terms of mapping algebras, and shows how they may be used to identify the André–Quillen obstructions to realizing a $\Pi_{\mathcal{A}}$-algebra as suitable $(\mathcal{S}_*, \mathcal{O})$-obstructions (Theorem 10.11).

0.6. Notation and conventions

The category of pointed connected topological spaces will be denoted by $\mathcal{T}_*$, that of pointed sets by $\mathcal{S}et_*$, that of groups by $\mathcal{G}p$, and that of groupoids by $\mathcal{G}pd$. For any category $\mathcal{C}$, $s\mathcal{C}$ denotes the category of simplicial objects over $\mathcal{C}$. However, $s\mathcal{S}et$ is denoted by $\mathcal{S}$, $s\mathcal{S}et_*$ by $\mathcal{S}_*$, and $s\mathcal{G}p$ by $\mathcal{G}$. The full subcategory of *reduced* simplicial sets in $\mathcal{S}_*$ (with a single 0-simplex) will be denoted by $\mathcal{S}_{\mathrm{red}}$. Objects in $s\mathcal{C}$ will generally be written as $X_\bullet$, $Y_\bullet$, and so on. The constant simplicial object on an object $X \in \mathcal{C}$ is written as $c(X) \in s\mathcal{C}$.

If $\mathbf{\Delta}$ is the category of finite ordered sets $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ with order-preserving maps, then $s\mathcal{C} \cong \mathcal{C}^{\mathbf{\Delta}}$. We write $\tau_n\mathbf{\Delta}$ for the full subcategory of $\mathbf{\Delta}$ with objects $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}\}$, and the corresponding diagram category $\mathcal{C}^{\tau_n\mathbf{\Delta}}$ is called the category of *n-truncated* simplicial objects in $\mathcal{C}$, also denoted by $\tau_n s\mathcal{C}$. The inclusion $\iota_n : \tau_n\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}$ induces the *n-truncation* functor $\tau_n : s\mathcal{C} \to \tau_n s\mathcal{C}$. Its left adjoint (when it exists) induces the *n-skeleton* functor $\mathrm{sk}_n : s\mathcal{C} \to s\mathcal{C}$, and its right adjoint induces the *n-coskeleton* functor $\mathrm{csk}_n : s\mathcal{C} \to s\mathcal{C}$.

Given $A \in \mathcal{S}$ and an object X in a category $\mathcal{C}$ with coproducts, define $X\hat{\otimes}A \in s\mathcal{C}$ by $(X\hat{\otimes}A)_n := \coprod_{a\in A_n} X$, with face and degeneracy maps induced from those of A. For $Y \in s\mathcal{C}$, set $Y \otimes A := \mathrm{diag}(Y\hat{\otimes}A) \in s\mathcal{C}$.

The category of all small categories will be denoted by $\mathcal{C}at$. For any set $\mathcal{O}$, $\mathcal{O}$-$\mathcal{C}at$ denotes the subcategory of $\mathcal{C}at$ consisting of the categories having $\mathrm{Obj}(\mathcal{C}) = \mathcal{O}$, with functors which are the identity on objects.

1. Model categories and cohomology

We first describe the model category framework needed to define the cohomology theories, and study the realization problems described above:

Assumption 1.1. We assume throughout this paper that our model categories are pointed, cofibrantly generated, simplicial (see [43, II, Section 1]), and right proper (that is, the pullback of a weak equivalence along a fibration is a weak equivalence).

For simplicity of treatment we will assume that all objects in $\mathcal{C}$ are fibrant (although many of our constructions make use of the category $\mathcal{S}$ of simplicial sets, where this does not hold). Note that we may take $\mathcal{C} = \mathcal{G}$ if we want such a model category for the homotopy theory of pointed connected topological spaces.

First, in order to provide an appropriate setting for resolutions, we shall need to deal with simplicial objects over our model category $\mathcal{C}$, for which we have the following:

Definition 1.2. Let $\mathcal{C}$ be a model category as above, with a set $\mathcal{A}$ of spherical objects (Section 0.1). In order to define a model category structure on $s\mathcal{C}$, we choose the set $\tilde{\mathcal{A}} := \{A \otimes \mathbf{\Delta}[n]/(A \otimes \partial\mathbf{\Delta}[n])\}_{n\in\mathbb{N}, A\in\mathcal{A}}$ (see Section 0.6) as the collection of spherical objects for $s\mathcal{C}$. We think of $A \otimes \mathbf{\Delta}[n]/(A \otimes \partial\mathbf{\Delta}[n])$ as the *simplicial* suspension of A; we reserve the notation Σ^k for (internal) suspension in $\mathcal{C}$.

Extending the simplicial structure from $\mathcal{C}$ to $s\mathcal{C}$ in the usual way (cf. [43, II, Section 4]), we set $[X_\bullet, Y_\bullet]_{\mathrm{ho}\,s\mathcal{C}} := \pi_0 \mathrm{map}_{s\mathcal{C}}(X_\bullet, Y_\bullet)$ for $X_\bullet, Y_\bullet \in s\mathcal{C}$. We write $\pi_n^{\#}(X_\bullet)$ for the $\mathcal{A}$-graded group $[A\hat{\otimes}\mathbf{S}^n, X_\bullet]_{\mathrm{ho}\,s\mathcal{C}}$ $(A \in \mathcal{A})$. These are called the *natural* homotopy groups of $X_\bullet$.

We now define the *resolution model category structure* on $s\mathcal{C}$ determined by $\mathcal{A}$, by letting a simplicial map $f : X_\bullet \to Y_\bullet$ be:

(i) a *weak equivalence* if $\pi_*^{\#}(f)$ is a weak equivalence of $\mathcal{A}$-graded simplicial groups.
(ii) a *cofibration* if it is (a retract of) a map with the following property: for each $n \geq 0$, there is a cofibrant object W_n in $\mathcal{C}$ which is weakly equivalent to a coproduct of objects from $\mathcal{A}$, and a map $\varphi_n : W_n \to Y_n$ in $\mathcal{C}$ inducing a trivial cofibration $(X_n \amalg_{L_nX_\bullet} L_nY_\bullet) \amalg W_n \to Y_n$. Here $L_nY_\bullet$ is the n-th latching object for $Y_\bullet$ (cf. [17, Section 2.1]).
(iii) a *fibration* if it is a Reedy fibration (cf. [39, 15.3]) and $\pi_{\mathcal{A}}f$ (Section 0.3) is a fibration of $\mathcal{A}$-graded simplicial groups.

See [24,34].

Applying $\pi_{\mathcal{A}}$ in each simplicial dimension to any $X_\bullet \in s\mathcal{C}$ yields a simplicial $\Pi_{\mathcal{A}}$-algebra $\pi_{\mathcal{A}}X_\bullet$. By taking the usual homotopy groups of the underlying $\mathcal{A}$-graded simplicial group in each degree, we obtain the $\mathbb{N}$-graded $\Pi_{\mathcal{A}}$-algebra $\pi_*\pi_{\mathcal{A}}X_\bullet$. This is related to natural homotopy groups by a *spiral long exact sequence* (cf. [35, 8.1]):

$$\begin{aligned} \cdots \to \Omega\pi_{n-1}^{\#}(X_\bullet) \xrightarrow{s_n} \pi_n^{\#}(X_\bullet) \xrightarrow{h_n} \pi_n\pi_{\mathcal{A}}X_\bullet \xrightarrow{\partial_n} \Omega\pi_{n-2}^{\#}(X_\bullet) \\ \xrightarrow{s_{n-1}} \pi_{n-1}^{\#}(X_\bullet) \to \cdots \to \pi_0^{\#}(X_\bullet) \xrightarrow{\cong} \pi_0\pi_{\mathcal{A}}X_\bullet \end{aligned} \tag{1.3}$$

It follows that a map $f : X_\bullet \to Y_\bullet$ in $s\mathcal{C}$ is a weak equivalence if and only if the map of simplicial $\Pi_{\mathcal{A}}$-algebras $f_* : \pi_{\mathcal{A}}X_\bullet \to \pi_{\mathcal{A}}Y_\bullet$ is a weak equivalence in the resolution model category $s\Pi_{\mathcal{A}}$-$\mathcal{A}lg$.

1.4. Examples of resolution model categories

(a) Let $\mathcal{C} = \mathcal{S}p$ with the *trivial* model category structure, and $\mathcal{A} := \{\mathbb{Z}\}$. The resulting resolution model category structure on $\mathcal{G} := s\mathcal{C}$ is the usual one.
(b) More generally, let $\mathcal{C} = \Theta\text{-}\mathcal{A}lg$ be a category of universal algebras (with an underlying group structure), represented by a theory Θ (cf. [1, Section 1]), such as $\Pi_A\text{-}\mathcal{A}lg$. In this case we let $\mathcal{A}$ be the collection of free monogenic algebras.
(c) We can iterate the process by taking $\mathcal{G}$ for $\mathcal{C}$, and letting $\mathcal{A}$ consist of the $\mathcal{G}$-spheres. We thus obtain a resolution model category structure on $s\mathcal{G}$ (or on $s\mathcal{T}_*$), which is the original example of [34].
(d) If $\mathcal{C}$ is a resolution model category and I is some small category, the category $\mathcal{C}^I$ of I-diagrams in $\mathcal{C}$ also has a resolution model category structure, in which the spherical objects are certain free I-diagrams (cf. [17, Section 1]).

In order to define cohomology groups in our model category, it is convenient to consider the following setting:

Definition 1.5. A model category $\mathcal{C}$ is called *semi-spherical* (see [18, Section 2.23]) if:

(a) It is equipped with a *coefficient category* $\mathrm{Coef}(\mathcal{C})$, together with a functor $\hat{\pi}_1 : \mathcal{C} \to \mathrm{Coef}(C)$.
(b) For each $n \geq 2$,, it is equipped with a functor $\pi_n : \mathcal{C} \to \hat{\pi}_1(-)\text{-}\mathcal{M}od$ taking $Z \in \mathcal{C}$ into the category of *modules* over $\hat{\pi}_1 Z$ (that is, abelian group objects in $\mathrm{Coef}(C)/\hat{\pi}_1 Z$).
(c) Each $Z \in \mathcal{C}$ has a functorial *Postnikov tower* of fibrations under Z:

$$Z \cdots \to P_n Z \xrightarrow{p^{(n)}} P_{n-1} Z \xrightarrow{p^{(n-1)}} \cdots \to P_0 Z\,,$$

with $Z \to \lim_n P_n Z$ a weak equivalence, and the usual properties for the structure maps $r^{(n)} : Z \to P_n Z$.
(d) For every $\Lambda \in \mathrm{Coef}(\mathcal{C})$, there is a functorial *classifying object* $B\Lambda \in \mathcal{C}$, unique up to homotopy, with $B\Lambda \simeq P_1 B\Lambda$ and $\hat{\pi}_1 B\Lambda \cong \Lambda$.
(e) Given $\Lambda \in \mathrm{Coef}(\mathcal{C})$ and a Λ-module M, for each $n \geq 1$ there is a functorial *Eilenberg–Mac Lane object* $E = E^{\Lambda}(G, n)$ in $\mathcal{C}$, unique up to homotopy, equipped with a section s for $r^{(1)} : E \to P_1 E \simeq B\Lambda$, such that $\pi_n E \cong M$ as Λ-modules and $\pi_k E = 0$ for $k \neq 0, 1, n$.
(f) For every $n \geq 1$, there is a functor that assigns to each $Z \in \mathcal{C}$ a homotopy pullback square:

$$\begin{array}{ccc} P_{n+1}Z & \xrightarrow{p^{(n+1)}} & P_n Z \\ \downarrow & \boxed{\mathrm{PB}} & \downarrow{\scriptstyle k_n} \\ B(\hat{\pi}_1 Z) & \longrightarrow & E^{\hat{\pi}_1 Z}(\pi_{n+1} Z, n+2)\,. \end{array} \tag{1.6}$$

The map k_n is called the n-th *k-invariant* for Z.

Examples 1.7. The motivating example is the category $\mathcal{T}_*$ of pointed topological spaces.

In addition, all the resolution model categories of Section 1.4 are semi-spherical (see [17, Section 3]). We note that for the "algebraic" categories $\mathcal{C} = s\Theta\text{-}\mathcal{A}lg$ of simplicial universal algebras (Section 1.4(b)), $\hat{\pi}_1 X_\bullet$ is just $\pi_0 X_\bullet$,and $\mathrm{Coef}(\mathcal{C})$ is $\Theta\text{-}\mathcal{A}lg$ itself. Thus a module over a Θ-algebra Λ is just an abelian group object in $\Theta\text{-}\mathcal{A}lg/\Lambda$ (cf. [12]).

Definition 1.8. Let $\mathcal{C}$ be a semi-spherical simplicial model category, and assume given $\Lambda \in \mathrm{Coef}(\mathcal{C})$, a Λ-module M, and an object $Z \in \mathcal{C}$ equipped with a *twisting map* $p : \hat{\pi}_1 Z \to \Lambda$. Following [43, II, Section 5], we define the n-th *cohomology group* of Z with coefficients in M to be

$$H^n(Z/\Lambda; M) := [Z, E^{\Lambda}(M, n)]_{\mathcal{C}/B\Lambda} = \pi_0 \,\mathrm{map}_{\mathcal{C}/B\Lambda}(Z, E^{\Lambda}(M, n)),$$

where $\mathrm{map}_{\mathcal{C}/A}(Z, Y)$ is the sub-simplicial set of the mapping space $\mathrm{map}_{\mathcal{C}}(Z, Y)$ in $\mathcal{C}$ consisting of maps over a fixed base A.

Typically, we have $\Lambda = \hat{\pi}_1 Z$, with p a weak equivalence; if in addition $Z \simeq B\Lambda$, we denote $H^n(Z/\Lambda; M)$ simply by $H^n(\Lambda; M)$.

Remark 1.9. There is also a relative version, for a cofibration $i : X \hookrightarrow Y$ in $\mathcal{C}/B\Lambda$: If Z is the cofiber of i in $\mathcal{C}/B\Lambda$ — that is, the homotopy pushout of

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle p}\downarrow & \boxed{\mathrm{PO}} & \downarrow{\scriptstyle q} \\ B\Lambda & \longrightarrow & Z\,, \end{array} \tag{1.10}$$

then

$$H^n((Y, X)/\Lambda; M) := [(Z, B\Lambda), (E^{\Lambda}(M, n), B\Lambda)]_{\mathcal{C}/B\Lambda}$$

(cf. [32, Section 2.1]). Again if $\Lambda = \hat{\pi}_1 Y$ we write simply $H^n(Y, X; M)$.

1.11. The module $\Omega\Lambda$

We close this section with the following important example of a module in the category of $\Pi_{\mathcal{A}}$-algebras:

Given a $\Pi_{\mathcal{A}}$-algebra Λ, we define the $\Pi_{\mathcal{A}}$-algebra $\Omega_+\Lambda$ as an $\mathcal{A}$-graded group by $(\Omega_+\Lambda)\{A\} := \Lambda\{\Sigma A \vee A\}$. We identify the $\Pi_{\mathcal{A}}$-algebra structure on $\Omega_+\Lambda$ as follows:

Given $f : B \to A$ in $\Pi_{\mathcal{A}}$, define $\nabla f : B \to A \vee A$ to be $-i_2 \circ f + (i_1 + i_2) \circ f$, using the cogroup structure on B (where $i_1, i_2 : A \to A \vee A$ are the two inclusions). If $j : A \vee A \hookrightarrow A$ is $* \vee \mathrm{Id}$, then $j \circ \nabla f \sim *$, with a nullhomotopy $H : CB \to A$. Now let I_*X denote the reduced cylinder in $\mathcal{C}$ and let G be the composite of

$$I_*B \xrightarrow{I_*\nabla f} I_*(A \vee A) = I_*A \vee I_*A \xrightarrow{q \vee p_0} \Sigma A \vee A,$$

where $q : I_*A \to \Sigma A$ is the quotient map and $p_0 : I_*A \to A$ is the projection. If we identify ΣB with the pushout $CB \cup_B I_*B \cup_B CB$ (under the two inclusions of B into I_*B), we define $E(\nabla f) : \Sigma B \to \Sigma A \vee A$ to be the map given on the pushout by (H, G, H), and call it the *partial suspension* of ∇f. Because $[\Sigma B,\ \Sigma A \vee A]$ is an abelian group, this is independent of H. See [4, Section 3] for more details, including explicit rules for applying the partial suspension to maps among wedges of spheres.

Since Λ is contravariant, the map $(E\nabla f,\ i_2 \circ f) : \Sigma B \vee B \to \Sigma A \vee A$ induces the required map $f^* : \Omega_+\Lambda\{A\} \to \Omega_+\Lambda\{B\}$. We thus have a split exact sequence of $\Pi_{\mathcal{A}}$-algebras:

$$* \longrightarrow \Omega\Lambda \longrightarrow \Omega_+\Lambda \underset{p}{\longrightarrow} \Lambda \longrightarrow *,$$

where $\Omega\Lambda := \mathrm{Ker}\,(p)$. This gives $\Omega_+\Lambda$ the structure of a module over Λ — or equivalently, a natural system on $\Pi_{\mathcal{A}}$ (see Section 2.4 below). Note that $(\Omega\Lambda)\{A\} \cong \Lambda\{\Sigma A\}$, for all $A \in \Pi_{\mathcal{A}}$, but the operation $f^* : \Omega\Lambda\{A\} \to \Omega\Lambda\{B\}$ described as above for $f : B \to A$ is *not* in general $(\Sigma f)^* : \Lambda\{\Sigma A\} \to \Lambda\{\Sigma B\}$.

