Standard Homological Properties for Quantum $GL_n$

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For some time, we have been studying Schur algebras and related algebras by a technique of descent from algebraic groups [9–13]. In the case of a Schur algebra, one can then, by a further technique of descent (described in [17, Chap. 6]), go on to study the group algebra of the corresponding symmetric group. The main inputs from algebraic group theory are the various homological properties of the induction functor $\text{Ind}_G^H$ (where $G$ is a reductive group with Borel subgroup $B$), particularly Kempf’s vanishing theorem. It is often difficult or awkward to realize the consequences of these properties directly within the context of finite dimensional algebras. We now wish to study the $q$-Schur algebra, introduced by Dipper and James [6], and Hecke algebras of type $A$, in a similar manner by descent from quantum $GL_n$. By way of preparation for this endeavour we here establish the standard homological properties of the induction functor in the quantum setting. We shall work throughout with the quantum version of $GL_n$ introduced by Dipper and Donkin [5]. For the Manin quantization, Parshall and Wang [26] have given proofs of these properties (as well as some applications to the $q$-Schur algebra) valid under the restriction that $q$ is an odd root of unity, by viewing quantum $GL_n$ as a covering of “classical” $GL_n$. The results we give here are valid for all $q \neq 0$ and (except for Proposition 3.10) are obtained without invoking the corresponding results in the classical case. In particular, our proof of Kempf’s vanishing theorem (valid also in the classical case $q = 1$) does not rely on properties of the cohomology of projective varieties; instead, it is based on the Koszul resolution defined by a maximal parabolic...

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subgroup. Another feature of the proofs (in keeping with the programme referred to above) is that no use is made of Hecke algebras.

In Section 1, we establish notation and summarize the general homological properties we shall need. In Section 2, we discuss the representation theory of various subgroups of quantum $GL_n$. Section 3 is the kernel of the paper, and we here prove Kempf vanishing and various other standard homological properties of the induction functor. Further representation theoretic properties are discussed in Section 4.

The results proved here are employed in the companion paper [15], whose contents we now describe. We obtain a basis of bideterminants of the coordinate algebra of the quantum monoid and of the modules for the quantum group induced from one dimensional modules for the Borel (quantum) subgroup (given by Carter and Lusztig [1] in the classical case $q = 1$). We determine a formula for the characters of the irreducible representations of the quantum general linear monoid at $q = 0$. We show that the “Ringel dual” of a $q$-Schur algebra is again a $q$-Schur algebra and give applications to decomposition numbers of $q$-Schur algebras and Hecke algebras (in type $A$).

1. PRELIMINARIES

We begin by describing some terminology and notation and some of the homological algebra arising from the induction functor for quantum groups. We fix a field $k$. We adopt the philosophy of Parshall and Wang [26] in regarding the category of quantum groups as the dual of the category of $k$-Hopf algebras and identifying the category of modules for a quantum group with the category of comodules for the corresponding Hopf algebra. We now explain this philosophy in detail.

We shall use the expression “let $G$ be a quantum $k$-monoid” (or “let $G$ be a quantum monoid over $k$,” or simply “let $G$ be a quantum monoid”) to indicate that we have in mind a bialgebra over $k$, called the coordinate algebra of $G$ and denoted $k[G]$. Similarly, we use the expression “let $G$ be a quantum $k$-group” (or “let $G$ be a quantum group over $k$,” or simply “let $G$ be a quantum group”) to indicate that we have in mind a Hopf algebra over $k$, called the coordinate algebra of $G$ and denoted $k[G]$. If $G$ is a quantum monoid we denote by $\delta_G: k[G] \to k[G] \otimes k[G]$ and $\epsilon_G: k[G] \to k$ comultiplication and augmentation maps of the bialgebra $k[G]$. If $G$ is a quantum group, we denote by $\sigma_G$ the antipode of the Hopf algebra $k[G]$. Let $G$ and $H$ be quantum monoids (resp. quantum groups). We shall use the expression “$\phi: G \to H$ is a morphism of quantum monoids” (resp. “$\phi$ is a morphism of quantum groups”) to indicate that we have in mind a bialgebra map (resp. Hopf algebra map) $\phi: k[H] \to k[G]$, called the
comorphism of $\phi$. We shall use the expression "$H$ is a quantum sub-
monoid of $G$," or simply "$H$ is a submonoid of $G$," to indicate that we
have in mind a bideal $I_H$ of $k[G]$ and that $k[H] = k[G]/I_H$. Similarly we
use the expression "$H$ is a quantum subgroup of $G$," or simply "$H$ is a
subgroup of $G$," to indicate that we have in mind a Hopf ideal $I_H$ of $k[G]$ 
and that $k[H] = k[G]/I_H$. Thus if $H$ is a submonoid (resp. subgroup) of $G$
we have the morphism $\phi: H \to G$ whose comorphism $\phi^*: k[G] \to k[H]$
is the natural map. We call $\phi$ the inclusion map and call $\phi^*$ the restriction
map. If $H_1, H_2$ are subgroups of the quantum monoid (resp. quantum
subgroup $G$) then $H_1 \cap H_2$ denotes the quantum monoid (resp. subgroup)
defined by $I_{H_1 \cap H_2} = I_{H_1} + I_{H_2}$. For quantum monoids (resp. quantum
groups) $G_1, G_2$ we denote by $G_1 \times G_2$ the quantum monoid (resp. quantum
group) defined by $k[G_1 \times G_2] = k[G_1] \otimes k[G_2]$. If $H_1 \leq G_1$ and $H_2 \leq G_2$ are quantum submonoids (resp. quantum subgroups), we identify
$H_1 \times H_2$ with a subgroup of $G_1 \times G_2$ in the obvious way.

We shall say that a quantum monoid or group $G$ is finite if its
coordinate algebra $k[G]$ is a finite dimensional $k$-space.