Remark 1.12. A canonical identification of $(A \otimes \mathbf{S}^1)/(* \otimes \mathbf{S}^1)$ with $\Sigma A \vee A$ in any pointed model category is given in [10], such that

$$\begin{array}{ccc} (B \otimes \mathbf{S}^1)/(* \otimes \mathbf{S}^1) & \xrightarrow{(f \otimes \mathbf{S}^1)/(* \otimes \mathbf{S}^1)} & (A \otimes \mathbf{S}^1)/(* \otimes \mathbf{S}^1) \\ \simeq \downarrow & & \simeq \downarrow \\ \Sigma B \vee B & \xrightarrow[(E\nabla f,\, i_2 \circ f)]{} & \Sigma A \vee A \end{array}$$

commutes up to homotopy for any $f : B \to A$. Thus our definition of $\Omega_+\Lambda$ agrees with that of [35, Section 9.4].

2. Track categories and natural systems

The first approach to realization problems in Section 0.3(a), developed in [11] (cf. [7, Sections 2–3]), concentrates on secondary homotopy structure, in the following sense:

Definition 2.1. A *track category* is a category $\mathcal{E}$ enriched in groupoids. It thus consists of two categories $\mathcal{E}_0$ and $\mathcal{E}_1$, with the same objects, and two functors $s, t : \mathcal{E}_1 \to \mathcal{E}_0$ which are the identity on objects. Here $\mathcal{E}_0$ is the ordinary category underlying $\mathcal{E}$, while $\mathcal{E}_1(X, Y)$ is a groupoid, with maps $f : X \to Y$ in $\mathcal{E}_0$ as objects, and a set $\mathcal{E}_1(f, g)$ of morphisms (called 2-*cells*) from $s(H) = f : X \to Y$ to $t(H) = g : X \to Y$, written as $H : f \Rightarrow g$. The groupoid operation is denoted by $H \square H'$, when defined.

There is natural equivalence relation on maps in $\mathcal{E}_0$ induced by the 2-cells, and the quotient category $\mathrm{ho}\,\mathcal{E}$ is called the *homotopy category* of $\mathcal{E}$. See [5, VI, Section 3] for further details.

Example 2.2. The motivating example of a track category is obtained from a model category $\mathcal{C}$ by letting $\mathcal{E}_0 = \mathcal{C}_{\mathrm{cf}}$ (the full subcategory of fibrant and cofibrant objects), with $\mathcal{E}_1(X, Y)$ the groupoid of tracks (homotopy classes of homotopies) between maps from X to Y in $\mathcal{C}$. This is called the *homotopy track category* of $\mathcal{C}$.

Remark 2.3. There is a model category structure on the category $\mathcal{T}rk$ of (small) track categories, with (strict) track functors, in which the weak equivalences are bi-essential surjections $F : \mathcal{E} \to \mathcal{E}'$ which induced equivalences of categories $F : \mathcal{E}_1(X, Y) \xrightarrow{\cong} \mathcal{E}'_1(FX, FY)$ (see [41]).

Definition 2.4. For any category $\mathcal{K}$, the *category of factorizations* of $\mathcal{K}$ is the category $\mathrm{Fac}\,\mathcal{K}$ having as objects $\mathrm{Arr}\,\mathcal{K}$ (the morphism set of $\mathcal{K}$) and as morphisms from f to g commuting squares of the form

$$\begin{array}{ccc} 1 & \xrightarrow{f} & 2 \\ \uparrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ 0 & \xrightarrow[g]{} & 3 \end{array}$$

with the obvious composition. A *natural system on* $\mathcal{K}$ with values in a category $\mathcal{M}$ is a functor $D : \mathrm{Fac}\,\mathcal{K} \to \mathcal{M}$. The category of such natural systems (with natural transformations as morphisms) will be denoted by $\mathcal{NS}_{\mathcal{K}}(\mathcal{M}) = \mathcal{M}^{\mathrm{Fac}\,\mathcal{K}}$. When $\mathcal{M} = \mathcal{A}b\mathcal{G}p$, D is called simply a *natural system on* $\mathcal{K}$ (see [11, Section 1]).

Example 2.5. For $\mathcal{A} \subseteq \mathcal{C}$ as in Section 0.1, we have a canonical natural system $\Omega\Pi_{\mathcal{A}}$ on $\Pi_{\mathcal{A}}$, defined for $f : B \to A$ in $\Pi_{\mathcal{A}}$ by $\Omega\Pi_{\mathcal{A}}(f) := \mathrm{Hom}_{\Pi_{\mathcal{A}}}(\Sigma B, A) = [\Sigma B, A]_{\mathrm{ho}\,\mathcal{C}}$. For $g : A \to A'$, the induced map $g_* : \Omega\Pi_{\mathcal{A}}(f) \to \Omega\Pi_{\mathcal{A}}(gf)$ is given by post-composition, while for $h : B' \to B$, $h^* : \Omega\Pi_{\mathcal{A}}(f) \to \Omega\Pi_{\mathcal{A}}(fh)$ is given by $(E\nabla h)^*(\alpha, f)$ (cf. Section 1.11).

2.6. Baues–Wirsching cohomology of a small category

If $N\mathcal{K}$ is the nerve of $\mathcal{K}$, define $\partial_{\max} : N_n\mathcal{K} \to N_1\mathcal{K}$ by $\partial_{\max}(\sigma) := d_1 d_2 \cdots d_{n-1}\sigma \in N_1\mathcal{K}$ (the composite of the corresponding composable sequence in $\mathcal{K}$), and set

$$N_n[f] := \{\sigma \in N_n\mathcal{K} : \partial_{\max}(\sigma) = f\} \quad \text{for any arrow } f \text{ in } \mathcal{K}.$$

This defines a collection of sets indexed by $\mathrm{Arr}\,\mathcal{K}$. Note that there is a forgetful functor from natural systems on $\mathcal{K}$ to $\mathrm{Arr}\,\mathcal{K}$-graded sets, whose left adjoint is the *free natural system* functor (cf. [21, Section 5.14]. Thus for each $n \geq 0$ we have a free natural system in sets on $\mathcal{K}$ denoted by $\tilde{N}_n\mathcal{K}$.

The face and degeneracy maps of $N\mathcal{K}$ induce maps of natural systems as follows:

(a) If $\phi = d_i : N_n\mathcal{K} \to N_{n-1}\mathcal{K}$ $(0 < i < n)$ or $\phi = s_j : N_n\mathcal{K} \to N_{n+1}\mathcal{K}$ $(0 \leq j \leq n)$, we define $\tilde{\phi} : \tilde{N}_n\mathcal{K} \to \tilde{N}_{n\pm1}\mathcal{K}$ to be $\mathcal{F}\phi$.
(b) Given $\sigma \in N_n\mathcal{K}$, define the map of natural systems $\tilde{d}_0 : \tilde{N}_n\mathcal{K} \to \tilde{N}_{n-1}\mathcal{K}$ by setting $\tilde{d}_0(\sigma) := (d_2 \cdots d_n\sigma)^*(d_0\sigma)$. This extends to all of $\tilde{N}_n\mathcal{K}$ by the adjointness of $\mathcal{U}$ and $\mathcal{F}$ above.
(c) We similarly define the n-th face map $\tilde{d}_n : \tilde{N}_n\mathcal{K} \to \tilde{N}_{n-1}\mathcal{K}$ by $\tilde{d}_n(\sigma) := (d_0 \cdots d_{n-2}\sigma)_*(d_n\sigma)$.

This makes $\tilde{N}_\bullet\mathcal{K} := (\tilde{N}_n\mathcal{K})_{n=0}^{\infty}$ into a simplicial object in the category $\mathcal{NS}_{\mathcal{K}}(\mathcal{S}et)$.

Finally, a natural system (in $\mathcal{A}b\mathcal{G}p$) on $\mathcal{K}$ can be thought of as an abelian group object $\mathcal{D}$ in $\mathcal{NS}_{\mathcal{K}}(\mathcal{S}et)$, so we can define a cosimplicial abelian group $C^\bullet(\mathcal{K}; D)$ by setting $C^n(\mathcal{K}; D) := \mathrm{Hom}_{\mathcal{NS}_{\mathcal{K}}(\mathcal{S}et)}(\tilde{N}_n\mathcal{K}, D)$. Its n-th cohomotopy group is defined to be the n-th *Baues–Wirsching cohomology group* of K with coefficients in D, written as $H^n_{\mathrm{BW}}(\mathcal{K}; D) := \pi^n(C^\bullet(\mathcal{K}; D))$.

The cochain complex $F^*(\mathcal{K}, D)$ used in [11] to define $H^n_{\mathrm{BW}}(\mathcal{K}; D)$is that associated with $C^\bullet(\mathcal{K}; D)$, so $H^n_{\mathrm{BW}}(\mathcal{K}; D) \cong H^nF^*(\mathcal{K}, D)$.

Definition 2.7. A *linear track extension* of a category $\mathcal{K}$ by a natural system D is a track category $\mathcal{E}$ with $\mathrm{ho}\,\mathcal{E} = \mathcal{K}$, for which $\mathrm{Aut}_{\mathcal{E}}(f)$ is naturally isomorphic to $D([f])$ for all maps f in $\mathcal{E}_0$. Such an extension is denoted by $D \to \mathcal{E} \to \mathcal{K}$.

Proposition 2.8 (*[5, VI, Theorem 3.15]*)**.** *The set of all linear track extensions of a category $\mathcal{K}$ by a given natural system D, up to (D-equivariant) weak equivalence, is in one-to-one correspondence with $H^3_{\mathrm{BW}}(\mathcal{K}; D)$.*

This can be interpreted as describing the homotopy equivalence classes in $\mathcal{T}rk/\,\mathrm{ho}\,\mathcal{E}$ (as in Section 2.3).

Remark 2.9. If $\mathcal{A}$ is a set of spherical objects in a model category $\mathcal{C}$, let $\tilde{\mathcal{C}}_{\mathcal{A}}$ be a sub-track category of the homotopy track category of $\mathcal{C}$ with $\mathrm{ho}\,\tilde{\mathcal{C}}_{\mathcal{A}} \cong \Pi_{\mathcal{A}}$. This is a linear track extension $\Omega\Pi_{\mathcal{A}} \to \tilde{\mathcal{C}}_{\mathcal{A}} \to \Pi_{\mathcal{A}}$, and one can describe an explicit cocycle representing the corresponding cohomology class $\chi_{\tilde{\mathcal{C}}_{\mathcal{A}}}$ in $H^3_{\mathrm{BW}}(\Pi_{\mathcal{A}}; \Omega\Pi_{\mathcal{A}})$ as follows:

Choose an arbitrary fixed representative $s\phi : 0 \to 1$ in $\mathcal{C}$ for each 1-simplex $\phi : 0 \to 1$ in $\mathcal{N}_1(\Pi_{\mathcal{A}})$, and a fixed track $H_{(\phi,\psi)} : s\phi \circ s\psi \simeq s(\phi\psi)$ for each 2-simplex $0 \xrightarrow{\psi} 1 \xrightarrow{\phi} 2$ in $\mathcal{N}_2(\Pi_{\mathcal{A}})$. Now we associate with each 3-simplex $0 \xrightarrow{\phi_1} 1 \xrightarrow{\phi_2} 2 \xrightarrow{\phi_3} 3$ in $\mathcal{N}_3(\Pi_{\mathcal{A}})$ the element

$$H_{(\phi_3,\phi_2\circ\phi_1)}\Box(\phi_3)_*H_{(\phi_2,\phi_1)}\Box(\phi_1)^*H^{-1}_{(\phi_3,\phi_2)}\Box H^{-1}_{(\phi_3\circ\phi_2,\phi_1)} \tag{2.10}$$

in $\mathrm{Aut}(s(\phi_3 \circ \phi_2 \circ \phi_1)) \cong (\Omega\Pi_{\mathcal{A}})(\phi_3 \circ \phi_2 \circ \phi_1)$.

3. $(\mathcal{S}_*, \mathcal{O})$-categories and $(\mathcal{S}_*, \mathcal{O})$-cohomology

For the second approach to the realization problem of Section 0.3, due to Dwyer and Kan, we use the framework of simplicially enriched categories:

Definition 3.1. For a fixed set $\mathcal{O}$, a category $\mathcal{Z}$ enriched in simplicial sets with object set $\mathcal{O}$ will be called an $(\mathcal{S}, \mathcal{O})$-*category*, and the category of all such will be denoted by $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$. Equivalently, such a category $\mathcal{Z}$ can be thought of as a simplicial object in $\mathcal{O}$-$\mathcal{C}at$ (Section 0.6): this means that $\mathcal{C}$ has a fixed object set $\mathcal{O}$ in each dimension, and all face and degeneracy functors the identity on objects.

More generally, if $(\mathcal{V}, \otimes)$ is any monoidal category, a $(\mathcal{V}, \mathcal{O})$-*category* is a small category $\mathcal{C} \in \mathcal{O}$-$\mathcal{C}at$ enriched over $\mathcal{V}$. The category of all such categories will be denoted by $(\mathcal{V}, \mathcal{O})$-$\mathcal{C}at$. Examples for $(\mathcal{V}, \otimes)$ include $\mathcal{T}$, $\mathcal{G}p$, $\mathcal{G}pd$, and $\mathcal{S}$, with $\otimes = \times$ (Cartesian product), or the category $\mathcal{S}et^{\square}$ of *cubical sets* with its monoidal enrichment $\otimes$ (see [18, Section 1.5]).

The main example that we shall be working with is $\mathcal{V} = \mathcal{S}_*$, with $\otimes = \wedge$ (smash product). Again we can identify an $(\mathcal{S}_*, \mathcal{O})$-category with a simplicial pointed $\mathcal{O}$-category.

3.2. $(\mathcal{S}_*, \mathcal{O})$-categories

In [29, Section 1], Dwyer and Kan define a simplicial model category structure on $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$, also valid for $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$ (cf. [40, Prop. 1.1.8]), in which a map $f : \mathcal{X} \to \mathcal{Y}$ is a fibration (or a weak equivalence) if for each $a, b \in \mathcal{O}$, the induced map $f_{(a,b)} : \mathcal{X}(a, b) \to \mathcal{Y}(a, b)$ is such.

The cofibrations in $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$ or $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$ are not easy to describe. However, if $\mathcal{K} \in \mathcal{O}$-$\mathcal{C}at$ is any category with object set $\mathcal{O}$, then $c(\mathcal{K}) \in s\mathcal{O}$-$\mathcal{C}at \cong (\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$ has a cofibrant replacement defined as follows:

There is a forgetful functor $U : \mathcal{C}at \to \mathcal{D}i\mathcal{G}$ to the category of directed graphs, whose left adjoint $F : \mathcal{D}i\mathcal{G} \to \mathcal{C}at$ is the free category functor (cf. [29, Section 2.4] and [26, Section 2]). Both U and F are the identity on objects. A canonical cofibrant replacement for the constant simplicial category $c(\mathcal{K}) \in s\mathcal{O}$-$\mathcal{C}at$ is provided by the simplicial category $\mathcal{F}_\bullet\mathcal{K}$, obtained by iterating the comonad $FU : \mathcal{O}$-$\mathcal{C}at \to \mathcal{O}$-$\mathcal{C}at$ (so $\mathcal{F}_n\mathcal{K} := (FU)^{n+1}\mathcal{K}$). The augmentation $\mathcal{F}_\bullet\mathcal{K} \to \mathcal{K}$ induces a weak equivalence $\mathcal{F}_\bullet\mathcal{K} \simeq c(\mathcal{K})$ in $s\mathcal{O}$-$\mathcal{C}at \approx (\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$. If $\mathcal{K}$ is pointed, $\mathcal{F}_\bullet\mathcal{K}$ is an $(\mathcal{S}_*, \mathcal{O})$-category.

Both $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$ and $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$ are semi-spherical (Section 1.5), with coefficient category $(\mathcal{G}pd, \mathcal{O})$-$\mathcal{C}at$ (=track categories with object set $\mathcal{O}$).

The *fundamental track category* of a (fibrant) $(\mathcal{S}, \mathcal{O})$- or $(\mathcal{S}_*, \mathcal{O})$-category $\mathcal{Z}$ is obtained by applying the fundamental groupoid functor $\hat{\pi}_1 : \mathcal{S} \to \mathcal{G}pd$ to each mapping space $\mathcal{Z}(a, b)$, noting that $\hat{\pi}_1$ commutes with cartesian products, and thus extends to $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$ (and to $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$, too, since in the pointed case the composition factors through $\wedge$). For each $n \geq 2$ we obtain a $\hat{\pi}_1 Z$-module by applying $\pi_n(-)$ to each mapping space of Z (again, π_n preserves products).

The usual Postnikov tower functor, classifying space, and Eilenberg–Mac Lane functors for $\mathcal{S}$ or $\mathcal{S}_*$ similarly preserve products, and thus extend to $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$ and $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$. For the functorial k-invariants, use the construction of [16, Section 6].

Notation 3.3. We write $H^*_{\mathrm{SO}}(\mathcal{Z}/\Lambda; M)$ (or just $H^*_{\mathrm{SO}}(\mathcal{Z}; M)$) for the cohomology groups of an $(\mathcal{S}_*, \mathcal{O})$-category $\mathcal{Z}$, as defined in Section 1.8. Similarly, we write $H^*_{\mathrm{SO}}((\mathcal{Z}, \mathcal{Y})/\Lambda; M)$ (or just $H^*_{\mathrm{SO}}(\mathcal{Z}, \mathcal{Y}; M)$) for the relative cohomology of Section 1.9. We call this the $(\mathcal{S}_*, \mathcal{O})$-*cohomology* (compare [32]).

Definition 3.4. A cubical version of the free simplicial category $\mathcal{F}_\bullet\mathcal{K}$ on a category $\mathcal{X} \in \mathcal{O}$-$\mathcal{C}at$ is provided by the *bar construction* of Boardman and Vogt: this is a category $W\mathcal{K}$ enriched in the monoidal category $(\mathcal{S}et^{\square}, \otimes)$ of cubical sets (Section 0.6). For $a, b \in \mathcal{O} = \mathrm{Obj}\,\mathcal{K}$, the cubical mapping complex $W\mathcal{K}(a_{n+1}, a_0)$ has an n-cube $I^n_{(f_\bullet)}$ for each sequence

$$f_\bullet = \left(a_{n+1} \xrightarrow{f_{n+1}} a_n \xrightarrow{f_n} a_{n-1} \dots a_1 \xrightarrow{f_1} a_0 \right) \tag{3.5}$$

of $(n+1)$ composable maps in $\mathcal{K}$.

The i-th 0-face d^0_i of $I^n f_\bullet$ is identified with $I^{n-1} f_1 \circ \cdots \circ (f_i \cdot f_{i+1}) \circ \cdots f_{n+1}$, that is, we carry out (in $\mathcal{K}$) the i-th composition in the sequence $f_\bullet$.

The cubical composition

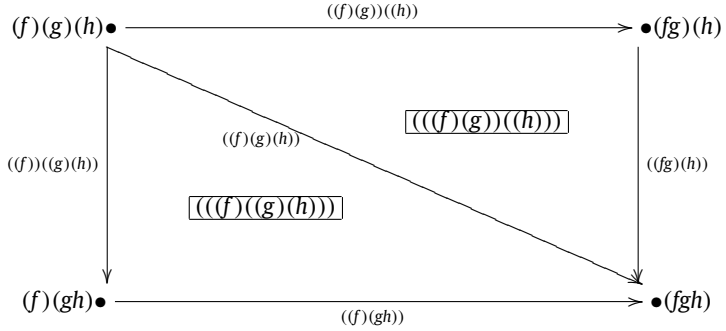
$$W\mathcal{K}(a_0, a_i) \otimes W\mathcal{K}(a_i, a_{n+1}) \to W\mathcal{K}(a_0, a_{n+1}) = W\mathcal{K}(a, b)$$

identifies the "product" $(n-1)$-cube $I^i f_0 \circ \cdots \circ f_i \otimes I^{n-i-1} f_{i+1} \circ \cdots \circ f_{n+1}$ with the i-th 1-face d^1_i of $I^n f_\bullet$. See [22, III, Section 1] or [18, Section 3.1] for further details.

Lemma 3.6. *For any small category $\mathcal{K}$, the simplicial category $\mathcal{F}_\bullet\mathcal{K}$ is a natural triangulation of $W\mathcal{K}$.*

Proof. The n-cube $I^n f_\bullet$ is subdivided into $n!$ n-simplices by fully parenthesizing $(f_1, \dots, f_{n+1})$ in all possible ways, with the i-th face map defined by omitting the i-th level of parentheses (cf. [20, Section 2.21]). □

Example 3.7. For $n = 2$, given three composable maps $0 \xrightarrow{h} 1 \xrightarrow{g} 2 \xrightarrow{f} 3$, we have



(3.8)

Remark 3.9. If D is a natural system on a category $\mathcal{K}$, with $\mathcal{O} = \mathrm{Obj}(\mathcal{K})$, it can be thought of as an abelian group object on $\mathcal{O}$-$\mathcal{C}at/\mathcal{K}$. Moreover, $\mathcal{K}$ itself is the (discrete) fundamental groupoid of the homotopically trivial simplicial category $\mathcal{F}_\bullet\mathcal{K} \simeq \mathcal{K}$ in $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$. (or $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$, if $\mathcal{K}$ is pointed). Thus D is just a module M over $\mathcal{K}$.

Theorem 3.10. *If D is a natural system on a small pointed category $\mathcal{K}$, the n-th Baues–Wirsching cohomology group $H^n_{\mathrm{BW}}(\mathcal{K}; D)$ is naturally isomorphic to the $(n-1)$-st $(\mathcal{S}_*, \mathcal{O})$-cohomology group $H^{n-1}_{\mathrm{SO}}(\mathcal{K}; D)$, for each $n \geq 1$.*

In [31, Theorem 5.3], Dwyer and Kan prove a similar result, using a different definition of the cohomology of a small category, which they call the *Hochschild–Mitchell* cohomology.

Proof. The $(\mathcal{S}_*, \mathcal{O})$-cohomology groups $H^n_{\mathrm{SO}}(\mathcal{K}/\mathcal{G}; D) \cong [\mathcal{F}_\bullet\mathcal{K},\ E^G(M,n)]_{(\mathcal{S}_*,\mathcal{O})\text{-}\mathcal{C}at/B\mathcal{G}}$ of Section 1.8 may be computed as the cohomotopy groups of the cosimplicial abelian group $E^\bullet := \mathrm{Hom}_{(\mathcal{S}_*,\mathcal{O})\text{-}\mathcal{C}at/\mathcal{K}}(\mathcal{F}_\bullet\mathcal{K}, D)$ (cf. [19, Proposition 3.11]).

In order to compare $E^\bullet$ with $C^\bullet(\mathcal{K}, D)$ of Section 2.6, note that for $n \geq 1$, there is an obvious one-to-one correspondence between the n-cubes of $W\mathcal{K}$ (Section 3.4) and the $(n+1)$-simplices of the nerve $\mathcal{N}(\mathcal{K})$. Moreover, for $n \geq 2$ this extends to the face maps, if we omit the d^1_i-faces with $1 < i < n$ – that is, those which are cubical products of two lower dimensional cubes. There are $2n - (n-2) = n+2$ remaining $(n-1)$-facets, of which two are the Cartesian products $I^0_{(f_0)} \times I^n_{(f_1,\dots,f_{n+1})}$ and $I^n_{(f_0,\dots,f_n)} \times I^0_{(f_n)}$ (corresponding to $d_0\sigma_{(f_\bullet)}$ and $d_{n+1}\sigma_{(f_\bullet)}$, respectively), and the others are obtained from the adjacent compositions as for $d_i\sigma_{(f_\bullet)}$ $(i = 1, \dots, n)$. Note that the facets that we have omitted are not relevant for the coboundary of a cubical $(n-1)$-chain.

Finally, the cubical cochain complex $C^\bullet_c := \mathrm{Hom}_{(\mathcal{S}et^\square,\mathcal{O})\text{-}\mathcal{C}at/\mathcal{K}}(W\mathcal{K}, D)$ has the same cohomology as $E^\bullet$ by the Lemma 3.6 and the Acyclic Model Theorem (cf. [36]), and clearly has the same cohomology as $C^\bullet(\mathcal{K}, D)$ by the correspondence described above. □

Remark 3.11. Using the triangulation of Lemma 3.6, we can realize correspondence between the n-cubes of $W\mathcal{K}$ and the $(n+1)$-simplices of the nerve $\mathcal{N}(\mathcal{K})$ simplicially in the barycentric subdivision B of the nerve, as follows:

Consider the triangulated n-cube $I^n_{(f_\bullet)}$ indexed by the composable sequence (3.5) as a subcomplex of $\mathcal{F}_\bullet\mathcal{K}$. and let $B_{(f_\bullet)}$ denote the barycentric subdivision of the corresponding $(n+1)$-simplex $\sigma^{n+1}_{(f_\bullet)}$ of $\mathcal{N}(\mathcal{K})$.

Note that for $i \geq 1$, the i-simplices of $\sigma^{n+1}_{(f_\bullet)}$ are labeled by sub-sequences of $f_\bullet$, with a single level of parenthesization (indicating where compositions, if any, have been carried out) – for example, $(f_2f_3)(f_4)(f_5f_6)$. These also label the corresponding vertices of $B_{f_\bullet}$ (ignoring those which come from the vertices of $\sigma^{n+1}_{(f_\bullet)}$), and each k-simplex of $B_{(f_\bullet)}$ corresponds to an ascending "flag" of $k+1$ inclusions of faces of $\sigma^{n+1}_{(f_\bullet)}$.

Now let $C_{(f_\bullet)}$ denote the set of vertices of $B_{(f_\bullet)}$ which are labeled by (one-level) parenthesizations of the *full* sequence $(f_1, \dots, f_{n+1})$ (corresponding to the simplices of $\sigma^{n+1}_{f_\bullet}$ which have both a_0 and a_{n+1} as vertices), and let $E_{(f_\bullet)}$ be the subcomplex of $B_{(f_\bullet)}$ spanned by $C_{(f_\bullet)}$. A k-simplex of $E_{(f_\bullet)}$ thus corresponds to a sequence of $k+1$ parenthesizations of $(f_1, \dots, f_{n+1})$, each obtained from the next by coalescing a neighbouring pair of parentheses (since this describes the only face maps of $\mathcal{N}(\mathcal{K})$ which remain inside $C_{(f_\bullet)}$). Therefore, such a $(k+1)$-flag can be labeled by a single $(k+1)$-level parenthesization of $(f_{(1)}, \dots, f_{(n+1)})$, just like the $(k-1)$-simplices of $I^n_{(f_\bullet)}$. Thus $I^n_{(f_\bullet)}$ is isomorphic as a simplicial complex to $E_{(f_\bullet)}$.

4. André–Quillen cohomology of Π_A-algebras

Since the category $s\Pi_A\text{-}\mathcal{A}lg$ of simplicial Π_A-algebras is a semi-spherical model category (Section 1.5), we can use Section 1.8 to define the cohomology groups of a Π_A-algebra Λ with coefficients in a Λ-module M (see Section 1.7).

Notation 4.1. In such algebraic settings, this is traditionally called the *André–Quillen cohomology*, since it can be computed via a cotangent complex, as in [3,44]. We therefore denote it by $H^*_{\mathrm{AQ}}(\Lambda; M) := H^*(B\Lambda; M)$.

We would like to compare this with the $(\mathcal{S}_*, \mathcal{O})$-cohomology of a suitable $(\mathcal{S}_*, \mathcal{O})$-category (cf. Section 3.2), for which we need the following framework:

Definition 4.2. Given a set $\mathcal{A}$ of spherical objects in a model category $\mathcal{C}$, we let $\mathcal{C}_\mathcal{A}$ denote the smallest full subcategory of $\mathcal{C}$ containing $\mathcal{A}$ and closed under weak equivalences and arbitrary coproducts.

Using (0.2), we see that the functor $\pi_\mathcal{A} : \mathrm{ho}\,\mathcal{C} \to \Pi_\mathcal{A}\text{-}\mathcal{A}lg$ induces an equivalence of categories between the corresponding subcategory $\mathrm{ho}\,\mathcal{C}_\mathcal{A}$ of the homotopy category $\mathrm{ho}\,\mathcal{C}$ and the category $\mathcal{F}_\mathcal{A}$ of *free* $\Pi_\mathcal{A}$-algebras in $\Pi_\mathcal{A}\text{-}\mathcal{A}lg$ (namely, those which are isomorphic to $\pi_\mathcal{A}B$ for $B \in \mathcal{C}_\mathcal{A}$). Moreover, we can extend any $\Pi_\mathcal{A}$-algebra $\Lambda : \Pi_\mathcal{A} \to \mathcal{S}et_*$ to a functor $\mathrm{ho}\,\mathcal{C}_\mathcal{A} \to \mathcal{S}et_*$ taking (arbitrary) coproducts to products.

A *small* $\mathcal{F}_\mathcal{A}$*-variant* is a full small subcategory $\mathcal{D}$ of $\mathcal{F}_\mathcal{A}$ (or $\mathrm{ho}\,\mathcal{C}_\mathcal{A}$) containing an isomorphic copy of $\Pi_\mathcal{A}$: in other words, $\mathrm{Obj}\,\mathcal{D}$ must contain all *finite* coproducts of objects from $\mathcal{A}$, up to isomorphism.

Given a $\Pi_\mathcal{A}$-algebra Λ and a small $\mathcal{F}_\mathcal{A}$-variant $\mathcal{D}$ with $\mathcal{O} := \mathrm{Obj}(\mathcal{D})$, we let $\mathcal{D}^+$ denote the category with object set $\mathcal{O}^+ := \mathcal{O} \cup \{\star\}$, where

$$\mathrm{Hom}_{\mathcal{D}^+}(A,B) = \begin{cases} \mathrm{Hom}_\mathcal{D}(A,B) & \text{if } a, b \in \mathcal{O} \\ \mathrm{Hom}_{\Pi_\mathcal{A}\text{-}\mathcal{A}lg}(A, \Lambda) = \Lambda\{A\} & \text{if } A \in \mathcal{O} \text{ and } B = \star \\ \{\mathrm{Id}_\star, *\} & \text{if } A = B = \star \\ \{*\} & \text{otherwise.} \end{cases} \tag{4.3}$$

That is, all maps out of $\star$ are trivial. Thus we have a full and faithful embedding of $\mathcal{D}$ in $\mathcal{D}^+$, and $\star$ is a weakly terminal object in $\mathcal{D}^+$. We call $(\mathcal{D}^+, \mathcal{D})$ a Λ*-pair* (in $\mathrm{ho}\,\mathcal{C}$).

Equivalently, if we embed $\mathcal{D}$ in $\mathcal{O}^+\text{-}\mathcal{C}at$ (making all maps into $\star$ trivial), we can think of a Λ-pair (in $\mathrm{ho}\,\mathcal{C}$) as an $\mathcal{O}^+$-category under $\mathcal{D}$ (and require only the last three conditions of (4.3)).

Example 4.4. Let $\mathcal{D}$ be the subcategory of ho $\mathcal{C}$ whose objects are of the form $\coprod_{i\in S} A_i$ with $A_i \in \mathcal{A}$ $(i \in S)$ and cardinality $\mathrm{Card}(S) \leq \max\{\aleph_0, \mathrm{Card}(U\Lambda)\}$. This is a small $\mathcal{F}_{\mathcal{A}}$-variant. We can think of $\mathcal{D}^+$ as a subcategory of $\Pi_{\mathcal{A}}$-$\mathcal{A}lg$, by identifying $\star$ with Λ.

It turns out that the relative $(\mathcal{S}_*, \mathcal{O})$-cohomology of such a pair (cf. Section 1.9) has an algebraic interpretation:

Theorem 4.5. *Let Λ be a $\Pi_{\mathcal{A}}$-algebra, M a Λ-module, and $(\mathcal{D}^+, \mathcal{D})$ a Λ-pair. Then for any $n \geq 1$, the n-th André–Quillen cohomology group $H^n_{\mathrm{AQ}}(\Lambda; M)$ is naturally isomorphic to the n-th relative $(\mathcal{S}_*, \mathcal{O})$-cohomology group $H^n_{\mathrm{SO}}(\mathcal{D}^+, \mathcal{D}; M)$.*

Proof. Let $V_\bullet \to \Lambda$ be the canonical free simplicial resolution (in the resolution model category on $s\Pi_{\mathcal{A}}$-$\mathcal{A}lg$ of Section 1.4(b)) produced by the "free on underlying" comonad $\mathcal{F} = FU$, and let $\mathcal{E}_\bullet$ be the analogous free $(\mathcal{S}_*, \mathcal{O})$-resolution for $\mathcal{D}^+ = \mathcal{D} \cup \{\star\}$ as in Section 4.4. Thus $\pi_0\mathcal{E}_\bullet$ is $\mathcal{D}^+ \in (\mathcal{S}_*, \mathcal{O}^+)$-$\mathcal{C}at$. The relative version $\hat{\mathcal{E}}_\bullet$ is obtained from $\mathcal{E}_\bullet$ by "excision of $\mathcal{D}$" – that is, we define the simplicial mapping spaces for $\hat{\mathcal{E}}_\bullet$ by

$$\hat{\mathcal{E}}_\bullet(A, B) := \begin{cases} \mathcal{E}_\bullet(A, B) & \text{if } B = \star \\ c(\mathrm{Hom}_{\mathcal{F}_{\mathcal{A}}}(A, B)) & \text{if } B \in \mathcal{F}^+_{\mathcal{A}}. \end{cases}$$

The twisting map $p : \mathcal{E}_\bullet \to c(\mathcal{K})$ induces an $(\mathcal{S}_*, \mathcal{O})$-functor $\rho : \mathcal{E}_\bullet \to \hat{\mathcal{E}}_\bullet$.

Note that $\pi_0 V_\bullet \cong U\Lambda$ and $UV_\bullet \cong \coprod_{\varphi\in\pi_0 V_\bullet} V[\varphi]$, where $V[\varphi]$ is the component of $\varphi \in \Lambda\{A\}$ for some $A \in \Pi_{\mathcal{A}}$ (Section 0.1). Then each $V[\varphi]$ is isomorphic to the component of $\varphi : A \to \Lambda$ in the simplicial mapping space $\mathcal{E}_\bullet(A, \Lambda) = \hat{\mathcal{E}}_\bullet(A, \Lambda)$ (so in simplicial dimension n, $V[\varphi]_n$ consists of depth n parenthesizations of composable sequences of morphisms in $\mathcal{K}$, with composite φ). Because $V_\bullet$ is a simplicial $\Pi_{\mathcal{A}}$-algebra, for any $\theta : A' \to A$ in $\mathcal{F}'_{\mathcal{A}}$ we have a simplicial map

$$\theta^* : V[\varphi] \to V[\varphi \circ \theta] \tag{4.6}$$

defining an action of $\mathcal{F}'_A$ on the simplicial sets $V[-]$.

Since the category $s\Pi_{\mathcal{A}}$-$\mathcal{A}lg$ is semi-spherical, for each Λ-module M and $n \geq 1$ we have an Eilenberg–Mac Lane object $E^\Lambda(M, n)$ in $s\Pi_{\mathcal{A}}$-$\mathcal{A}lg/B\Lambda$, as well as an object $E^\Lambda(D, n)$ in $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at/\mathcal{K}$. Moreover, we can assume that both are strict abelian group objects in their respective categories (see [17, Section 3.14]).

Any map of simplicial $\Pi_{\mathcal{A}}$-algebras $f : V_\bullet \to E^\Lambda(M, n)$ (over $B\Lambda$) defines an $(\mathcal{S}_*, \mathcal{O})$-map $\hat{f} : \hat{\mathcal{E}}_\bullet \to E^\Lambda(D, n)$, which is defined on the simplicial mapping spaces $\mathcal{E}_\bullet(A, \Lambda)$ via the above identification with the components $V[\varphi]$ of $V_\bullet$. These fit together to define an $(\mathcal{S}_*, \mathcal{O})$-map, because of the action (4.6).