If $X$ is a set we write $\text{id}_X$ (or simply $\text{id}$) for the identity map $X \to X$. We
fix a field $k$. For a $k$-coalgebra $A$, we denote by $\text{Comod}(A)$ the category of
right $A$-comodules. We write $V \in \text{Comod}(A)$ to indicate that $V$ is a right
$A$-comodule. We shall often write $\tau_V$ for the structure map $V \to V \otimes A$, for $V \in \text{Comod}(A)$. We write $\text{comod}(A)$ for the full subcategory of
$\text{Comod}(A)$ whose objects are the finite dimensional right $A$-comodules.
For $X \in \text{Comod}(A)$ and a vector space $V$, we write $[V] \otimes X$ for the vector
space $V \otimes X$ regarded as a right $A$-comodule via the structure map
$\tau_{[V] \otimes X} = \text{id} \otimes \tau_X: V \otimes X \to V \otimes X \otimes A$. Let $(A, \delta, \epsilon)$ and $(A', \delta', \epsilon')$ be
$k$-coalgebras and let $\phi: A \to A'$ be a morphism of coalgebras. We have functors $\phi_0: \text{Comod}(A) \to \text{Comod}(A')$ and $\phi^0: \text{comod}(A') \to \text{comod}(A)$,
described in [7].

For $V \in \text{Comod}(A)$ with structure map $\tau: V \to V \otimes A$, $\phi_0(V)$ is the
$k$-space $V$ regarded as an $A'$-comodule via the structure map $(\text{id} \otimes \phi)\tau: A \to V \otimes A'$. If $\alpha: V \to W$ is a morphism of $A$-comodules, $\phi_0(\alpha): \phi_0(V) \to \phi_0(W)$ is the $k$-map $\alpha$ regarded as a morphism of $A'$-comodules.

For $W \in \text{Comod}(A)$, $\phi^0(W)$ is the $A$-subcomodule of $[W] \otimes A$ consisting of the elements $f$ such that $(\tau \otimes \text{id})(f) = (\text{id} \otimes (\phi \otimes \text{id})\delta)(f)$. The map $(\text{id} \otimes \epsilon)$ restricts to an $A$-comodule morphism $\eta: \phi_0(\phi^0(W)) \to W$ and for any $V \in \text{Comod}(A)$ and $A'$-comodule homomorphism $\alpha: \phi_0(V)$
$\to W$ there is a unique $A$-comodule map $\tilde{\alpha}: V \to \phi^0(W)$ such that
$\eta \circ \tilde{\alpha} = \alpha$. In this way, we get a $k$-space isomorphism $\text{Hom}_A(\phi_0(V), W)$
$\to \text{Hom}(V, \phi^0(W))$ (Frobenius reciprocity).

Let $G$ be a quantum monoid over $k$. By a left (resp. right) $G$-module, we
mean a right (resp. left) $k[G]$-comodule. The coordinate algebra $k[G]$ 
viewed as a $G$-module via $\delta_G: k[G] \to k[G] \otimes k[G]$ is called the regular
$G$-module. We set $\text{Mod}(G) = \text{Comod}(k[G])$ (resp. $\text{mod}(G) = \text{comod}(k[G])$
and write \( V \in \text{Mod}(G) \) (resp. \( V \in \text{mod}(G) \)) to indicate that \( V \) is a (left) \( G \)-module (resp. finite dimensional (left) \( G \)-module). If \( G \) is a quantum group and \( V \in \text{mod}(G) \) we denote by \( V^* \) the dual module. If \( V \) has basis \( v_1, \ldots, v_n \) and \( \alpha_1, \ldots, \alpha_n \) is the dual basis of \( V^* = \text{Hom}_k(V, k) \) and \( \tau_\alpha(v_i) = \sum v_j \otimes f_{ij} \) then \( \tau_{\alpha^*}(\alpha_i) = \sum \alpha_j \otimes \alpha^*_j(f_{ij}), \) for \( 1 \leq i \leq n. \) The regular \( G \)-module is injective and every \( G \)-module embeds in a direct sum of copies of the regular module. The category \( \text{Mod}(G) \) has enough injectives.

For \( V_1, V_2 \in \text{Mod}(G) \) and \( i \geq 0, \) we denote by \( \text{Ext}^i_G(V_1, V_2) \) the \( i \)th derived functor of \( \text{Hom}_G(V_1, -) \) evaluated at \( V_2. \) For \( V \in \text{Mod}(G), \) we write \( H^i(G, V) \) for \( \text{Ext}^i_G(k, V), \) for \( i \geq 0, \) where \( k \) denotes the trivial one dimensional module.

Let \( \phi: G \to H \) be a morphism of quantum monoids. We set \( \phi^* = (\hat{\phi})_G: \text{Mod}(H) \to \text{Mod}(G) \) and \( \phi_* = (\hat{\phi})^G_H: \text{Mod}(H) \to \text{Mod}(G). \) Then \( \phi^* \) is exact, \( \phi_* \) is left exact and \( \phi_* \) takes injective modules to injective modules. These properties follow from the corresponding properties for \( \phi \) and \( \phi^G_H. \) We shall say that \( \phi^* \) (resp. \( \phi_* \)) is \( \phi \)-restriction (resp. \( \phi \)-induction) or simply restriction (resp. induction) if \( \phi \) is inclusion and write \( \text{Res}^G_H \) (resp. \( \text{Ind}^G_H \)) for \( \phi^* \) (resp. \( \phi_* \)). For \( H \) a subgroup of \( G \) and \( V \in \text{Mod}(G) \) we shall often write \( V|_H, \) or simply \( V, \) for \( \text{Res}^G_H V \) and say we are regarding \( V \) as an \( H \)-module. Various other notations will be used from time to time. For a morphism of quantum monoids \( \phi: G \to H \) and \( V \in \text{Mod}(H), \) we will sometimes write \( V^\phi \) for \( \phi^* V. \) If \( \phi: k[H] \to k[G] \) is injective, we will say that \( \phi \)-restriction is inflation and write \( \phi^* V \) as \( \text{Inf}^G_H V, \) for \( V \in \text{Mod}(H). \)