Precomposing this with $\rho : \mathcal{E}_\bullet \to \hat{\mathcal{E}}_\bullet$ yields an element in the relative $(\mathcal{S}_*, \mathcal{O})$-cohomology group $H^n_{\mathrm{SO}}(\mathcal{D}^+, \mathcal{D}; M)$. The converse direction is treated similarly. □

5. Diagrams and $(\mathcal{S}, \mathcal{O})$-categories

We now explain the approach of Dwyer et al. (see [29,30,32,33]) to realizing a homotopy commutative diagram $X : \mathcal{K} \to \mathrm{ho}\,\mathcal{T}$, based on the concepts introduced in Section 3.

Definition 5.1. A *diagram up to homotopy* in a simplicial model category $\mathcal{C}$ is a functor $X : \mathcal{K} \to \mathrm{ho}\,\mathcal{C}$ from some small indexing category $\mathcal{K}$. By definition, one can choose a functor $X_0 : \mathrm{sk}_0 F(\mathcal{K}) \to \mathcal{C}$ lifting X (sometimes called a 0-*realization* of X). An extension of any such X_0 to a simplicial functor $X_\infty : F(\mathcal{K}) \to \mathcal{C}$ makes X ∞-*homotopy commutative*.

A classical result of Boardman and Vogt (compare [33, Corollary 2.5]) says:

Theorem 5.2 (*[22, Cor. 4.21 & Thm. 4.49]*). *A diagram $X : \mathcal{K} \to \mathrm{ho}\,\mathcal{T}$ can be rectified (i.e., lifted to $\hat{X} : \mathcal{K} \to \mathcal{T}$) if and only if X can be made ∞-homotopy commutative.*

Notation 5.3. When we want to emphasize that we are thinking of a simplicial model category $\mathcal{C}$ just as a simplicially enriched category, we denote it by ${}^s\mathcal{C}$.

Remark 5.4. Theorem 5.2 implies that the rectification of a homotopy commutative diagram $X : \mathcal{K} \to \mathrm{ho}\,\mathcal{C}$ can be described purely in terms of the simplicially enriched category ${}^s\mathcal{C}$ – in fact, we can restrict to an $(\mathcal{S}_*, \mathcal{O})$-category ${}^s\mathcal{C}_X$ the sub-simplicially enriched category of ${}^s\mathcal{C}$ with function complex map ${}_{s\mathcal{C}_X}(u, v) := \mathrm{map}_{s\mathcal{C}}(Xu, Xv)$ for each $u, v \in \mathcal{O} := \mathrm{Obj}\,\mathcal{K}$.

Note that a choice of a 0-realization $X_0 : \Gamma \to \mathcal{T}_*$ is equivalent to choosing basepoints in each ${}^s\mathcal{C}_X(u, v)$, though of course this cannot be done coherently unless X is rectifiable.

5.5. The obstruction theory

Given $X : \mathcal{K} \to \mathrm{ho}\,\mathcal{C}$ as above, the (possibly empty) moduli space $\mathrm{hc}\,X$ of all rectifications of X is homotopy equivalent to the space $\mathrm{hc}_\infty X := \mathrm{map}_{\mathcal{O}\text{-}\mathcal{C}at}(F(\mathcal{K}), {}^s\mathcal{C}_X)$ of all functors making X ∞-homotopy commutative, which in turn is the (homotopy) inverse limit of the tower

$$\mathrm{hc}_\infty X \to \cdots \to \mathrm{hc}_n X \to \mathrm{hc}_{n-1} X \cdots \to \mathrm{hc}_1 X\,,$$

where $\mathrm{hc}_n \mathcal{A} := \mathrm{map}_{\mathcal{O}\text{-}\mathcal{C}at}(F(\mathcal{K}), P_{n-1}\,{}^s\mathcal{C}_X)$. Therefore, the realization problem can be solved if one can successively lift $\hat{X}_1 \in \mathrm{hc}_1 X$ through the tower.

The components of $\mathrm{hc}_\infty X$ are not in general determined by those of the spaces $\mathrm{hc}_n X$ (cf. [33, 3.4]). Because each $\mathrm{hc}_n X$ is a mapping space, we can use successive liftings $\hat{X}_n \in \mathrm{hc}_n X$ to pull back the $(n-1)$-st k-invariant for ${}^s\mathcal{C}_X$ to a map

$h_n : \mathcal{F}_\bullet\mathcal{K} \to K^G(\pi_n\, {}^s\mathcal{C}_X, n+1)$, and Dwyer, Kan, and Smith show:

Proposition 5.6 ([33, Proposition 3.6]). *The map $\hat{X}_n$ lifts to $\hat{X}_{n+1} \in \mathrm{hc}_{n+1} X$ if and only if $[h_n] \in H^{n+1}_{\mathrm{SO}}(\mathcal{K}; \pi_n\, {}^s\mathcal{C}_X)$ vanishes.*

5.7. A relative version

There is also a relative version of this obstruction theory, in which, given $X : \mathcal{K} \to \mathrm{ho}\,\mathcal{C}$ as above, we assume that we have a subcategory $\mathcal{L}$ of $\mathcal{K}$ equipped with a lift $\hat{Y} : \mathcal{L} \to \mathcal{C}$ of $X|_{\mathcal{L}}$. This defines a map from the pushout

$$\begin{array}{ccc} \mathcal{F}_\bullet\mathcal{L} & \xrightarrow{\ i\ } & \mathcal{F}_\bullet\mathcal{K} \\ {\scriptstyle p}\downarrow & \boxed{\mathrm{PO}} & \downarrow \\ \mathcal{L} & \longrightarrow & \mathcal{F}_\bullet(\mathcal{K}, \mathcal{L}) \end{array}$$

(compare (1.10)) into $P_0\, {}^s\mathcal{C}_X$, which lifts (non-canonically) to $P_1\, {}^s\mathcal{C}_X$ because X is homotopy commutative.

Again, we can use each of the successive liftings $\hat{X}_n \in \mathrm{hc}_n X$ to pull back the $(n-1)$-st k-invariant for ${}^s\mathcal{C}_X$ to a map $h_n : \mathcal{F}_\bullet(\mathcal{K}, \mathcal{L}) \to K^{\mathcal{K}}(\pi_n\, {}^s\mathcal{C}_X, n+1)$, representing an $(\mathcal{S}_*, \mathcal{O})$-cohomology class $[h_n \in H^{n+1}_{\mathrm{SO}}((\mathcal{K}, \mathcal{L})/\mathcal{K}; \pi_n\, {}^s\mathcal{C}_X)$, and the relative version of Proposition 5.6 clearly holds.

6. The first obstruction

Given a natural system D on a category $\mathcal{K}$, one can always construct a trivial linear track category with D as its (abelian) fundamental groupoid. Moreover, by Proposition 2.8, the linear track extensions of $\mathcal{K}$ by D are classified up to weak equivalence by $H^3_{\mathrm{BW}}(\mathcal{K}; D)$. When $\mathcal{K} = \Pi_{\mathcal{A}}$ for some set $\mathcal{A}$ of spherical objects in a model category $\mathcal{C}$, the cohomology class determining the extension is represented by the explicit cocycle of Section 2.9. We now show how this is reflected in $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$. For this purpose, we need an $(\mathcal{S}_*, \mathcal{O})$-version of Definition 4.2:

Definition 6.1. Let $\mathcal{C}$ be a simplicial model category with spherical objects $\mathcal{A}$. A *small ${}^s\mathcal{C}_{\mathcal{A}}$-variant* is a full (necessarily small) fibrant sub-simplicial category ${}^s\mathcal{C}'_{\mathcal{A}}$ of ${}^s\mathcal{C}_{\mathcal{A}}$ (Section 5.3), such that $\pi_0\, {}^s\mathcal{C}'_{\mathcal{A}}$ is a small $\mathcal{F}_{\mathcal{A}}$-variant (Section 4.2). This just means that $\mathcal{O} := \mathrm{Obj}\ {}^s\mathcal{C}'_{\mathcal{A}}$ contains all finite coproducts of objects of $\mathcal{A}$, up to weak equivalence.

We assume that all objects in $\mathcal{C}'_{\mathcal{A}}$ are cofibrant, and for simplicity we also assume that $\mathcal{O}$ contains a *canonical* copy of $\mathrm{Obj}\ \Pi_{\mathcal{A}}$.

Example 6.2. A *minimal* small ${}^s\mathcal{C}_{\mathcal{A}}$-variant is any skeletal subcategory $\mathfrak{X}$ of ${}^s\mathcal{C}_{\mathcal{A}}$ with $\pi_0\mathfrak{X} = \Pi_{\mathcal{A}}$ (Section 0.1). In particular, we denote by ${}^s\mathcal{C}^{\min}_{\mathcal{A}}$ the *canonical* minimal small ${}^s\mathcal{C}_{\mathcal{A}}$-variant, whose objects consist of a (functorial) fibrant and cofibrant replacement for each non-isomorphic finite coproduct of objects from $\mathcal{A}$.

More generally, if $\mathcal{D}$ is any small $\mathcal{F}_{\mathcal{A}}$-variant, choose any embedding $i : \mathcal{D} \hookrightarrow \mathrm{ho}\,\mathcal{C}$ for which $i(a)$ is fibrant and cofibrant for each $a \in \mathcal{O} := \mathrm{Obj}\,\mathcal{D}$. We then obtain a fibrant small ${}^s\mathcal{C}_{\mathcal{A}}$-variant ${}^s\mathcal{C}'_{\mathcal{A}}$ by setting $\mathrm{map}_{{}^s\mathcal{C}'_{\mathcal{A}}}(a, b) := \mathrm{map}_{\mathcal{C}_{\mathcal{A}}}(i(a), i(b))$.

6.3. The 0-th k-invariant

In general, it makes no sense to speak of the 0-th k-invariant of an $(\mathcal{S}_*, \mathcal{O})$-category $\mathcal{X}$, since $\hat{\pi}_1\mathcal{X}$ is not an abelian group object over $\mathcal{K} := \pi_0\mathcal{X}$ – even though we do have a pullback square of the form (1.6) for $n = 0$, too. However, k_0 is a well-defined cohomology class in the following specialized situation:

Assumption 6.4. Let $\mathcal{A}$ be a collection of spherical objects in a simplicial model category ${}^s\mathcal{C}$, let ${}^s\mathcal{C}'_{\mathcal{A}}$ be a small ${}^s\mathcal{C}_{\mathcal{A}}$-variant – so that $\mathcal{D} := \pi_0\, {}^s\mathcal{C}'_{\mathcal{A}}$ is a small $\mathcal{F}_{\mathcal{A}}$-variant, the track category $\mathcal{E}$ of ${}^s\mathcal{C}'_{\mathcal{A}}$ is linear (Section 2.2), and $\Omega\mathcal{D}$ is a natural system on $\mathcal{D} := \pi_0\, {}^s\mathcal{C}'_{\mathcal{A}}$ (cf. Section 1.11 and Section 2.5). We let $\mathcal{O} := \mathrm{Obj}\ {}^s\mathcal{C}'_{\mathcal{A}}$.

With these assumptions we find:

Theorem 6.5. *The 0-th k-invariant for ${}^s\mathcal{C}'_{\mathcal{A}}$ corresponds to the cohomology class $\chi_{\mathcal{E}}$ classifying the linear track extension $\Omega\mathcal{D} \to \mathcal{E} \to \mathcal{D}$ (cf. Section 2.9) under the natural isomorphism of Theorem 3.10.*

Proof. Set $\mathcal{X} := {}^s\mathcal{C}'_{\mathcal{A}}$, and consider the following square of the form (1.6) in $(\mathcal{S}_*, \mathcal{O})$:

$$\begin{array}{ccccc} P_1\mathcal{X} & \xrightarrow{\ i\ } & \mathcal{Y} & \xleftarrow[\simeq]{\ \xi\ } & \mathcal{F}_\bullet\mathcal{D} = B\mathcal{D} \\ {\scriptstyle p}\downarrow & \boxed{\mathrm{PO}} & \downarrow{\scriptstyle q} & & \\ P_0\mathcal{X} & \xrightarrow[j]{} & \mathcal{Z}\,, & & \end{array}$$

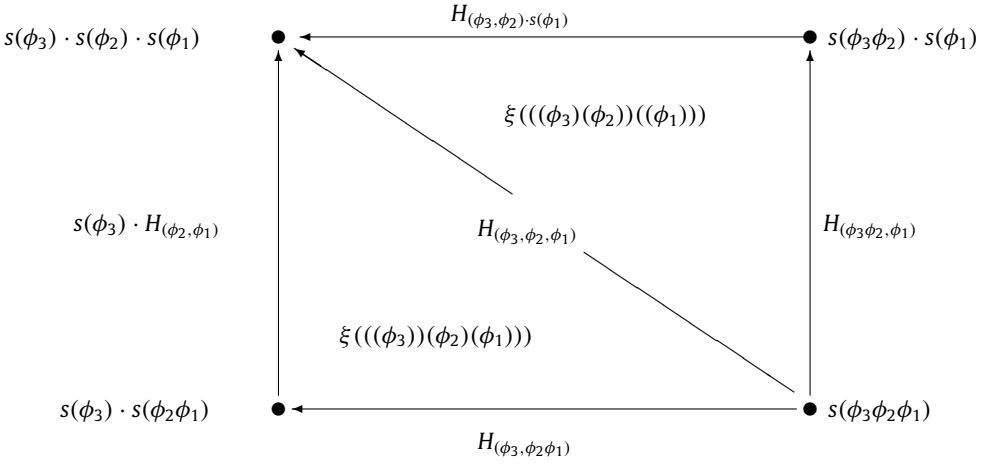
in which the homotopy pushout $\mathcal{Z}$ satisfies $P_2\mathcal{Z} \simeq E_{\mathcal{D}}(\Omega\mathcal{D}, 2) \simeq E_{\mathcal{D}}(\pi_1\mathcal{X}, 2)$ by [16, Proposition 6.4], and thus if $r^{(2)} : \mathcal{Z} \to P_2\mathcal{Z}$ is the structure map of Section 1.5(c), the 0-th k-invariant for $\mathcal{X}$ is $k_0 := r^{(2)} \circ q \sim r^{(2)} \circ j$ by construction. We use Kan's original model for the Postnikov system, so $(P_k X)_n$ consists of $\sim_k$-equivalence classes of n-simplices in $\mathcal{X}$, where $\sigma \sim_k \tau \Leftrightarrow \mathrm{sk}_k\,\sigma = \mathrm{sk}_k\,\tau$ (cf. [38, VI, Section 2]). We assume that $\mathcal{X}$ is fibrant (so each mapping space $X(u, v)$ is a Kan complex).

Factor $p = p^{(1)} : P_1\mathcal{X} \to P_0\mathcal{X}$ as a cofibration $i : P_1\mathcal{X} \to \mathcal{Y}$ followed by a weak equivalence, so the pushout above is a homotopy pushout, as required. Thus $Y_i = X_i$ for $i \le 1$, while $Y_2 = (X_2/\sim) \amalg \bar{Y}$, where $\bar{Y}$ has a "fill-in" 2-simplex $T = T_{(\sigma_0,\sigma_1,\sigma_2)}$ for every triple of 1-simplices $(\sigma_0, \sigma_1, \sigma_2)$ in X_1, with matching faces, having $d_iT = \sigma_i$. The pushout $\mathcal{Z}$ thus consists of the reduction via $\sim_0$ of the copy of $\mathcal{X}$ in $\mathcal{Y}$, with $\bar{Y}$ unaffected. The 2-simplices $K_{(\sigma,0,0)}$ for non-nullhomotopic σ represent $\hat{\pi}_1\mathcal{X}$ in $P_2\mathcal{Z} \simeq E_{\mathcal{D}}(\hat{\pi}_1\mathcal{X}, 2)$. We shall not need the description of $\mathcal{Y}$ or $\mathcal{Z}$ in higher dimensions.

Let $\mathcal{F}_\bullet\mathcal{D}$ be the cofibrant replacement for $P_0\mathcal{X}$ constructed as in Section 3.2. The weak equivalence $\xi : \mathcal{F}_\bullet\mathcal{D} \to \mathcal{Y}$ is then defined as follows:

Every 0-simplex $(\phi) \in \mathcal{F}_\bullet\mathcal{D}$ corresponds to a homotopy class $\phi \in [Xu, Xv]_{\mathrm{ho}\,\mathcal{C}}$, and $\xi(\phi)$ is a choice of a representative $s(\phi)$ in $(P_0\mathcal{X})_0 = X(u, v)_0$. For a (non-composite) 1-simplex $\sigma = ((\phi_k) \dots (\phi_1))$ in $(FU)^2\mathcal{D}$, $\xi(\sigma)$ is a choice of a homotopy $H_{(\phi_k,\dots,\phi_1)}$ between $s(\phi_k) \cdot \dots \cdot (\phi_1)$ and $s(\phi_k \cdot \dots \cdot \phi_1)$, which exists since $\mathcal{D} = \mathrm{ho}\,\mathcal{E}$. Finally, the faces of any 2-simplex $\tau \in (FU)^3\mathcal{D}$ form a triple of matching 1-simplices, so their image under ξ has a canonical fill-in $T \in \bar{Y}$, and we set $\xi(\tau) = T$.

Now either of the two maps from $P_0\mathcal{X}$ to $P_2\mathcal{Z}$ represents k_0; using the cofibrant model $\mathcal{F}_\bullet\mathcal{D}$ for the source, it is enough to identify the map on 2-simplices — or, using the identification of simplicial and cubical cohomology mentioned in the proof of Theorem 3.10, on the (triangulated) square $I^2_{(\phi_3\circ\phi_2\circ\phi_1)}$ as in (3.8). By the descriptions of ξ and $\mathcal{Y}$ above, this maps to



which is just the cocycle of (2.10), under the isomorphism of Theorem 3.10. □

Corollary 6.6. *Under the assumptions of Section 6.4, the equivalence classes of linear track extensions $\Omega\mathcal{D} \to \mathcal{E} \to \mathcal{D}$ are in one-to-one correspondence with one-stage Postnikov systems of $(\mathcal{S}_*, \mathcal{O})$-categories $\mathcal{Y}$ (that is, those satisfying $\mathcal{Y} \simeq P_1\mathcal{Y}$) such that $\pi_0\mathcal{Y} \cong \mathcal{D}$, and $\pi_1\mathcal{Y} \cong \Omega\mathcal{D}$ as $(\mathcal{K}\text{-}\mathcal{M}od, \mathcal{O})$-categories.*

6.7. A relative version

Now assume that ${}^s\mathcal{C}'_{\mathcal{A}} \in (\mathcal{S}_*, \mathcal{O})\text{-}\mathcal{C}at$ as in Section 6.4 extends to an $(\mathcal{S}_*, \mathcal{O}^+)$-subcategory $\mathcal{X}$ of ${}^s\mathcal{C}$, obtained by adding a single new object $Y \in \mathcal{C}$. Thus $\mathcal{O}^+ := \mathcal{O} \cup \{Y\}$, $\mathcal{X}|_{\mathcal{O}} = {}^s\mathcal{C}'_{\mathcal{A}}$, and we omit all non-trivial maps out of Y, so $\mathrm{map}_{\mathcal{X}}(Y, Y) = c(\{\mathrm{Id}_Y, *\})$ and $\mathrm{map}_{\mathcal{X}}(Y, B) = c(\{*\})$ for all $B \in \mathcal{O}$ (see Section 8.1 below).

In this case we can extend the track category $\mathcal{E}$ of ${}^s\mathcal{C}'_{\mathcal{A}}$ to a track category $\mathcal{E}^+$ for $\mathcal{X}$, which is still linear (since all non-trivial maps are out of homotopy cogroup objects). If $\mathcal{D}^+ := \pi_0\mathcal{X}$, then $(\mathcal{D}^+, \mathcal{D})$ is a Λ-pair, for $\Lambda := \pi_{\mathcal{A}}Y$ (Section 4.2), and $\Omega\mathcal{D}^+$ is a natural system on $\mathcal{D}^+$. Therefore, Theorem 6.5 applies in this situation, too: that is, the 0-th k-invariant for $\mathcal{X}$ corresponds to the cohomology class classifying the linear track extension $\Omega\mathcal{D}^+ \to \mathcal{E}^+ \to \mathcal{D}^+$.

Note that the inclusion of categories ${}^s\mathcal{C}'_{\mathcal{A}} \hookrightarrow \mathcal{X}$, and the corresponding inclusion of object sets $\mathcal{O} \hookrightarrow \mathcal{O}^+$, induces natural transformations in the Baues–Wirsching cohomology and $(\mathcal{S}_*, \mathcal{O})$-cohomology fitting into long exact sequences with the relative versions, with all vertical maps being isomorphisms by Theorem 3.10:

$$\begin{array}{ccccc} \dots H^n_{\mathrm{BW}}(\mathcal{D}^+; \Omega\mathcal{D}^+) & \xrightarrow{i^*} & H^n_{\mathrm{BW}}(\mathcal{D}; \Omega\mathcal{D}) & \xrightarrow{\delta^n} & H^{n+1}_{\mathrm{BW}}(\mathcal{D}^+, \mathcal{D}; \Omega\mathcal{D}^+) \dots \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \dots H^{n-1}_{\mathrm{SO}}(\mathcal{X}; \Omega\mathcal{D}^+) & \xrightarrow{i^*} & H^{n-1}_{\mathrm{SO}}({}^s\mathcal{C}'_{\mathcal{A}}; \Omega\mathcal{D}) & \xrightarrow{\delta^{n-1}} & H^n_{\mathrm{SO}}(\mathcal{X}, {}^s\mathcal{C}'_{\mathcal{A}}; \Omega\mathcal{D}^+) \dots \end{array} \tag{6.8}$$

Lemma 6.9. *The class $\delta^3(\chi_{\mathcal{E}})$ in $H^4_{\mathrm{SO}}(\mathcal{D}^+, \mathcal{D}; \Omega\mathcal{D}^+)$ is the obstruction to realizing Λ by a track category $\mathcal{E}^+$ inside that of $\mathcal{C}$.*

Proof. The class $\delta^3(\chi_{\mathcal{E}})$ vanishes if and only if $\chi_{\mathcal{E}}$ is in the image of i^* in the top row of (6.8) — that is, if and only if $\mathcal{E}$ extends to a linear track category $\mathcal{E}^+$ realizing Λ. □

From the ladder of isomorphisms (6.8) we deduce:

Corollary 6.10. *The class $\delta^2(\chi_{\tilde{\mathcal{D}}})$ maps under the isomorphism of Theorem 3.10 to the relative k-invariant $\delta^2(k_0)$ in $H^3_{\mathrm{SO}}(\mathcal{X},\ {}^{s}\mathcal{C}'_{\mathcal{A}};\ \Omega\mathcal{D}^+)$, which is the obstruction to realizing Λ as a one-stage Postnikov system in the $(\mathcal{S}_*, \mathcal{O})$-category for $\mathcal{C}$.*

7. Realizing $\Pi_{\mathcal{A}}$-algebras

The approach of [35,35,16] to realizing Π-algebras can be generalized somewhat (see [17]), but it still does not apply to arbitrary resolution model categories (for example, it does not even apply to topological spaces if $\mathcal{A}$ consists of mod-p Moore spaces — see [13, Section 4.6]). We therefore restrict to the following setting:

Definition 7.1. If $\mathcal{C}$ is a semi-spherical resolution model category equipped with a set of spherical objects $\mathcal{A}$, the resolution model category $s\mathcal{C}$ (Section 1.2) is called a *strict E^2-model category* if the inclusion $c(-) : \mathcal{C} \to s\mathcal{C}$ has a left adjoint $R : s\mathcal{C} \to \mathcal{C}$, called the *realization* functor for $s\mathcal{C}$, such that for all $A_0 \in \mathcal{A}$, the natural map induced by the unit $\varepsilon_{X_\bullet} : X_\bullet \to c(RX_\bullet)$

$$\varepsilon_* : \|\operatorname{map}_{\mathcal{C}}(A_0, X_\bullet)\| \to \operatorname{map}_{\mathcal{C}}(A_0, R(X_\bullet)) \quad \text{is a weak equivalence} \tag{7.2}$$

as long as $X_\bullet \in s\mathcal{C}$ is cofibrant in the resolution model category structure on $s\mathcal{C}$ determined by $\mathcal{A}_0 := \{\Sigma^k A_0\}_{k=0}^{\infty}$. Here $\|Q_\bullet\|$ is the diagonal of a bisimplicial set $Q_\bullet \in s\mathcal{S}$.

Example 7.3. The main example that we have in mind is $\mathcal{C} = \mathcal{T}_*$ with $\mathcal{A} = \{\mathbf{S}^k\}_{k=1}^{\infty}$, and R the usual geometric realization. In this case the cofibrancy condition on $X_\bullet$ implies that each X_n is $(k-1)$-connected, when $A_0 = \mathbf{S}^k$, so (7.2) holds by [42, Theorem 12.3] (see also [2]).

In [17, Theorems 3.15–3.19], it was shown that all the examples of Section 1.4 are E^2-model categories, which satisfy a somewhat weaker set of axioms (see [17, Definition 3.12]). However, there are a number of additional examples satisfying these stricter conditions — $\mathcal{T}_*$ can be replaced by $\mathcal{S}_{\mathrm{red}}$ or $\mathcal{G}$, or various categories of spectra, or DG-categories; or we can take diagrams in these categories. We can also use localized or truncated spheres. In order to cover all these cases we have therefore stated the conditions needed in axiomatic form. This also permits them to be dualized more readily (Section 0.4).

In this context the obstruction theory of [16] can be stated using the following:

Definition 7.4. A *quasi-Postnikov tower* for a $\Pi_{\mathcal{A}}$-algebra Λ is a tower of fibrations

$$\cdots \xrightarrow{p^{(n+1)}} X\langle n+1\rangle_\bullet \xrightarrow{p^{(n)}} X\langle n\rangle_\bullet \xrightarrow{p^{(n-1)}} \cdots \xrightarrow{p^{(0)}} X\langle 0\rangle_\bullet \simeq B\Lambda$$

in $s\mathcal{C}/B\Lambda$ such that $\pi_{\mathcal{A}} X\langle n\rangle_\bullet \simeq E^{\Lambda}(\Omega^{n+1}\Lambda, n+2)$ (as for the usual Postnikov system of a realization of $B\Lambda$ in $s\mathcal{C}$ — see [17, Section 5.8]). The object $X\langle n\rangle_\bullet \in s\mathcal{C}$ will be called an *n-th quasi-Postnikov section* for Λ.

The following is shown in [16, Section 9] and [17, Theorems 5.6–5.7]:

Theorem 7.5. *Let $\mathcal{C}$ be an E^2-model category with a set of spherical objects $\mathcal{A}$. A $\Pi_{\mathcal{A}}$-algebra Λ is realizable if and only it has a quasi-Postnikov tower in $s\mathcal{C}/B\Lambda$. Moreover, if such a tower exists in degrees $\le n-1$, then:*

(a) *Up to homotopy, there is a unique $X\langle n\rangle_\bullet \in s\mathcal{C}$ with $P_{n-1}X\langle n\rangle_\bullet = X\langle n-1\rangle_\bullet$,*

$$\pi_k^{\#}(X\langle n\rangle_\bullet) \cong \begin{cases} \Omega^k\Lambda & \text{for } 0 \le k \le n, \\ 0 & \text{otherwise} \end{cases} \tag{7.6}$$

(see Section 1.11).

(b) *This $X\langle n\rangle_\bullet$ is an n-th quasi-Postnikov section for Λ if and only if the $(n+2)$-nd k-invariant for $\pi_{\mathcal{A}}X\langle n\rangle_\bullet$ vanishes in $H^{n+3}_{\mathrm{AQ}}(\Lambda; \Omega^{n+1}\Lambda)$.*

(c) *In that case, the different choices for the map $p^{(n)} : X\langle n+1\rangle_\bullet \to X\langle n\rangle_\bullet$ are in one-to-one correspondence with elements of $H^{n+2}_{\mathrm{AQ}}(\Lambda; \Omega^{n+1}\Lambda)$.*

Note that from the spiral exact sequence (1.3) we can deduce from (7.6) that

$$\pi_k\pi_{\mathcal{A}}X\langle n\rangle_\bullet \cong \begin{cases} \Lambda & \text{for } k=0 \\ \Omega^{n+1}\Lambda & \text{for } k=n+2, \\ 0 & \text{otherwise.} \end{cases} \tag{7.7}$$

The vanishing of the $(n+2)$-nd k-invariant for $\pi_{\mathcal{A}}X\langle n\rangle_\bullet$ is equivalent to the latter being an Eilenberg–Mac Lane object $E^{\Lambda}(\Omega^{n+1}\Lambda, n+2)$ in $s\Pi_{\mathcal{A}}$-$\mathcal{A}lg$.

8. Mapping algebras

In order to compare the approaches of Sections 5 and 7, we need to recast the problem of realizing $\Lambda \in \Pi_{\mathcal{A}}$-$\mathcal{A}lg$ as one of rectifying a suitable homotopy commutative diagram – or more precisely, of lifting a diagram through the Postnikov system of an $(\mathcal{S}_*, \mathcal{O})$-category.

The obvious first choice is to consider a diagram $X : \mathcal{K} \to \mathrm{ho}\,\mathcal{C}$ for $\mathcal{K} := \mathcal{F}_{\mathcal{A}}^{+}$ (Section 4.2). Unfortunately, there are two problems with this:

(a) We do not actually have such a diagram X to begin with, since the putative value of $X(\star) \in \mathrm{ho}\,\mathcal{C}$ is precisely the realization of the $\Pi_{\mathcal{A}}$-algebra Λ in $\mathcal{C}$ that we are looking for.
(b) Moreover, we do not expect a rectification $\hat{X} : \mathcal{F}_{\mathcal{A}}^{+} \to \mathcal{C}$ to exist (unless the model category $\mathcal{C}$ is "formal"), since commuting diagrams in $\mathrm{ho}\,\mathcal{C}$ do not generally lift to $\mathcal{C}$.

In order to solve the second problem, we introduce the following concept:

Definition 8.1. Let ${}^{s}\mathcal{C}'_{\mathcal{A}}$ be a small ${}^{s}\mathcal{C}_{\mathcal{A}}$-variant (Section 6.1) with object set $\mathcal{O}$, and let $\mathcal{O}^{+} := \mathcal{O} \cup \{\star\}$. An *$\mathcal{A}$-mapping algebra (based on ${}^{s}\mathcal{C}'_{\mathcal{A}}$)* is an $(\mathcal{S}_*, \mathcal{O}^{+})$-category $\mathfrak{X}$ with mapping spaces as follows (compare (4.3)):

$$\mathrm{map}_{\mathfrak{X}}(B,C) \;=\; \begin{cases} \mathrm{map}_{s\mathcal{C}_{\mathcal{A}}}(B,C) & \text{if } B, C \in \mathcal{O} \\ c(\{*, \mathrm{Id}_{\star}\}) & \text{if } B = C = \star \\ c(\{*\}) & \text{otherwise.} \end{cases} \tag{8.2}$$

The category of all $\mathcal{A}$-mapping algebras based on ${}^{s}\mathcal{C}'_{\mathcal{A}}$ will be denoted by $\mathcal{M}_{\mathcal{A}}^{\mathcal{C}'_{\mathcal{A}}}$ (or simply $\mathcal{M}_{\mathcal{A}}$, when ${}^{s}\mathcal{C}'_{\mathcal{A}}$ is understood from the context). Elements in $\mathcal{M}_{\mathcal{A}}$ will be written as $\mathfrak{X}$, $\mathfrak{Y}$, etc, and we denote $\mathrm{map}_{\mathfrak{X}}(B,\star)$ by $\mathfrak{X}\{B\}$ for all $B \in \mathcal{O}$. If we embed $(\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$ in $(\mathcal{S}_*, \mathcal{O}^{+})$-$\mathcal{C}at$ by making $\mathrm{map}(B,\star) = \{*\}$ for all $B \in \mathcal{O}^{+}$ (as in Section 4.2), then we can think of an $\mathcal{A}$-mapping algebra based on ${}^{s}\mathcal{C}'_{\mathcal{A}}$ as an $(\mathcal{S}_*, \mathcal{O}^{+})$-category under ${}^{s}\mathcal{C}'_{\mathcal{A}}$, subject to last two conditions of (8.2). Thus $\mathcal{M}_{\mathcal{A}}$ inherits a simplicial model category structure from $(\mathcal{S}_*, \mathcal{O}^{+})$-$\mathcal{C}at$.

Note that if we set $\mathcal{D}^{+} := \pi_0\mathfrak{X}$, we obtain a Λ-pair $(\mathcal{D}^{+}, \mathcal{D})$ for $\mathcal{D} := \pi_0\,{}^{s}\mathcal{C}'_{\mathcal{A}}$, where the $\Pi_{\mathcal{A}}$-algebra Λ is defined by $\Lambda\{A\} := \pi_0\mathfrak{X}\{A\}$ for all $A \in \mathcal{A}$. Thus we can think of an $\mathcal{A}$-mapping algebra as an enriched version of a $\Pi_{\mathcal{A}}$-algebra.

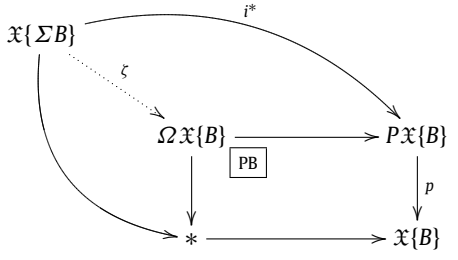
Example 8.3. Given a small ${}^{s}\mathcal{C}_{\mathcal{A}}$-variant ${}^{s}\mathcal{C}'_{\mathcal{A}} \subseteq {}^{s}\mathcal{C}_{\mathcal{A}}$, the motivating example of an $\mathcal{A}$-mapping algebra $\mathfrak{X}$ based on ${}^{s}\mathcal{C}'_{\mathcal{A}}$ is obtained by choosing any $X \in \mathcal{C}$, and setting $\mathrm{map}_{\mathfrak{X}}(A,\star) := \mathrm{map}_{s\mathcal{C}}(A,X)$. We denote this $\mathcal{A}$-mapping algebra by $\mathfrak{M}_{\mathcal{A}}^{{}^{s}\mathcal{C}'_{\mathcal{A}}}X$ (or simply $\mathfrak{M}_{\mathcal{A}}X$, when ${}^{s}\mathcal{C}'_{\mathcal{A}}$ is understood from the context). Clearly $\pi_0(\mathfrak{M}_{\mathcal{A}}X) \cong \pi_{\mathcal{A}}X$. We say that an $\mathcal{A}$-mapping algebra $\mathfrak{Y}$ is *realizable* (by $X \in \mathcal{C}$) if $\mathfrak{Y} \cong \mathfrak{M}_{\mathcal{A}}X$. Since any $Y \in \mathcal{C}$ is fibrant, $\mathfrak{M}_{\mathcal{A}}X$ is always fibrant.

Remark 8.4. Recall that the *path object* $PK \in \mathcal{S}_*$ for a fibrant pointed simplicial set K has $(PK)_n := \{x \in K_{n+1} : d_1 \dots d_{n+1}x = *\}$, with re-indexed face and degeneracy maps, and the universal fibration $p : PK \to K$ is induced by d_0 (cf. [27, Section 2.9]). We denote the *path fibration* functor $K \mapsto (PK \xrightarrow{p} K)$ by $\rho : \mathcal{S}_* \to \mathcal{S}_*^{T}$, where $\mathcal{S}_*^{T}$ is the category of diagrams in $\mathcal{S}_*$ indexed by $T = (0 \to 1)$. Because ρ commutes with products, it extends to a functor $\rho : \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}}^{T}$.