**Proposition 1.1.** Let \( \phi: G \to G' \) be a morphism of quantum groups. For \( V \in \text{Mod}(G) \) and \( U \in \text{Mod}(G') \) we have a Grothendieck spectral sequence with \( E_2 \)-term \( \text{Ext}^{i-1}_G(U, \text{Res}^G_{G'} V), \) converging to \( \text{Ext}^{i}_{G'}(\phi^* U, V). \) In particular, if \( H \) is a subgroup of a quantum group \( G \) and \( V \in \text{Mod}(H) \) then we have a Grothendieck spectral sequence, with \( E_2 \)-term \( H^2(G, \text{Res}^G_H V), \) converging to \( H^* \).

**Proof.** Frobenius reciprocity gives an isomorphism of functors \( \text{Hom}_{G'}(U, -) \circ \phi_* \to \text{Hom}_G(\phi^* U, -). \) The functors are left exact and \( \phi_* \) takes injectives to injectives (hence \( \text{Hom}_{G'}(U, -) \)-acyclics). Hence, we get a Grothendieck spectral sequence as described.

Note that, for morphisms \( \phi_1: G_1 \to G_2, \psi: G_2 \to G_3 \) of quantum monoids we have \( (\psi \circ \phi)_* \equiv \psi_* \circ \phi_* , \) by [7, Sect. 3, (d), (ii)], and since \( \phi_* \) takes injective objects to injective (hence \( \psi_* \)-acyclic) objects we get the following.

**Proposition 1.2.** For \( V \in \text{Mod}(G_1) \), we have a Grothendieck spectral sequence, with \( E_2 \)-term \( (R\psi_\eta) \circ (R\phi_\eta) V, \) converging to \( R^*(\psi \circ \phi)_* V. \) In particular, if \( H \leq J \leq G \) are subgroups and \( V \in \text{Mod}(H), \) we have a
Grothendieck spectral sequence, with $E_2$-term $R^1\text{ind}_I^G \ast R^1\text{ind}_I^G V$, converging to $R^1\text{ind}_I^G V$.

We have the generalized tensor identity (see also Parshall and Wang [26, (2.7.1)]).

**Proposition 1.3.** (i) Let $G$ be a quantum group and $X, I \in \text{Mod}(G)$ with $I$ injective. Then $X \otimes k[G] \cong |X| \otimes k[G]$ and $X \otimes I$ is injective. Suppose that $\sigma_G$ is bijective. Then we have $k[G] \otimes X \cong k[G] \otimes |X|$ and $I \otimes X$ is injective.

(ii) Let $\phi: G_1 \to G_2$ be a morphism of quantum groups and let $V \in \text{Mod}(G_1), W \in \text{Mod}(G_2)$. We have $R\phi_\ast^\ast(W \otimes V) \cong V \otimes R\phi_\ast W$, for all $i \geq 0$. In particular, if we have a subgroup $H$ of a quantum group $G$ and $V \in \text{Mod}(G), W \in \text{Mod}(H)$ then $R^1\text{ind}_I^G(V \otimes W) \cong V \otimes R^1\text{ind}_I^G W$, for $i \geq 0$.

Suppose the antipode $\sigma_G$ is bijective. Then we have $R\phi_\ast^\ast(W \otimes \phi^\ast V) \cong R\phi_\ast W \otimes V$, for all $i \geq 0$. In particular, if we have a subgroup $H$ of a quantum group $G$ and $V \in \text{Mod}(G), W \in \text{Mod}(H)$ then $R^1\text{ind}_I^G(W \otimes V) \cong R^1\text{ind}_I^G W \otimes V$, for $i \geq 0$.

(iii) Let $H$ be a subgroup of a quantum group $G$ and suppose that $\sigma_G$ is bijective. For $U \in \text{Mod}(H)$ and $i \geq 0$ we have a $k$-space isomorphism $R^1\text{ind}_I^G U \to H^\ast(H, U \otimes k[G])$.

**Proof.** Let $\phi, V, W$ be as in (ii). The $k$-linear map $V \otimes W \otimes k[G_1] \to V \otimes W \otimes k[G_2]$ taking $v \otimes w \otimes a$ to $\sum v_i \otimes w \otimes f_i a$ (where $v \in V, w \in W, a \in k[G]$ and $\tau_i(v) = \sum v_i \otimes f_i$) restricts to a $G_2$-module isomorphism $V \otimes \phi_\ast W \to \phi_\ast(\phi^\ast V \otimes W)$, by the proof of [7, Prop. 3(h)]. Thus we have the first part of (ii) in degree 0. We therefore have, in (i), $X \otimes k[G] = X \otimes \text{Ind}_I^G G \cong \text{Ind}_I^G (X \otimes k[G])$, as required. It follows that $X \otimes I$ is injective, using the criterion that a $G$-module is injective if and only if it is a direct summand of a direct sum of copies of the regular module $k[G]$.

If $\sigma_G$ is bijective with inverse $\pi$, say, then the $k$-linear map $\alpha: W \otimes k[G_2] \otimes V \to W \otimes V \otimes k[G_2]$ defined by $\alpha(w \otimes a \otimes v) = \sum w \otimes v_i \otimes af_i$ is a linear isomorphism with inverse $\beta: W \otimes V \otimes k[G_2] \to W \otimes k[G_2] \otimes V$ given by $\beta(w \otimes v \otimes a) = \sum w \otimes a \pi(f_i) \otimes v_i$ (for $w \in W, v \in V, a \in k[G]$ with $\tau_i(v) = \sum v_i \otimes f_i$). It is easy to check that $\alpha$ restricts to a $G_2$-module isomorphism $\phi_\ast(W) \otimes V \to \phi_\ast(W \otimes \phi^\ast V)$, giving the second part of (ii) in degree 0. One obtains the second part of (i) as above.

Now let $0 \to W \to I_0 \to I_1 \to \cdots$ be an injective resolution of $W$. Then $0 \to \phi^\ast V \otimes W \to \phi^\ast V \otimes I_0 \to \phi^\ast V \otimes I_1 \to \cdots$ is an injective resolution of $\phi^\ast V \otimes W$. Applying $\phi_\ast$ and using [7, Prop. 3(h)], we obtain a commu-
tative diagram

\[ 0 \to \phi_s(\phi^*V \otimes W) \to \phi_s(\phi^*V \otimes I_0) \to \phi_s(\phi^*V \otimes I_1) \to \cdots \]

\[ \downarrow \]

\[ 0 \to V \otimes \phi_s W \to V \otimes \phi_s I_0 \to V \otimes \phi_s I_1 \to \cdots \]

where the vertical maps are isomorphisms. Hence we have \( R^i\phi_s(\phi^*V \otimes W) = V \otimes R\phi_s W, \) for all \( i \geq 0, \) proving the first part of (i) in arbitrary degree. The proof the second part is similar.

We now prove (iii). By Proposition 1.1, we have a spectral sequence with \( E_2 \)-term \( H^*(H, U \otimes k[G]) \) converging to \( H^*(H, U \otimes k[G]). \)

We have \( R^i\Ind_{G}^{H}(U \otimes k[G]) \equiv R^i\Ind_{G}^{H}(U \otimes k[G]) \) by (i) so that \( H^*(H, R^i\Ind_{G}^{H}(U \otimes k[G])) = 0 \) for \( i > 0 \) by (i). Thus the spectral sequence degenerates and we have \( H^*(H, U \otimes k[G]) \equiv H^0(H, R^0\Ind_{G}^{H}(U \otimes k[G])) \) for all \( i \geq 0. \)

Let \( R^i\Ind_{G}^{H}(U \otimes k[G]) \equiv R^i\Ind_{G}^{H}(U \otimes k[G]) \) by (i), and we get \( H^0(H, R^i\Ind_{G}^{H}(U \otimes k[G])) \equiv R^0\Ind_{G}^{H}(U \otimes k[G]) \), giving \( H^*(H, U \otimes k[G]) \equiv R^0\Ind_{G}^{H}(U), \) as required.

**Corollary 1.4.** Let \( \phi: G_1 \to G_2 \) be a morphism of quantum groups and let \( W \in \text{Mod}(G_1) \). If there exists \( 0 \neq V \in \text{Mod}(G_2) \) such that \( \phi^*(V) \otimes W \) is injective then \( R^i\phi_s W = 0 \) for all \( i > 0. \) In particular, if \( H \) is a subgroup of a quantum group \( G, \) \( W \in \text{Mod}(H) \) and there exists \( 0 \neq V \in \text{Mod}(G) \) such that \( V \mid_H \otimes W \) is injective then \( R^i\Ind_{G}^{H}(W) = 0, \) for all \( i > 0. \)

We say that a quantum group \( \overline{G} \) is a factor group of a quantum group \( G \) if \( k[\overline{G}] \) is a subHopf algebra of \( k[G]. \) If \( \overline{G} \) is a factor group of \( G \) and \( k[\overline{G}] \) is central in \( k[G] \) we get a subgroup \( G_1 \) of \( G \) with defining ideal \( I_{\overline{G}_1} = \langle \text{Ker}(\epsilon_{\overline{G}}) \cap k[\overline{G}] \rangle \). We shall use the following two results of Parshall and Wang [26, (2.10.2), (2.10.11)].

**Proposition 1.5.** Let \( \overline{G} \) be a factor group of the quantum group \( G \) such that \( k[G] \) is central in \( k[\overline{G}]. \) Let \( \pi: G \to \overline{G} \) be the quotient map and let \( G_1 \) be the corresponding subgroup of \( G. \) Suppose further that \( k[G] \) is a faithfully flat \( k[\overline{G}] \)-module.

(i) If \( V \in \text{Mod}(G) \) then \( V^G_{G_1} \) is a \( G \)-submodule of \( V \) and we have \( V_{G_1}^G = \pi^*W \) for some \( W \in \text{Mod}(\overline{G}). \)

(ii) \( k[G]^G = k[\overline{G}]. \)

(iii) \( I_{G_1}^G \) is injective for any injective \( G \)-module \( I. \)

Note that, in the situation of Proposition 1.5, the statement \( V_{G_1}^G = \pi^*W \) for some \( W \in \text{Mod}(\overline{G}) \) amounts to the fact that \( \tau_{G_1}(V_{G_1}^G) \leq V_{G_1}^G \otimes k[\overline{G}]. \)

Thus \( V_{G_1}^G \) is naturally a \( \overline{G} \)-module and we obtain a left exact functor \( F: \text{Mod}(G) \to \text{Mod}(\overline{G}), \) defined by \( F(V) = V_{G_1}^G, \) for \( V \in \text{Mod}(G), \) (and where \( F(\alpha) \) is given by restriction, for a morphism of \( G \)-modules \( \alpha: V \to V' \)). By
Proposition 1.5(iii), an injective resolution of $G$-modules is also an injective resolution of $G_1$-modules so we get a $k$-space isomorphism $RF^iV \cong H^i(G_1, V)$, for $V \in \text{Mod}(G)$ and $i \geq 0$. We shall simply write $H^i(G_1, V)$ for the $i$th derived functor of $F$ evaluated at $V \in \text{Mod}(G)$. For $V \in \text{Mod}(G)$ we have $V^G = (V^{G_1})^G$ and, by Proposition 1.5(ii), the functor $-^{G_1}$ takes injective $G$-modules to injective $G$-modules. Thus we get part (i) of the following: the Lyndon–Hochschild–Serre spectral sequence.