Note that

$$\rho\,\mathrm{map}_{s\mathcal{C}}(A,Y) \quad \text{is induced by the inclusion } i : A \hookrightarrow CA. \tag{8.5}$$

If we define the suspension ΣX in $\mathcal{C}$ as the cofiber of $i : X \hookrightarrow CX$, where CX is the reduced cone, then for any fibrant $\mathcal{A}$-mapping algebra $\mathfrak{X}$ and every $B \in {}^{s}\mathcal{C}'_{\mathcal{A}}$ we have a natural map ζ to the pullback (in $\mathcal{S}_*$), as indicated:



(8.6)

Similarly, if $B = \coprod_{i\in I} B_i$ for $B_i \in {}^{s}\mathcal{C}'_{\mathcal{A}}$, we have a natural map

$$\mathfrak{X}\{B\} \xrightarrow{\theta} \prod_{i\in I} \mathfrak{X}\{B_i\}. \tag{8.7}$$

Definition 8.8. An $\mathcal{A}$-mapping algebra $\mathfrak{X}$ based on ${}^{s}\mathcal{C}'_{\mathcal{A}}$ will be called *realistic* if whenever there are weak equivalences

$$A' \simeq \Sigma A \quad \text{and} \quad B \simeq \coprod_{i\in I} B_i, \tag{8.9}$$

in ${}^{s}\mathcal{C}'_{\mathcal{A}}$, the maps ζ in (8.6) and θ in (8.7) are weak equivalences.

Lemma 8.10. *Any realizable $\mathcal{A}$-mapping algebra is realistic.*

Proof. This holds since both ζ in (8.6) and θ in (8.7) map into homotopy limits. □

Note that if $\mathfrak{X} := \mathfrak{M}_{\mathcal{A}} Y$ and one of the maps in (8.9) is an isomorphism, then so is the corresponding map ζ or θ.

Lemma 8.11. *Any map $f : X \to X'$ in $\mathcal{C}$ induces a morphism of $\mathcal{A}$-mapping algebras $f_* : \mathfrak{M}_{\mathcal{A}} Y \to \mathfrak{M}_{\mathcal{A}} Y'$, and f is an $\mathcal{A}$-equivalence (Section 0.1) if and only if $f_* : \mathfrak{M}_{\mathcal{A}} Y\{A\} \to \mathfrak{M}_{\mathcal{A}} Y'\{A\}$ is a weak equivalence in $\mathcal{S}_*$ for each $A \in \mathcal{A}$.* □

Definition 8.12. A *free* $\mathcal{A}$-mapping algebra based on ${}^s\mathcal{C}'_{\mathcal{A}}$ is one of the form $\mathfrak{M}_{\mathcal{A}} B$ for $B \in {}^s\mathcal{C}'_{\mathcal{A}}$.

Lemma 8.13. *If $\mathcal{Y}$ is an $\mathcal{A}$-mapping algebra based on ${}^s\mathcal{C}'_{\mathcal{A}}$ and $B \in {}^s\mathcal{C}'_{\mathcal{A}}$, there is a natural isomorphism* $\mathrm{map}_{\mathcal{M}_{\mathcal{A}}}(\mathfrak{M}_{\mathcal{A}} B, \mathcal{Y}) \cong \mathcal{Y}\{B\}$.

Proof. This follows from the enriched Yoneda Lemma (cf. [28]). □

Definition 8.14. If $\mathfrak{X}$ is a $\mathcal{A}$-mapping algebra based on ${}^s\mathcal{C}'_{\mathcal{A}}$, for any $n \geq 0$ we obtain its *n-th Postnikov section* $P_n\mathfrak{X}$ by setting $(P_n\mathfrak{X})\{B\} := P_n(\mathfrak{X}\{B\})$ for any $B \in \mathcal{O} := \mathrm{Obj}\, {}^s\mathcal{C}'_{\mathcal{A}}$. This is well-defined, since when we compose the composition map $\gamma : \mathrm{map}_{\mathfrak{X}}(B, A) \times \mathrm{map}_{\mathfrak{X}}(A, \star) \to \mathrm{map}_{\mathfrak{X}}(B, \star) = \mathfrak{X}\{B\}$ of the simplicial enrichment with Postnikov fibration $p : \mathfrak{X}\{B\} \to P_n(\mathfrak{X}\{B\})$, the result factors as

$$\begin{aligned} \mathrm{map}_{{}^s\mathcal{C}'_{\mathcal{A}}}(B, A) \times \mathrm{map}_{\mathfrak{X}}(B, \star) &\to P_n \,\mathrm{map}_{{}^s\mathcal{C}'_{\mathcal{A}}}(B, A) \times (P_n\mathfrak{X})\{A\} \\ &= P_n\left(\mathrm{map}_{{}^s\mathcal{C}'_{\mathcal{A}}}(B, A) \times \mathfrak{X}\{A\}\right) \xrightarrow{P_n\gamma} (P_n\mathfrak{X})\{B\}. \end{aligned}$$

A map of $\mathcal{A}$-mapping algebras $\Phi : \mathfrak{X} \to \mathcal{Y}$ is called an *n-equivalence* if it induces a weak equivalence of n-th Postnikov sections. A map $f : X \to Y$ in $\mathcal{C}$ is an *n-stage $\mathcal{A}$-equivalence* if $\mathfrak{M}_{\mathcal{A}} f : \mathfrak{M}_{\mathcal{A}} X \to \mathfrak{M}_{\mathcal{A}} Y$ is an *n-equivalence* of $\mathcal{A}$-mapping algebras.

Example 8.15. For $n = 0$, we can replace $P_0\mathfrak{X}$ by $\pi_0\mathfrak{X}$, using the fact that the composition in any simplicially enriched category $\mathcal{X}$ factors through its homotopy category $\pi_0\mathcal{X}$. In particular, this shows that if ${}^s\mathcal{C}'_{\mathcal{A}}$ is a small ${}^s\mathcal{C}_{\mathcal{A}}$-variant and $\mathcal{D} := \pi_0\, {}^s\mathcal{C}'_{\mathcal{A}}$, then any Λ-pair $(\mathcal{D}^+, \mathcal{D})$ can be enriched by an $\mathcal{A}$-mapping algebra $\mathfrak{X}_\Lambda$ based on ${}^s\mathcal{C}'_{\mathcal{A}}$ with $\pi_0\mathfrak{X}_\Lambda \cong \Lambda$.

Remark 8.16. The tower $(P_n\mathfrak{M}_{\mathcal{A}} X)_{n=0}^{\infty}$ may be the best approximation to an $\mathcal{A}$-Postnikov tower available, since the category $\mathcal{C}$ itself may not have such towers – e.g., when $\mathcal{C} = \mathcal{T}_*$ and $\mathcal{A}$ consist of mod-p Moore spaces (see [15, Section 3.10]).

9. The Stover category

We now specialize to a specific small ${}^s\mathcal{C}_{\mathcal{A}}$-variant, which defines $\mathcal{A}$-mapping algebras with various useful properties:

Definition 9.1. Let $\mathcal{C}$ be an E^2-model category with spherical objects $\mathcal{A}$. We assume for simplicity that

$$\mathcal{A} = \{\Sigma^k A_0\}_{k=0}^{\infty} \quad \text{for some strict cogroup object } A_0. \tag{9.2}$$

An *elementary Stover object* in $\mathcal{C}$ is one of the form

$$B := \mathrm{colim}\left(A \xrightarrow{\mathrm{inc}} (CA_{(j)})_{j\in T}\right), \tag{9.3}$$

where $A \in \mathcal{A}$, and the colimit is that of the diagram consisting of A, together with an inclusion $A \hookrightarrow CA_{(j)}$ into the cone on $A_{(j)}$ (a copy of A) for each $j \in T$. The set T is called the *null set* for B. Note that B is still in $\mathcal{C}_{\mathcal{A}}$, and is still a cogroup object in $\mathcal{C}$.

A *Stover object* is any coproduct $B = \coprod_{i\in I} B_{(i)}$ of elementary Stover objects $\{B_{(i)}\}_{i\in I}$.

The *Stover category*, denoted by ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$, is the full sub-simplicial category of ${}^s\mathcal{C}_{\mathcal{A}}$ consisting of all Stover objects such that the cardinalities of the indexing set I for the coproduct, and of the null sets $T_{(i)}$ for each coproduct summand $B_{(i)}$, are bounded by a fixed limit cardinal κ (see Remark 9.18 below).

Evidently, ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$ is a small ${}^s\mathcal{C}_{\mathcal{A}}$-variant (Section 6.1). Any $\mathcal{A}$-mapping algebra based on ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$ will be called a *Stover mapping algebra*, and the realizable Stover mapping algebra for any $Y \in \mathcal{C}$ will be denoted by $\mathfrak{M}^{\mathrm{St}}_{\mathcal{A}} Y$. The category of all Stover mapping algebras will be denoted by $\mathcal{M}^{\mathrm{St}}_{\mathcal{A}}$.

Similarly, any $\mathcal{A}$-mapping algebra based on the canonical minimal small ${}^s\mathcal{C}_{\mathcal{A}}$-variant ${}^s\mathcal{C}^{\mathrm{min}}_{\mathcal{A}}$ (Section 6.2) will be called a *minimal $\mathcal{A}$-mapping algebra*, and the minimal $\mathcal{A}$-mapping algebra for Y will be denoted by $\mathfrak{M}^{\mathrm{min}}_{\mathcal{A}} Y$.

Lemma 9.4. *For any $Y \in \mathcal{C}$, the mapping spaces of the Stover mapping algebra $\mathfrak{X}^{\mathrm{St}} = \mathfrak{M}^{\mathrm{St}}_{\mathcal{A}} Y$ are canonically determined by the minimal $\mathcal{A}$-mapping algebra $\mathfrak{X}^{\mathrm{min}} = \mathfrak{M}^{\mathrm{min}}_{\mathcal{A}} Y$.*

Proof. For $A \in \mathcal{A}$, set $\mathfrak{X}^{\mathrm{St}}\{A\} := \mathfrak{X}^{\mathrm{min}}\{A\}$. If B is an elementary Stover object as in (9.3) (with $T \neq \emptyset$), we define $\mathfrak{X}^{\mathrm{St}}\{B\}$ to be the pullback in $\mathcal{S}_*$:

$$\begin{array}{ccc} \mathfrak{X}^{\mathrm{St}}\{B\} & \longrightarrow & \prod_{j\in T} P\mathfrak{X}^{\mathrm{min}}\{A_{(j)}\} \\ {\scriptstyle f}\downarrow & \boxed{\mathrm{PB}} & \downarrow{\scriptstyle \prod_j p_j} \\ \mathfrak{X}^{\mathrm{min}}\{A\} & \xrightarrow{\Delta} & \prod_{j\in T} \mathfrak{X}^{\mathrm{min}}\{A_{(j)}\} \end{array} \tag{9.5}$$

(where $P\mathfrak{X}$ is the path functor of Section 8.4 and Δ is the diagonal).

If $B = \coprod_{I\in I} B_{(i)}$ is a coproduct of elementary Stover objects, we set

$$\mathfrak{X}^{\mathrm{St}}\{B\} := \prod_{I\in I} \mathfrak{X}^{\mathrm{St}}\{B_{(i)}\}. \quad \square \tag{9.6}$$

Remark 9.7. If $\mathfrak{X}$ is any fibrant $\mathcal{A}$-mapping algebra based on the minimal small ${}^s\mathcal{C}_{\mathcal{A}}$-variant ${}^s\mathcal{C}^{\min}_{\mathcal{A}}$, we may use (9.5) and (9.6) to define the mapping spaces of the corresponding $\mathcal{A}$-mapping algebra $\mathfrak{X}^{\mathrm{St}}$ based on ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$. Of course, this does not determine the action of ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$ on $\mathfrak{X}^{\mathrm{St}}$.

Lemma 9.8. *If an $\mathcal{A}$-mapping algebra $\mathfrak{X}$ based on ${}^s\mathcal{C}^{\min}_{\mathcal{A}}$ is realistic (Section 8.8), so is the corresponding $\mathcal{A}$-mapping algebra $\mathfrak{X}^{\mathrm{St}}$ based on ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$.*

Proof. Since the right vertical map in (9.5) is a fibration, so is $f : \mathfrak{X}^{\mathrm{St}} \to \mathfrak{X}^{\min}$, and (9.5) is a homotopy pullback. Thus if ζ in (8.6) and θ in (8.7) are weak equivalences for $\mathfrak{X}$ whenever the maps in (8.9) are, the same is true for $\mathfrak{X}^{\mathrm{St}}$. $\square$

Corollary 9.9. *Under assumption (9.2), all the mapping spaces of a realistic Stover mapping algebra $\mathfrak{X}^{\mathrm{St}}$ are determined up to weak equivalence by the single simplicial set $\mathfrak{X}\{A_0\}$.*

In the dual case (Section 0.4), when we have homotopy group objects $\{W_n\}_{n=1}^{\infty}$ in $\mathcal{C}$ with each $W_n = \Omega W_{n+1}$, it is not enough to know the single mapping space $\mathrm{map}_{\mathcal{C}}(X, W_1)$; in this case we need its Ω^∞-structure.

Definition 9.10. Let $\mathfrak{X}$ be an $\mathcal{A}$-mapping algebra based on a small $\mathcal{C}_{\mathcal{A}}$-variant ${}^s\mathcal{C}'_{\mathcal{A}}$, and $B \in {}^s\mathcal{C}'_{\mathcal{A}}$. For each $\phi \in \mathfrak{X}\{A\}_0$ we call the pullback N^ϕ (in $\mathcal{S}_*$)

$$\begin{array}{ccc} N^\phi & \longrightarrow & P\mathfrak{X}\{B\} \\ \downarrow & \boxed{\mathrm{PB}} & \downarrow p \\ \phi & \xrightarrow{\mathrm{inc}} & \mathfrak{X}\{B\} \end{array}$$

the *space of nullhomotopies* for ϕ. (It will be empty if ϕ is not nullhomotopic.)

If ${}^s\mathcal{C}'_{\mathcal{A}}$ is any small ${}^s\mathcal{C}_{\mathcal{A}}$-variant containing $\mathcal{A}$ itself, and $\mathcal{Y}$ is any $\mathcal{A}$-mapping algebra based on ${}^s\mathcal{C}'_{\mathcal{A}}$, the *Stover construction* on $\mathcal{Y}$ is the Stover object given by

$$K\mathcal{Y} := \coprod_{A\in\mathcal{A}} \coprod_{\phi\in\mathcal{Y}\{A\}_0} \mathrm{colim}\left(A_{(\phi)} \xrightarrow{\mathrm{inc}} (CA_{(\Phi)})_{\Phi\in N_0^\phi}\right). \tag{9.11}$$

This defines a functor $K : \mathcal{M}^{\mathrm{St}}_{\mathcal{A}} \to \mathcal{C}$.

Proposition 9.12. *The composite $L := K \circ \mathfrak{M}^{\mathrm{St}}_{\mathcal{A}} : \mathcal{C} \to \mathcal{C}$ is a comonad on $\mathcal{C}$.*

Proof. Note that $K\mathcal{Y}$ depends only on the 0-simplices $\rho_0\mathcal{Y} := (P\mathcal{Y}_0 \to \mathcal{Y}_0)$ of the path fibration ρ (Section 8.4). Because ρ is a functor, any map of $\mathcal{A}$-mapping algebras $\Psi : \mathcal{Y} \to \mathcal{Z}$ induces a map of the indexing categories for the colimit (9.11). Again, this depends only on $\rho_0\Psi$. This in turn induces a map $K\Psi : K\mathcal{Y} \to K\mathcal{Z}$. Thus we have defined a functor $K_0 : \rho_0\mathcal{M}^{\mathrm{St}}_{\mathcal{A}} \to \mathcal{C}$. We show that the functor K_0 is left adjoint to $\rho_0\mathfrak{M}^{\mathrm{St}}_{\mathcal{A}} : {}^s\mathcal{C} \to \rho_0\mathcal{M}^{\mathrm{St}}_{\mathcal{A}}$:

Given $f : K\mathcal{Y} \to X$ in $\mathcal{C}$, we define $\hat{f} : \rho_0\mathcal{Y} \to \rho_0\mathfrak{M}^{\mathrm{St}}_{\mathcal{A}}X$ by sending $\phi \in \mathcal{Y}\{A\}_0$ to $f|_{A_{(\phi)}} \in (\mathfrak{M}^{\mathrm{St}}_{\mathcal{A}}X)\{A\}_0 = \mathrm{map}_{s\mathcal{C}}(A, X)_0$, and similarly for $\Phi \in \mathcal{Y}\{A\}_1$ with $d_0\Phi = 0$ and $d_1\Phi = \phi$ (using (8.5)).

Conversely, given $\psi : \rho_0\mathcal{Y} \to \rho_0(\mathfrak{M}^{\mathrm{St}}_{\mathcal{A}}X)$ in $\rho_0\mathcal{M}^{\mathrm{St}}_{\mathcal{A}}$, we define $\tilde{\psi} : K\mathcal{Y} \to X$ using the fact that $K\mathcal{Y}$ is defined by the colimit (9.11), so it is enough to define a map of diagrams, given by $\psi(\phi) : A_{(\phi)} \to X$ for $\phi \in \mathcal{Y}\{A\}_0$ and $\psi(\Phi) : CA_{(\Phi)} \to X$ for $\Phi \in (P\mathcal{Y}\{A\})_0$, again using (8.5).

Since we can factor $L := K \circ \mathfrak{M}^{\mathrm{St}}_{\mathcal{A}}$ as the composite $K_0 \circ (\rho_0\mathfrak{M}^{\mathrm{St}}_{\mathcal{A}})$ of an adjoint pair of functors, the functor $L : \mathcal{C} \to \mathcal{C}$ is a comonad (cf. [23, Section 4]). $\square$

Remark 9.13. Let $\mathfrak{X}$ be a fibrant Stover mapping algebra, and assume that each $A \in \mathcal{A}$ is a strict cogroup object in $\mathcal{C}$. Thus $\mathfrak{X}\{A\}$ is the underlying simplicial set of a simplicial group. Moreover, since the structure maps in (9.5)and (9.6) are all maps of simplicial groups (see Section 8.4), the same is true of $\mathfrak{X}\{B\}$ for $B \in {}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$. (Of course, the composition maps in ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$ need not be homomorphisms, so $\mathfrak{X}$ is not necessarily enriched in $\mathcal{G}$.)