**Proposition 1.6.** Assume the hypotheses of Proposition 1.5.

(i) We have a Grothendieck spectral sequence with $E_2$ term $H^i(G_1, H^j(G_1, V))$ converging to $H^{i+j}(G, V)$.

(ii) If $W$ is a $G$-module which is trivial as a $G_1$-module (i.e., $W^{G_1} = W$) then $H^i(G_2, V \otimes W) \cong H^i(G_1, V) \otimes W$, as $\overline{G}$-modules, for all $i \geq 0$.

(iii) Suppose $G = G_1 \times G_2$, for quantum groups $G_1, G_2$. If $V$ is trivial as a $G_1$-module then $H^i(G_2, V)$ is trivial as a $G_2$-module for all $i \geq 0$.

For (ii), one checks directly that $H^0(G_1, V \otimes W) = H^0(G_1, V) \otimes W$ and obtains the result in general by dimension shifting. (Note that for a $G$-module $I$, which is injective as a $G_1$-module, $I \otimes W$ is a direct sum of copies of $I$ and hence injective as a $G_1$-module.) For (iii), again the result is clear in degree 0 and follows in general by dimension shifting.

2. STANDARD SUBGROUPS OF QUANTUM $\text{GL}_n$

We are concerned here with the quantum general linear group introduced in [5] and its subgroups. For positive integers $a, b$ with $a \leq b$, we denote by $[a, b]$ the set of all integers $r$ such that $a \leq r \leq b$. Let $n$ be a positive integer. Let $R$ be a commutative ring and $q \in R$. Let $A_{R, q}(n)$ be the $R$-algebra generated by $c_{ij}$, $1 \leq i, j \leq n$ subject to the relations:

\begin{align*}
&\text{AI} \quad c_{is}c_{ir} = c_{ir}c_{is} \quad \text{for all } 1 \leq i, r, s \leq n; \\
&\text{AII} \quad c_{jr}c_{is} = qc_{ir}c_{jr} \quad \text{for all } 1 \leq i < j \leq n, \quad 1 \leq r \leq s \leq n; \\
&\text{AIII} \quad c_{js}c_{ir} = c_{ir}c_{js} + (q - 1)c_{is}c_{jr} \quad \text{for all } 1 \leq i < j \leq n, \quad r < s .
\end{align*}

Note that AIII, in the presence of AI, AII is equivalent to

\begin{align*}
&\text{AIII}' \quad c_{is}c_{js} + c_{jr}c_{is} = c_{js}c_{ir} + c_{is}c_{jr} \\
&\quad \text{for all } 1 \leq i \leq j \leq n, \quad 1 \leq r < s \leq n.
\end{align*}
In what follows we work over a field \( k \) and put \( A = A(n) = A_{k,q}(n) \). There are algebra maps \( \delta : A \to A \otimes A \), \( \epsilon : A \to k \) given on the generators by \( \delta(c_{ij}) = \sum_{r=1}^{\infty} c_{ir} \otimes c_{rj} \), \( \epsilon(c_{ij}) = \delta_{ij} \) (Kronecker delta), for \( 1 \leq i, j \leq n \) and \( (A, \delta, \epsilon) \) is a bialgebra. We denote by \( M(n) \), or simply \( M \), the quantum monoid with \( k[M(n)] = A_q(n) \).

The quantum determinant \( d \) is defined by the equation

\[
d = \sum_{\pi} \text{sgn}(\pi)c_{1,1}^{\pi_1}c_{2,2}^{\pi_2} \cdots c_{n,n}^{\pi_n}
\]

where \( \pi \) runs over permutations of \([1, n]\) and \( \text{sgn}(\pi) \) denotes the sign of \( \pi \). We assume from now on that \( q \) is not zero. The set \( \{1, d, d^2, \ldots\} \) is an Ore set and bialgebra structure on \( A_q(n) \) extends to a Hopf algebra structure on the localized algebra \( A_q(n)_d \). We denote by \( G \) the quantum group with \( k[G] = A_q(n)_d \).

The above relations are homogeneous so that the coordinate algebra of \( M \) has a grading \( k[M] = \bigoplus_{r=0}^{\infty} A(n, r) \), where \( A(n, 1) \) is the \( k \)-span of the elements \( c_{ij} \), \( 1 \leq i, j \leq n \). For \( a = (a_{11}, a_{12}, \ldots, a_{nn}) \in \mathbb{N}_0^n \) we set \( c^a = c_{11}^{a_{11}}c_{12}^{a_{12}} \cdots c_{nn}^{a_{nn}} \). The elements \( c^a, a \in \mathbb{N}_0^n \), form a \( k \)-basis of \( k[M] \); see [5, 11.8 Theorem and 4.3.2 Lemma]. Thus the dimension of \( A(n, r) \) is \((n^2 + r - 1)\). We let \( V \) be the subspace \( A(n, 1) + kd^{-1} \) of \( k[G] \). Putting \( V^m = V \cdots V \) \((m\text{-times})\) we have \( V^m = \sum_{r=0}^{m+r} A(n, r)d^{-r} \), since \( d \) normalizes \( A(n, 1) \); see [5, 4.1.9]. Now \( V \) generates the algebra \( k[G] \) which therefore has polynomial growth. Hence any subgroup of \( G \) has polynomial growth and we get the following from [14, Theorem].

**Proposition 2.1.** If \( J \) is a subgroup of \( G \) and \( H \) is a finite subgroup of \( J \) then \( k[J]_H \) is isomorphic to a direct sum of copies of \( k[H] \).

**Remark.** A similar argument applies to the Manin quantization of \( \text{GL}_n \) (and indeed to the two parameter quantization due to Takeuchi [29]) so subgroups of these groups also have the property that the restriction of the regular module to a finite subgroup \( H \) is a direct sum of copies of \( k[H] \).

**Remark 2.2.** If \( H \) is a subgroup of \( G \) then \( \sigma_H^2 \) is bijective, since \( \sigma_H^2 \) is conjugation by the unit \( d_{11}^H \); see [5, 4.3.15 Theorem].

We obtain submonoids of \( M \) and subgroups \( G \) by setting certain of the coordinate functions \( c_{ij} \) equal to 0.

**Definition.** A subset \( \Psi \) of \([1, n]^2\) is called **costandard** if the following conditions are satisfied.

(i) \( \Psi \) contains no diagonal element \((i, i), 1 \leq i \leq n\).

(ii) Whenever \((i, r) \in \Psi \) and \( k \in [1, n] \) we have \((i, k) \in \Psi \) or \((k, r) \in \Psi \).
(iii) Whenever \((i, r), (j, s) \in [1, n]^2\) with \(i < j\) and \(r < s\) and \((i, r)\) and \((j, r)\) belong to \(\Psi\) then \((i, s)\) or \((j, r)\) belongs to \(\Psi\).

We call \(\Psi \subseteq [1, n]^2\) standard if the complement of \(\Psi\) in \([1, n]^2\) is costandard.

**Remarks.** (1) Clearly if \(\Psi_1\) and \(\Psi_2\) are costandard subsets of \([1, n]^2\) then \(\Psi_1 \cup \Psi_2\) is costandard.

(2) Let \(\Psi\) be a costandard subset of \([1, n]^2\) and define \(\overline{\Psi} = \{(r, i) : (i, r) \in \Psi\}\). It is easy to check that \(\overline{\Psi}\) is also costandard.

(3) Suppose \(1 < a < n\) and let \(\Psi = \{(i, r) : i \leq a\) and \(r > a\). It is easy to check that \(\Psi\) is costandard and hence also \(\overline{\Psi} = \{(i, r) : i > a\) and \(r \leq a\) is costandard.

(4) Let \(\lambda\) be a composition of \(n\), i.e., a sequence of non-negative integers \((a_1, a_2, \ldots, a_m)\) with \(a_1 + \cdots + a_m = n\). Let \(\Psi\) be the set of elements \((i, r)\) such that, for some \(h\) we have \(i \leq a_1 + \cdots + a_h\) and \(r > a_1 + \cdots + a_h\). Then \(\Psi\) is a union of sets of the form described in (3) and hence costandard.

**Proposition 2.3.** Let \(\Psi\) be a costandard subset of \([1, n]^2\) and let \(I(\Psi)\) be the \(k\)-span of the monomials \(c^a\) with \(a = (a_1, a_2, \ldots, a_m)\) and \(a_i \neq 0\) for some \((i, r) \in \Psi\). Then \(I(\Psi)\) is a biideal of \(A\).

**Proof.** For \(a = (a_1, a_2, \ldots, a_m) \in \mathbb{N}_0^2\) we put \(|a| = \Sigma_{i,j} a_{ij}\). For \(m \geq 0\) we write \(\mathcal{B}_m\) for the set of all monomials \(c^a\) with \(|a| = m\) and write \(\mathcal{B}_m(\Psi)\) for the set of all monomials \(c^a\) with \(|a| = m\) and \(a_i > 0\) for some \((i, r) \in \Psi\). We write \(I_m(\Psi)\) for the \(k\)-span of \(\mathcal{B}_m(\Psi)\). We first show that \(I(\Psi)\) is a left ideal. We claim that for every \(m \geq 0\), we have

\[
c_{ir} y \in I_{m+1}(\Psi)
\]

for all \((i, r) \in [1, n]^2\), \(y \in \mathcal{B}_m(\Psi)\) and for all \((i, r) \in \Psi\), \(y \in \mathcal{B}_m\). Assume for a contradiction that this is not the case and let \(m\) be minimal for which \((*)\) fails. Let \((j, s)\) be minimal in the lexicographic order such that either \((j, s) \in \Psi\) and \(c_{js} y \notin I_{m+1}(\Psi)\) for some \(y \in \mathcal{B}_m\) or \((j, s) \in [1, n]^2\) and \(c_{js} y \notin I_{m+1}(\Psi)\) for some \(y \in \mathcal{B}_m(\Psi)\). Clearly, we have \(m > 0\). We write \(y = c_{ir} y'\) where \(y' = c_{ir} r_1 \cdots c_{ir_m} r_m\) with \((i, r) \leq (i_1, r_1) \leq \cdots \leq (i_m, r_m)\). If \(j \leq i\) then \(c_{js} y \in \mathcal{B}_{m+1}(\Psi) \subset I_{m+1}(\Psi)\), a contradiction. If \(j > i\) and \(s \leq r\) then \(c_{js} y = c_{js} c_{ir} y' = q c_{ir} c_{js} y'\). But \((i, r) < (j, s)\) gives \(c_{js} y = c_{ir}(q c_{js} y') \in I_{m+1}(\Psi)\) by minimality, a contradiction. Thus we have \(j > i\) and \(s > r\). We have

\[
c_{js} y = c_{js} c_{ir} y' = c_{ir} c_{js} y' + (q - 1)c_{is} c_{jr} y'.
\]
Again we get \(c_{ir}c_{js}y' \in I_{m+1}(\Psi)\) by minimality so that we have \(c_{ir}c_{js}y' \notin I_{m+1}(\Psi)\). If \(y' \in \mathcal{B}_{m-1}(\Psi)\) then we get \(c_{ir}y' \in I_m(\Psi)\) by the minimality of \(m\) and so \(c_{ir}c_{js}y' \in I_{m+1}(\Psi)\), by the minimality of \((j, s)\), a contradiction. Thus \(y' \in \mathcal{B}_{m-1}(\Psi)\) and so \((i, r) \in \Psi\) or \((j, s) \in \Psi\). Now condition (iii) gives \((i, s) \in \Psi\) or \((j, r) \in \Psi\). It follows that \(c_{ir}c_{js}y' \in I_{m+1}(\Psi)\), by the minimality of \(m\) and \((j, s)\). Thus all cases lead to a contradiction and so (*) holds, in particular \(I(\Psi)\) is a left ideal.