If K^e is the zero-component of $K = \mathfrak{X}\{B\}$, we thus have two canonical short exact sequences of simplicial groups (respectively, groups):

$$1 \to K^e \to K \to \pi_0 K \to 1 \qquad\qquad 1 \to PK_0 \longrightarrow K_1^e \underset{s_0}{\overset{d_0}{\rightleftarrows}} K_0^e \to 1$$

This implies that K_1 is canonically determined *as a set* by $K_0, \pi_0 K$, and PK_0. In other words, $\rho_0\mathfrak{X}$ and $\pi_0\mathfrak{X}$ together determine $\mathrm{csk}_1\,\mathfrak{X}$ up to isomorphism (and of course conversely).

Definition 9.14. We call $L : \mathcal{C} \to \mathcal{C}$ the *Stover comonad* on $\mathcal{C}$.

The counit $\varepsilon : L \to \mathrm{Id}$ for L is the "tautological" natural transformation $\varepsilon_X : K(\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}X) \to X$, which sends the copy of A indexed by $\phi \in (\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}X)\{A\}_0 = \mathrm{Hom}_C(A, X)$ in (9.11) to X by ϕ, and similarly for the cones $CA_{(\Phi)}$.

The comultiplication $\mu : L \to L^2$ is induced by the natural inclusion $\nu : K\mathcal{Y} \to K(\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(K\mathcal{Y}))$, defined for any Stover mapping algebra $\mathcal{Y}$, which sends $A_{(\phi)}$ in $K\mathcal{Y}$ identically to the copy of A in $K(\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(K\mathcal{Y}))$ indexed by the inclusion $A_{(\phi)} \hookrightarrow K\mathcal{Y}$.

The *Stover resolution* of an object $Y \in \mathcal{C}$ is the simplicial resolution $Q_\bullet$ of Y, where $Q_n := L^{n+1}Y$ for each $n \geq 0$ (and the face and degeneracy maps are induced by η and μ).

Remark 9.15. If we extend K to a *simplicial* functor $\tilde{K} : \mathcal{M}_{\mathcal{A}}^{\mathrm{St}} \to {}^s\mathcal{C}$, it factors through $\bar{K} : \rho\mathcal{M}_{\mathcal{A}}^{\mathrm{St}} \to {}^s\mathcal{C}$, so $\mathrm{csk}_n \bar{K}$ depends on $\mathrm{csk}_n \rho\mathcal{M}_{\mathcal{A}}^{\mathrm{St}}$, which is determined in turn by $\mathrm{csk}_{n+1} \mathcal{M}_{\mathcal{A}}^{\mathrm{St}}$.

Proposition 9.16. *If $s\mathcal{C}$ is a strict E^2-model category with spherical objects $\mathcal{A}$, the Stover resolution defines a one-to-one correspondence between objects $Y \in \mathcal{C}$ up to $\mathcal{A}$-equivalence (Section 0.1) and weak equivalence of simplicial objects $Q_\bullet \in s\mathcal{C}$ with $\pi_{\mathcal{A}}Q_\bullet \simeq B\Lambda$ (where $\Lambda \cong \pi_{\mathcal{A}}Y$).*

Proof. By [34, Section 3.3], the simplicial object $Q_\bullet$ is cofibrant in the resolution model category structure on $s\mathcal{C}$, and by [46, Section 2], the map $\varepsilon : Q_\bullet \to c(Y)$ (induced by η) is a weak equivalence. Thus $\pi_{\mathcal{A}}Q_\bullet \simeq B\Lambda$ by (1.3). From (7.2) we see that $\| \mathrm{map}_{\mathcal{C}}(A_0, Q_\bullet)\| \simeq \mathrm{map}_{\mathcal{C}}(A_0, R(Q_\bullet))$. Applying the Bousfield–Friedlander spectral sequence of [25, Theorem B.5] to the bisimplicial set $\mathcal{M}_\bullet := \mathrm{map}_{\mathcal{C}}(A_0, Q_\bullet)$, with

$$E^2_{s,t} = \pi_s\pi_t\mathcal{M}_\bullet \;\Rightarrow\; \pi_{s+t}\|\mathcal{M}_\bullet\|, \tag{9.17}$$

we conclude that $\pi_*\|\mathcal{M}_\bullet\| \cong \Lambda$, and thus $\pi_{\mathcal{A}}(RQ_\bullet) \cong \pi_* \mathrm{map}_{\mathcal{C}}(A_0, R(Q_\bullet)) \cong \Lambda$. This is an isomorphism of $\Pi_{\mathcal{A}}$-algebras, since $\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Q_\bullet$ is a simplicial mapping algebra, and so applying $\| - \|$ to each bisimplicial set $\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Q_\bullet\{A\}$ ($A \in \mathcal{A}$) yields a mapping algebra, which is actually determined by $\|\mathcal{M}_\bullet\| = (\|\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Q_\bullet\|)\{A_0\}$ by Corollary 9.9. Thus $RQ_\bullet$ is $\mathcal{A}$-equivalent to Y. Functoriality of the Stover construction (and of the spectral sequence) shows that the correspondence of weak $\mathcal{A}$-homotopy types is one-to-one. □

Remark 9.18. We can now explain how the cardinal κ of Section 9.1 is chosen:

Given a $\Pi_{\mathcal{A}}$-algebra Λ, the collection of all homotopy types of objects $Y \in \mathrm{ho}\,\mathcal{C}$ with $\Lambda \cong \pi_{\mathcal{A}}Y$ is a set (as can be seen by considering all choices of k-invariants for cofibrant replacements of $c(Y)$ in $s\mathcal{C}$).

Define κ to be the smallest limit cardinal such that each such homotopy type Y has a Stover resolution in which each of the sets $\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(L^nY)\{A\}_0$ and N^ϕ for $\phi \in \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(L^nY)\{A\}_0$ in (9.11), for each $A \in \mathcal{A}$ and $n \geq 0$, has cardinality $\leq \kappa$.

9.19. Extending the Stover comonad

Applying the functor $\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}$ to the augmented simplicial object $Q_\bullet \to Y$ over $\mathcal{C}$ yields an augmented simplicial object $\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Q_\bullet \to \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Y$. We can think of this as coming from a monad $\mathfrak{L}$ on *realizable* Stover mapping algebras, given by $\mathfrak{L}(\mathcal{Y}) := \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(\tilde{K}\mathcal{Y})$, with counit $\eta := \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(\varepsilon)$ right inverse to the unit $\xi : \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Y \to \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(K(\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Y))$ (sending $\phi : A \to Y$ to the inclusion $A_{(\phi)} \hookrightarrow K(\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Y)$)). Because ε was a counit for L, the following square commutes:

$$\begin{array}{ccc} \mathfrak{L}\mathfrak{L}\mathcal{X} & \xrightarrow{\mu_{\mathcal{X}}} & \mathfrak{L}\mathcal{X} \\ {\scriptstyle \mathfrak{L}(\eta_{\mathcal{X}})}\downarrow & & \downarrow{\scriptstyle \eta_{\mathcal{X}}} \\ \mathfrak{L}\mathcal{X} & \xrightarrow[\eta_{\mathcal{X}}]{} & \mathcal{X} \end{array} \tag{9.20}$$

for $\mathcal{X} = \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Y$ (cf. [23, Section 4.1].

We observe that even though the simplicial functor $\mathfrak{M}_{\mathcal{A}}$ does not usually preserve coskeleta (even for $A = \mathbf{S}^1$ in $\mathcal{S}_*$), we deduce from Remark 9.15 that

$$\mathrm{csk}_n \mathfrak{L}\mathcal{X} \quad \text{is determined by } \mathrm{csk}_{n+1}\mathcal{X} \tag{9.21}$$

because $\tilde{K}$ actually lands in ${}^s\mathcal{C}_{\mathcal{A}}^{\mathrm{St}}$, so $\mathfrak{L}$ takes values in free Stover mapping algebras (Section 8.12).

Definition 9.22. A fibrant Stover mapping algebra $\mathcal{X}$ is called an $\mathfrak{L}$-*algebra* if it is equipped with a splitting $\eta_{\mathcal{X}} : \mathfrak{L}\mathcal{X} \to \mathcal{X}$ for $\xi : \mathcal{X} \to \mathfrak{L}\mathcal{X}$, such that (9.20) commutes.

Proposition 9.23. *Any realistic Stover mapping algebra $\mathcal{Y}$ can be realized, up to $\mathcal{A}$-equivalence.*

Proof. Iterating the functor $\mathfrak{L}$ on $\mathcal{Y}$ yields an augmented simplicial Stover mapping algebra $\mathcal{V}_\bullet \to \mathcal{Y}$, and since $\mathfrak{L} = \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}} \circ K$, in fact $\mathcal{V}_\bullet = \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Q_\bullet$. Here $Q_\bullet$ is the simplicial Stover object with $Q_0 = K\mathcal{Y}$ and $Q_n := K\mathcal{V}_{n-1}$ for $n \geq 1$. The extra face map $d_n : Q_n \to Q_{n-1}$ is $K\mathfrak{L}^{n-1}(\eta) : K\mathfrak{L}^{n-1}\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}\mathcal{K}\mathcal{Y} \to K\mathfrak{L}^{n-1}\mathcal{Y}$, where $\eta : \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}\mathcal{K}\mathcal{Y} \to \mathcal{Y}$ is the $\mathfrak{L}$-algebra structure map.

Proposition 2.6 of [46] shows that if $Q_\bullet \in s\mathcal{C}$ is the Stover resolution of $X \in \mathcal{C}$, then $\pi_*Q_\bullet$ is a free $\Pi_{\mathcal{A}}$-algebra resolution of $\pi_{\mathcal{A}}Y$. The proof does not in fact depend on the existence of X, but only on its mapping algebra $\mathcal{Y} := \mathfrak{M}_{\mathcal{A}}^{\min}X$. Here we

use the fact that $\mathcal{Y}$ is realistic. Thus we deduce that the simplicial $\Pi_{\mathcal{A}}$-algebra $G_\bullet := \pi_{\mathcal{A}} Q_\bullet \cong \pi_0 \mathcal{V}_\bullet$ is a free $\Pi_{\mathcal{A}}$-algebra resolution of $\Lambda = \pi_0 \mathcal{Y}$. Thus the spectral sequence of (9.17) collapses, showing that $RQ_\bullet$ realizes Λ.

Finally, by combining the weak equivalences of Section 8.8 with Lemma 9.4, we deduce that $\|\mathcal{V}_\bullet\|$ (the realization functor applied to each simplicial space $\mathcal{V}_\bullet\{B\}$ for $B \in {}^s\mathcal{C}_{\mathcal{A}}^{\mathrm{St}}$) is a Stover mapping algebra, which is weakly equivalent to $\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(RQ_\bullet)$,as well as to the original mapping algebra $\mathcal{Y}$. □

Since every realizable Stover mapping algebra is realistic, this shows:

Corollary 9.24. *The correspondence of Proposition 9.16 actually factors through the category of realistic $\mathfrak{L}$-algebras, up to weak equivalence.*

10. Realizing mapping algebras

In order to solve the first problem mentioned at the beginning of Section 8, we must reinterpret the inductive approach to realizing a $\Pi_{\mathcal{A}}$-algebra Λ described in Section 7 as an inductive process for realizing mapping algebras. For this we need:

Definition 10.1. A map of $\mathcal{A}$-mapping algebras $\mathfrak{f} : \mathfrak{X} \to \mathcal{Y}$ is called an *n-equivalence* if $P_n\mathfrak{f}$ is a weak equivalence of $\mathcal{A}$-mapping algebras. Similarly, a map $f : X \to Y$ in $\mathcal{C}$ is called an *n–$\mathcal{A}$-equivalence* if $\mathfrak{M}_{\mathcal{A}} f$ is an n-equivalence of $\mathcal{A}$-mapping algebras.

An $\mathfrak{L}$-algebra $\mathfrak{X}$ is called a *n-realistic $\mathfrak{L}$-algebra* if:

(a) $r^{(n)} : \mathfrak{X} \to P_n\mathfrak{X}$ is a weak equivalence of $\mathcal{A}$-mapping algebras.
(b) The map ζ in (8.6) is an $(n-1)$-equivalence in $\mathcal{S}_*$ whenever the first map in (8.9) is a weak equivalence.
(c) The map θ in (8.7) is an n-equivalence whenever the second map in (8.9) is a weak equivalence.

Remark 10.2. Note that we cannot expect to do better than (b) above, since $\Omega P^n K$ is just $P^{n-1}\Omega K$ for any $K \in \mathcal{S}_*$. Thus even under Assumption (9.2), where for a realistic Stover mapping algebra $\mathfrak{X}$, the simplicial set $\mathfrak{X}\{A_0\}$ determines $\mathfrak{X}\{B\}$ for any B in ${}^s\mathcal{C}_{\mathcal{A}}^{\mathrm{St}}$ up to weak equivalence, in the n-realistic case $\mathfrak{X}\{\Sigma A_0\}$ carries more information than $\Omega\mathfrak{X}\{A_0\}$ does.

We can now refine Corollary 9.24 as follows:

Proposition 10.3. *There is a one-to-one correspondence between n-realistic $\mathfrak{L}$-algebras $\mathfrak{X}$ with $\pi_0\mathfrak{X} \cong \Lambda \in \Pi_{\mathcal{A}}$-$\mathcal{A}lg$ and n-th quasi-Postnikov sections for Λ, up to weak equivalence.*

Proof. Let $\mathfrak{X}$ be an n-realistic $\mathfrak{L}$-algebra, so its structure map $\eta = \eta_{\mathfrak{X}}$ factors through $P_n\mathfrak{L}\mathfrak{X} \to \mathfrak{X} \simeq P_n\mathfrak{X} = \mathrm{csk}_{n+1}\mathfrak{X}$. We wish to construct the Stover resolution $\mathcal{V}_\bullet \to \mathfrak{X}$ as in the proof of Proposition 9.23. For all $k \geq 0$, the objects $\mathcal{V}_k := \mathfrak{L}^{k+1}\mathfrak{X}$ depend only on $\rho_0\mathfrak{X}$, which is determined by $P_0\mathfrak{X} = \mathrm{csk}_1\mathfrak{X}$. Similarly, all the degeneracy and face maps, in all simplicial dimensions, are determined by $\mathcal{V}_0 \in {}^s\mathcal{C}_{\mathcal{A}}^{\mathrm{St}}$, except for $d_k : \mathcal{V}_k \to \mathcal{V}_{k-1}$, which is $\mathfrak{L}^k\eta$. By (9.21), this map itself, as an arrow in $\mathcal{C}_{\mathcal{A}}^{\mathrm{St}} \subseteq \mathcal{C}$, depends only on $\mathrm{csk}_k\,\eta : \mathrm{csk}_k\,\mathcal{V}_0 \to \mathrm{csk}_k\,\mathfrak{X}$. Thus η determines the $n+1$-st truncation $\tau_{n+1}\mathcal{V}_\bullet$ of $\mathcal{V}_\bullet$, and thus $P_n\mathcal{V}_\bullet$.

Conversely, if we can construct $\tau_{n+1}\mathcal{V}_\bullet$ for $\mathfrak{X}$, this is equivalent (as in the proof of Proposition 9.23) to constructing $\tau_{n+1}Q_\bullet$ for the (putative) object $Y \in \mathcal{C}$ realizing $\mathfrak{X}$, with $\tau_{n+1}\mathcal{V}_\bullet := \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(\tau_{n+1}Q_\bullet)$. Thus we have an n-th quasi-Postnikov section for Λ, which we denote by $Q\langle n\rangle_\bullet \in s\mathcal{C}$ (see Definition 7.4). Applying the n-th Postnikov section functor $P_n : \tau_{n+1}s\mathcal{S}_* \to s\mathcal{S}_*$ to each $(n+1)$-truncated simplicial set $\tau_{n+1}\mathcal{V}_\bullet\{A\}$ yields the corresponding quasi-Postnikov section $\mathcal{V}\langle n\rangle_\bullet \in s\mathcal{M}_{\mathcal{A}}^{\mathrm{St}}$, with $\mathcal{V}\langle n\rangle_\bullet := \mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(Q\langle n\rangle_\bullet)$. This is because each $Q\langle n\rangle_k$ for $k \geq n+2$ is constructed as a matching object (cf. [35, Section 2.1]), which is a limit, so it commutes with mapping spaces.

In particular, $\pi_{\mathcal{A}} Q\langle n\rangle_k \cong (\pi_0\mathcal{V}\langle n\rangle_k\{A\})_{A\in\mathcal{A}}$ for all $k \geq 0$. Thus from (7.7) we see that

$$\pi_k\pi_0\mathcal{V}\langle n\rangle_\bullet\{A\} \;=\; \pi_k(\pi_{\mathcal{A}} Q\langle n\rangle_\bullet)\{A\} \cong \begin{cases} \Lambda\{A\} & \text{for } k=0 \\ (\Omega^{k+1}\Lambda)\{A\} & \text{for } k=n+2, \\ 0 & \text{otherwise} \end{cases} \tag{10.4}$$

for any $A \in \mathcal{A}$ (and thus, using Lemma 9.4, for any $B \in {}^s\mathcal{C}_{\mathcal{A}}^{\mathrm{St}}$). The Bousfield–Friedlander spectral sequence (9.17) for the bisimplicial set $\mathcal{V}\langle n\rangle_\bullet\{A\}$ converges to $\pi_*\|\mathcal{V}\langle n\rangle_\bullet\|$, because $\pi_0\mathcal{V}\langle n\rangle_\bullet\{A\}$ is a simplicial group. Moreover, the first possible differential is $d^{n+2} : (\Omega^{n+1}\Lambda)\{A\})_0 \to \Lambda\{A\}_{n+1}$, so $\pi_i\|\mathcal{V}\langle n\rangle_\bullet\{A\}\| \cong \Lambda\{A\}_i$ for $i \leq n$. By naturality we deduce that the map of Stover mapping algebras $\|\mathcal{V}\langle n\rangle_\bullet\| \to \mathfrak{X}$ is an n-equivalence, so $P_n\|\mathcal{V}\langle n\rangle_\bullet\| \simeq \mathfrak{X}$.