We now show that \(I(\Psi)\) is a right ideal. Assume for a contradiction that this is not the case and let \(m\) be minimal such that for some \((j, s) \in [1, n]^2\) and \(y \in \mathcal{B}_m(\Psi)\) we have \(yc_{js} \notin I_{m+1}(\Psi)\). Clearly we have \(m > 0\). Write \(y = c_{ir}y'\) with \(y' = c_{i_2}r_1 \cdots c_{i_{m-r_m}}\) and \((i, r) \leq (i_2, r_2) \leq \cdots \leq (i_m, r_m)\). If \(y' \in \mathcal{B}_{m-1}(\Psi)\) then we get \(y'c_{js} \in I_m(\Psi)\) by minimality and hence \(yc_{js} = c_{ir}y'c_{js} \in I(\Psi)\) since \(I(\Psi)\) is a left ideal. We must therefore have \((i, r) \in \Psi\) and hence \(yc_{js} = c_{ir}y'c_{js} \in I_{m+1}(\Psi)\) by (*).

Note that \(I(\Psi)\) is the ideal generated by the set \(\{c_{ir} : (i, r) \in \Psi\}\). Condition (i) implies that \(e(c_{ir}) = 0\) and condition (ii) implies that \(\delta(c_{ir}) \in I(\Psi) \otimes A + A \otimes I(\Psi)\) for all \((i, r) \in \Psi\). Hence \(I(\Psi)\) is biideal of \(A\).

**Corollary 2.4.** Let \(\Psi\) be a standard subset of \([1, n]^2\) and let \(\Psi'\) be the complement of \(\Psi\) in \([1, n]^2\). Then \(A/I(\Psi')\) has a k-basis consisting of the elements \(x^a + I(\Psi)\), where \(a\) runs over all \(a = (a_{11}, \ldots, a_{nn}) \in \mathbb{N}_0^n\) with \(a_{ir} = 0\) for \((i, r) \notin \Psi\).

**Notation.** If \(\Psi\) is a standard subset of \([1, n]^2\) with complement \(\Psi'\) we denote by \(M(\Psi)\) the submonoid of \(M(n)\) such that \(I_{M(\Psi)} = I(\Psi')\). We denote by \(G(\Psi)\) the subgroup of \(G\) whose defining Hopf ideal \(I_{G(\Psi)}\) is generated by \(I(\Psi')\). We call \(M(\Psi)\) the standard submonoid and \(G(\Psi)\) the standard subgroup defined by \(\Psi\).

It is easy to deduce the following.

**Corollary 2.5.** For standard subsets \(\Psi_1, \Psi_2\) of \([1, n]^2\), we have \(M(\Psi_1) \cap M(\Psi_2) = M(\Psi_1 \cap \Psi_2)\) and \(G(\Psi_1) \cap G(\Psi_2) = G(\Psi_1 \cap \Psi_2)\).

Thus an intersection of standard submonoids (resp. subgroups) is a standard submonoid (resp. subgroup).

Taking \(\Psi\) to be \(((i, j) : 1 \leq i \leq j \leq n)\) we obtain the positive Borel subgroup \(B^+\) of upper triangular matrices. Taking \(\Psi\) to be \(((i, j) : 1 \leq j \leq i \leq n)\) we obtain the negative Borel subgroup \(B\) of lower triangular matrices. Taking \(\Psi\) to be \(((i, i) : 1 \leq i \leq n)\) we obtain the torus of diagonal matrices \(T = B^+ \cap B\).

Now let \(a = (a_1, \ldots, a_m)\) be a composition of \(n\). We set \(\Psi(a) = [1, a_1]^2 \times [a_1 + 1, a_1 + a_2]^2 \times \cdots \times [a_1 + \cdots + a_{m-1} + 1, n]^2\). We abbreviate
2.4 that we have natural isomorphisms $M(a_1) \times M(a_2) \times \cdots \times M(a_m) \to M(a)$ and $G(a_1) \times G(a_2) \times \cdots \times G(a_m) \to G(a)$.

In particular, we have $G = G(n)$ and $T = G(1, 1, \ldots, 1)$. The coordinate algebra of $T$ is the Laurent polynomial algebra $k[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]$, where $T_i$ is the restriction of $c_{ii}$ to $T$, for $1 \leq i \leq n$. Thus $k[T]$ is commutative so that $T$ may be regarded as an affine group scheme in the usual sense and indeed, when so viewed, $T$ is the split torus of dimension $n$.

We put $\Psi^+ = ((i, j) : 1 \leq i < j \leq n)$ and $\Psi^- = ((i, j) : 1 \leq j < i \leq n)$. We put $\Psi^-(a) = \Psi_- \cup \Psi(a)$ and $\Psi^+(a) = \Psi^+ \cup \Psi(a)$. We put $P(a) = G(\Psi^-(a))$ and $P^+(a) = G(\Psi^+(a))$. Thus we have $P^+(n) = P^-(n) = G$, $P^+(1, 1, \ldots, 1) = B^+$ and $P^-(1, 1, \ldots, 1) = B$ and in general $B^\pm \leq P^+(a) \leq G$, $B \leq P(a) \leq G$ and $G(a) = P^+(a) \cap P(a)$. We call the subgroups $P(a)$ (resp. $P^+(a)$), as $a$ runs over compositions of $n$ the negative (resp. positive) parabolic subgroups. We have a natural morphism $\phi : P(a) \to G(a)$ (resp. $\phi^+ : P^+(a) \to G(a)$) given by $\phi(c_{ij} + I_{G(a)}) = c_{ij} + I_{P(a)}$ (resp. $\phi^+(c_{ij} + I_{G(a)}) = c_{ij} + I_{P^+(a)}$) for all $(i, r) \in [1, n]^2$. We shall frequently regard a $G(a)$-module as a $P(a)$-module via inflation. We define $B(a) = B \cap G(a)$, and call $B(a)$ the negative Borel subgroup of $G(a)$. Thus $B(n) = B$. We have a natural isomorphism $B(a_1) \times \cdots \times B(a_m) \to B(a)$.