In summary, each of $\mathcal{V}\langle n\rangle_\bullet \in s\mathcal{M}_{\mathcal{A}}^{\mathrm{St}}$, $Q\langle n\rangle_\bullet \in s\mathcal{C}$, and the n-realistic Stover mapping algebra $\mathfrak{X}$ determines the other two. □

Remark 10.5. Note that from the quasi-Postnikov section $Q\langle n\rangle_\bullet \in s\mathcal{C}$ we can also recover an object $Z\langle n\rangle := R(Q\langle n\rangle_\bullet) \in \mathcal{C}$ (using Definition 7.1), and we see that $[\Sigma^i A_0, Z\langle n\rangle]_{\mathcal{C}} \cong \pi_i\mathfrak{X}\{A_0\} \cong \Lambda\{\Sigma^i A_0\}$ for $0 \leq i \leq n$, by (7.2), since $Q\langle n\rangle_\bullet$ is $\mathcal{A}$-cofibrant and $\mathcal{A}$ is generated by A_0 by (9.2).

However, we can do more than this, by Remark 10.2: the inclusion of the subcollection of spherical objects $\mathcal{A}\langle k\rangle := \{\Sigma^k A_0, \Sigma^{k+1}A_0, \dots\}$ in $\mathcal{A}$ induces a forgetful functor $\mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}\langle k\rangle}$ (which omits the simplicial set $\mathfrak{X}\{A_0^i\}$ $(0 \leq i < k)$ from $\mathfrak{X}$). If we denote this by $\mathfrak{X} \mapsto \mathfrak{X}^{(k)}$, applying the procedure described in the proof of Proposition 10.3 to the $\mathcal{A}$-mapping

algebra $\mathfrak{X}^{(k)}$ (which is still n-realistic) yields a new simplicial object $Q\langle n\rangle_\bullet^{(k)} \in s\mathcal{C}$, and $Z\langle n\rangle^{(k)} := RQ\langle n\rangle_\bullet^{(k)}$ now realizes the $\Pi_{\mathcal{A}}$-algebra $\Omega^k\Lambda$ through degree n. Moreover, there is a natural $(n-1)$–$\mathcal{A}$-equivalence $\Omega Z\langle n\rangle^{(k+1)} \to Z\langle n\rangle^{(k)}$ for each $k \geq 0$, induced by the maps ζ of (8.6).

The collection of objects $\{Z\langle n\rangle^{(k)}\}_{k=0}^{\infty}$, equipped with these structure maps, thus form an *n-stem*, in the sense of [9]. In the case when $\mathcal{C} = \mathcal{T}_*$ and $\mathcal{A} = \{\mathbf{S}^k\}_{k=1}^{\infty}$, these behave like the collection $\{P_{n+k}X\langle k-1\rangle\}_{k=1}^{\infty}$ of $(k-1)$-connected covers of $(n+k)$-Postnikov sections of a (putative) space X.

Lemma 10.6. *Let $\mathfrak{X}$ be an n-realistic $\mathcal{A}$-mapping algebra, and let $Q\langle n\rangle_\bullet$ be the n-quasi-Postnikov section in $s\mathcal{C}$ corresponding to $\mathfrak{X}$ under Proposition 10.3, with $\Lambda := \pi_0\mathfrak{X}$. Then there is a natural isomorphism $\pi_i\mathfrak{X} \cong \pi_i^{\#}(Q\langle n\rangle_\bullet)$ as Λ-modules for all $i \geq 0$.*

Proof. From (10.4), (7.6), and (9.17), we see that

$$\pi_k\mathfrak{X} \cong \pi_k^{\#}(Q\langle n\rangle_\bullet) \cong \begin{cases} \Omega^k\Lambda & \text{for } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

To describe the natural identification, note that by Proposition 10.3 we know that $\mathfrak{X}$ is n-equivalent to $\|\mathfrak{V}\langle n\rangle_\bullet\| = \|\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(Q\langle n\rangle_\bullet)\|$. Since we assumed that each $A \in \mathcal{A}$ was a strict cogroup object in $\mathcal{C}$, $K := \|\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(Q\langle n\rangle_\bullet)\|\{A\}$ has the natural structure of a simplicial group. Therefore, an element in $\pi_k\mathfrak{X}\{A\}$ may be represented by a Moore k-cycle ϕ in

$$Z_kK \subseteq K_k = \mathrm{map}_{\mathcal{C}}(A, Q\langle n\rangle_k)_k = \mathrm{Hom}_{\mathcal{C}}(A \otimes \mathbf{\Delta}[k], Q\langle n\rangle_k) \subseteq \mathrm{Hom}_{s\mathcal{C}}(c(A)\hat{\otimes}\mathbf{\Delta}[k], Q\langle n\rangle_\bullet)$$

(see Section 0.6).

On the other hand, by [35, Proposition 5.8] we can represent an element of $\pi_k^{\#}(Q\langle n\rangle_\bullet)\{A\}$ by an element in $\mathrm{Hom}_{\mathcal{C}}(A, Z_kQ\langle n\rangle_\bullet)$ – that is by a map $f : A \to Q\langle n\rangle_k$ such that $d_if = *$ for all $0 \leq i \leq k$. If $\delta_k \in \mathbf{\Delta}[k]_k$ is the non-degenerate k-simplex of $\mathbf{\Delta}[k]$, we define $\hat{\phi} : A\hat{\otimes}\mathbf{\Delta}[k] \to Q\langle n\rangle_\bullet$ by sending $A \otimes \{\delta_k\}$ to $Q\langle n\rangle_k$ by f, and extend by zero to the other non-degenerate simplices of $\mathbf{\Delta}[k]$. □

Definition 10.7. For any $\mathcal{A}$-mapping algebra $\mathfrak{X}$, the *associated simplicial $\Pi_{\mathcal{A}}$-algebra* $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ is defined by requiring $\Pi_{\mathcal{A}}(\mathfrak{X})_n$ to be the $\Pi_{\mathcal{A}}$-algebra induced by the action of $\pi_0\,{}^s\mathcal{C}'_{\mathcal{A}}$ on each set of n-simplices $\mathfrak{X}\{A\}_n$ of $\mathfrak{X}\{A\} \in \mathcal{S}_*$. Note that $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ is itself an $\mathcal{A}$-mapping algebra, and the quotient map $h : \mathfrak{X} \to \Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ is a map of $\mathcal{A}$-mapping algebras.

For simplicity, let us denote the cofibrant object $Q_\bullet \in s\mathcal{C}$ associated by Proposition 9.23 with a realistic $\mathcal{A}$-mapping algebra $\mathfrak{X}$ by $Q\langle\infty\rangle_\bullet$.

Lemma 10.8. *Assume that $\mathfrak{X}$ is an n-realistic $\mathcal{A}$-mapping algebra, with $0 \leq n \leq \infty$, and $Q\langle n\rangle_\bullet$ is the object associated with $\mathfrak{X}$ by Proposition 9.23 (or Proposition 10.3). There is a natural isomorphism of $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ with the simplicial $\Pi_{\mathcal{A}}$-algebra $\pi_{\mathcal{A}}Q\langle n\rangle_\bullet$, and $h : \mathfrak{X} \to \Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ induces the Hurewicz homorphism $h_{\#} : \pi_*^{\#}(Q\langle n\rangle_\bullet) \to \pi_*\pi_{\mathcal{A}}Q\langle n\rangle_\bullet$ of (1.3).*

Proof. As in the proof of Lemma 10.6, we may replace $\mathfrak{X}$ by the n-equivalent $\mathcal{A}$-mapping algebra $\|\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(Q\langle n\rangle_\bullet)\|$, so that any element in $\mathfrak{X}\{A\}_k$ may be identified with a map $f : A \otimes \mathbf{\Delta}[k] \to Q\langle n\rangle_k$.

Since $Q := Q\langle n\rangle_k \in {}^s\mathcal{C}_{\mathcal{A}}^{\mathrm{St}}$, we may identify this with $f^*(\mathrm{Id}_Q)$, for $f \in (\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}Q)\{A\}$, so by definition of h we have $h(f) = [f]^*h(\mathrm{Id}_Q) = [f] \in [A, Q]_{\mathcal{C}} = (\pi_{\mathcal{A}}Q\langle n\rangle_k)\{A\}$. This identifies $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ with $\pi_{\mathcal{A}}Q\langle n\rangle_\bullet$. From the description of the Hurewicz homomorphism in [35, Section 5], we see that it coincides with $h_{\#}$. □

Proposition 10.9. *If $\mathfrak{X} = \mathfrak{X}\langle n\rangle$ is an n-realistic $\mathcal{A}$-mapping algebra with $\Lambda := \pi_0\mathfrak{X} \in \Pi_{\mathcal{A}}$-$\mathcal{A}lg$, the obstruction to extending $\mathfrak{X}$ to an $(n+1)$-realistic $\mathcal{A}$-mapping algebra $\mathfrak{X}\langle n+1\rangle$ (with $P_n\mathfrak{X}\langle n+1\rangle = \mathfrak{X}\langle n\rangle$) is the $(n+1)$-st k-invariant for $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$, i.e., $\tilde{k}_{n+1} \in H^{n+3}_{\mathrm{AQ}}(\Lambda; \Omega^{n+1}\Lambda)$.*

Proof. Again, let $Q\langle n\rangle_\bullet$ be the n-th quasi-Postnikov section for Λ corresponding to $\mathfrak{X}$ under Proposition 10.3, with $\mathfrak{X} \simeq P_n\|\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(Q\langle n\rangle_\bullet)\|$. By Lemma 10.8 and (7.7) we know that $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ has only two non-zero homotopy groups: $\pi_0\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet \cong \Lambda$ and $\pi_{n+2}\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet \cong \Omega^{n+1}\Lambda$.

If the extension $\mathfrak{X}\langle n+1\rangle$ exists, the fibration $p^{(n+1)} : \mathfrak{X}\langle n+1\rangle \to P_n\mathfrak{X}\langle n+1\rangle \simeq \mathfrak{X}\langle n\rangle$ induces $p_{\#}^{(n+1)} : \Pi_{\mathcal{A}}(\mathfrak{X}\langle n+1\rangle)_\bullet \to \Pi_{\mathcal{A}}(\mathfrak{X}\langle n\rangle)_\bullet$, which is the identity on $\pi_0\mathfrak{X}\langle n+1\rangle = \pi_0\mathfrak{X}\langle n\rangle \cong \Lambda$ (again by Lemma 10.8).

Since $\Pi_{\mathcal{A}}(\mathfrak{X}\langle n\rangle)_\bullet \simeq P_{n+1}\Pi_{\mathcal{A}}(\mathfrak{X}\langle n\rangle)_\bullet$, $p_{\#}^{(n+1)}$ factors via $P_{n+1}\Pi_{\mathcal{A}}(\mathfrak{X}\langle n+1\rangle)_\bullet = B\Lambda$, so the structure map $\tilde{p}^{(n+2)} : \Pi_{\mathcal{A}}(\mathfrak{X})_\bullet \to P_{n+1}\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet = B\Lambda$ has a section s. This is equivalent to the vanishing of the $(n+1)$-st k-invariant $\tilde{k}_{n+1} \in H^{n+3}_{\mathrm{AQ}}(\Lambda; \Omega^{n+1}\Lambda)$ for $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$.

Conversely, if the k-invariant $\tilde{k}_{n+1}$ for $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet \cong \pi_{\mathcal{A}}Q\langle n\rangle_\bullet$ vanishes, then $Q\langle n\rangle_\bullet$ extends to an $(n+1)$-st quasi-Postnikov section $Q\langle n+1\rangle_\bullet$ for Λ, by Theorem 7.5, so we obtain $\mathfrak{X}\langle n+1\rangle := \|\mathfrak{M}_{\mathcal{A}}^{\mathrm{St}}(Q\langle n+1\rangle_\bullet)\|$, as required by Proposition 10.3. □

Remark 10.10. Since the quotient map h of Section 10.7 is surjective, and $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet\{B\}$ has the underlying structure of a simplicial group for each $B \in {}^s\mathcal{C}_{\mathcal{A}}^{\mathrm{St}}$, h is a fibration in $\mathcal{M}_{\mathcal{A}}^{\mathrm{St}} \subseteq (\mathcal{S}_*, \mathcal{O})$-$\mathcal{C}at$. In fact, we may identify the long exact sequence in π_* for the fibration h with the spiral exact sequence (1.3), up to a re-indexing.

If we denote the fiber of h by $B'\mathfrak{X}$, we deduce from (7.6) and (7.7) that

$$\pi_i(\mathcal{B}'\mathfrak{X}) \cong \begin{cases} \Omega^i\Lambda & \text{for } 1 \leq i \leq n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Looping back the fibration sequence for h, for each $A \in \mathcal{A}$ we obtain

$$\mathfrak{X}\{\Sigma A\} = P_n\mathfrak{X}\{\Sigma A\} \xrightarrow{p^{(n)}} P_{n-1}\mathfrak{X}\{\Sigma A\} \xrightarrow{\zeta'} \Omega\mathfrak{X}\{A\} = P_{n-1}\Omega\mathfrak{X}\{A\} \xrightarrow{k'} E(\Omega^{n+1}\Lambda, n+1),$$

where ζ' is the weak equivalence of Section 10.1(b), and k' is the (looped) $(n-1)$-th k-invariant for $\mathfrak{X}\{\Sigma A\}$.

We can think of ζ' as the structure map for the n-stem $P_n\mathfrak{X}$, which is classified by $h : \mathfrak{X} \to \Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$. If we could produce a map $q : \Pi_{\mathcal{A}}(\mathfrak{X})_\bullet \to E(\Omega^{n+1}\Lambda, n+2)$ which is a π_{n+2}-isomorphism, then $q \circ h$ would be the n-th k-invariant for $\mathfrak{X} = \mathfrak{X}\langle n\rangle$, which would define an $(n+1)$-realistic $\mathcal{A}$-mapping algebra $\mathfrak{X}\langle n+1\rangle$, and thus an $(n+1)$-quasi-Postnikov section for Λ.

Now the inclusion of the homotopy fiber of q is a map $s : B\Lambda \to \Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$ which is a section for $p = p^{(n+2)} : \Pi_{\mathcal{A}}(\mathfrak{X})_\bullet \to P_{n+1}\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet = B\Lambda$. Moreover, q is then P_{n+2} applied to the pinch map to of the cofiber of s, so the existence of q is equivalent to the existence of a section s for p. Both are equivalent as above to the vanishing of the $(n+1)$-st k-invariant for $\Pi_{\mathcal{A}}(\mathfrak{X})_\bullet$.

Thus we can interpret this k-invariant, in the context of stems, as the obstruction to gluing the n-windows of an n-stem to produce an $(n+1)$-stem.

We can thus summarize the results of this section in the following:

Theorem 10.11. *Let $\mathcal{C}$ be a strict E^2-model category, with spherical objects $\mathcal{A}$ satisfying (9.2), and let Λ be a $\Pi_{\mathcal{A}}$-algebra. Proposition 10.9 then provides an inductively defined sequence of $(\mathcal{S}_*, \mathcal{O})$-cohomology classes $\tilde{k}_n \in H^{n+2}_{SO}(B\Lambda; \Omega^n\Lambda)$ $(n = 1, 2, \dots)$ for producing a realistic Stover mapping algebra realizing Λ — which is equivalent to realizing Λ in $\mathcal{C}$.* □

Remark 10.12. We now interpret the classes $\tilde{k}_n$ in the context of the Dwyer–Kan–Smith theory of Section 5. The homotopy commutative diagram which we are trying to rectify will be indexed by the Λ-pair $(\mathcal{D}, \mathcal{D}^+) := ((\mathcal{C}^{\mathrm{St}}_{\mathcal{A}})^+, \mathcal{C}^{\mathrm{St}}_{\mathcal{A}})$, defined as in Example 8.15. As in Section 5.7, identifying the (ordinary) category $\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$ as the zero-simplices of the (simplicially enriched) ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$ and applying degeneracies gives the required simplicial map $\tilde{X}_0 : c(\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}) \to {}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}}$.

However, the obstruction theory of Section 5 does not quite apply in our situation, since to begin with we do not have given a simplicially enriched category ${}^s\mathcal{C}_X$ (Section 5.4) extending ${}^s\mathcal{C}^{\mathrm{St}}_{\mathcal{A}} - sc.$ a (we hope realistic) Stover mapping algebra $\mathfrak{X}$. Instead, we construct $\mathfrak{X}$ by induction on its quasi-Postnikov tower $(\mathfrak{X}\langle n\rangle)_{n=0}^{\infty}$ of n-realistic Stover mapping algebras. At the beginning of the process we can always choose a 0-realistic Stover mapping algebra $\mathfrak{X}\langle 0\rangle$ realizing Λ, as well as an extension $X_0 : P_0(\mathcal{C}^{\mathrm{St}}_{\mathcal{A}})^+ \to \mathfrak{X}\langle 0\rangle$ for the given $\tilde{X}_0$.

In view of Proposition 10.9, we do not actually need to lift X_0 to the successive n-realistic Stover mapping algebras $\mathfrak{X}\langle n\rangle$, but only to their $\Pi_{\mathcal{A}}$-algebra versions $\Pi_{\mathcal{A}}(\mathfrak{X}\langle n\rangle)_\bullet$ These are Stover mapping algebras, though they are not n-realistic. Moreover, the Dwyer–Kan–Smith obstructions of Section 5.7 reduce in this case to the k-invariants $\tilde{k}_n$, as in the proof of Proposition 10.9.

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