We now consider the representation theory of some subgroups of $G$, beginning with $T$. We set $X(n) = \mathbb{Z}^n$. We call $X(n)$ the character group of $T$ and usually abbreviate it to $X$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X$, we have a 1-dimensional $T$-module $k_\lambda$ with structure map $k_\lambda \to k(T)$ taking $a \in k_\lambda$ to $a \otimes T_1^{\lambda_1} \cdots T_n^{\lambda_n} \in k(T)$. The $T$-modules $k_\lambda = \lambda \in X$ for a complete set of inequivalent irreducible $T$-modules. For $\lambda \in X$ and a $V \in \text{Mod}(T)$, we let $V^\lambda$ denote the sum of all submodules isomorphic to $k_\lambda$. The subspace $V^\lambda$ is called the $\lambda$ weight space of $V$. Every $T$-module is completely reducible, so we have $V = \bigoplus_{\lambda \in X} V^\lambda$, for any $V \in \text{Mod}(T)$. We denote by $ZX$ the ring which has a $\mathbb{Z}$-basis on symbols $(e^\lambda : \lambda \in X)$ and multiplication $e^\lambda e^\mu = e^{\lambda + \mu}$, for $\lambda, \mu \in X$. We define the formal character of $V \in \text{mod}(T)$ by $\text{ch} V = \sum_{\lambda \in X} (\dim V^\lambda)e^\lambda$.

We define $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the $i$th position) for $1 \leq i \leq n$. By a root, we mean an element of $X$ of the form $e_i - e_j$, with $1 \leq i, j \leq n$ and $i \neq j$. Let $\Phi$ be the set of roots. We call $e_i - e_j \in \Phi$ positive (resp. negative) if $i < j$ (resp. $i > j$). Let $\Phi^+$ (resp. $\Phi^-$) denote the set of positive (resp. negative) roots. The elements $e_1 - e_2, \ldots, e_{n-1} - e_n$ of $X$ are called simple and we denote the set of simple roots by $I$. We write $\alpha_i = e_i - e_{i+1}, 1 \leq r < n$ and $\omega_i = e_1 + \cdots + e_i$, for $1 \leq r \leq n$. Let $(,)$ denote the $\mathbb{Z}$-bilinear form on $X$ such that $(e_i, e_j) = \delta_{ij}, 1 \leq i, j \leq n$. 

$M(\Psi(a))$ (resp. $G(\Psi(a))$) to $M(a)$ (resp. $G(a)$). It follows from Corollary 2.4 that we have natural isomorphisms $M(a_1) \times M(a_2) \times \cdots \times M(a_m) \to M(a)$ and $G(a_1) \times G(a_2) \times \cdots \times G(a_m) \to G(a)$.
We now consider the negative Borel subgroup \( B \). For \( \lambda \in X \), we regard \( k_\lambda \) also as a \( B \)-module via inflation and as a \( B(a) \)-module, for any composition \( a \) of \( n \) by restriction.

**Lemma 2.6.** Let \( a \) be a composition of \( n \).

(i) \( \{ k_\lambda : \lambda \in X \} \) is a complete set of inequivalent irreducible \( B(a) \)-modules.

(ii) Let \( \text{Grot}(B(a)) \) denote the Grothendieck group of finite dimensional \( B(a) \)-modules. There is an isomorphic \( \overline{\text{Grot}}: \text{Grot}(B(a)) \to \mathbb{Z} X \) satisfying \( \overline{\text{Grot}}([V]) = \text{ch} V \) for any finite dimensional \( B(a) \)-module \( V \).

**Proof.** (ii) is a consequence of (i). We now prove (i). Clearly, \( k_\lambda \not\approx k_\mu \) for \( \lambda \neq \mu \). We must show that each irreducible \( B(a) \)-module to isomorphic to \( k_\lambda \) for some \( \lambda \in X \). We consider first the case \( B(a) = B \). Let \( \text{Mod}_d(B) \) denote the full subcategory of \( \text{Mod}(B) \) whose objects are the \( B \)-modules \( V \) such that every composition factor of \( V \) belongs to \( \{ k_\lambda : \lambda \in X \} \). Then \( \text{Mod}_d(B) \) is closed under submodules, quotient modules, sums and tensor products. We claim that \( \text{Mod}_d(B) = \text{Mod}(B) \) if \( V \) is a \( B \)-module then \( V \) embeds in a direct sum of copies of the regular module \( k[B] \).

Hence it suffices to show that \( k[B] \) belongs to \( \text{Mod}_d(B) \). Since the natural map \( k[G] \to k[B] \) is a \( B \)-module surjection, it suffices to show that \( k[G] \) belongs to \( \text{Mod}_d(B) \). Now if \( V \) is any finite dimensional \( B \)-submodule of \( k[G] \) then \( V \cap \mathfrak{k}[M(n)] \) for \( r \gg 0 \). The multiplication map \( V \cap \mathfrak{k}[M(n)] \to k[G] \) is a \( B \)-module map with image \( V \) and \( k\mathfrak{d} \) is isomorphic, as a \( B \)-module, to \( k[G] \). It therefore suffices to show that \( k[G] \) belongs to \( \text{Mod}_d(B) \). For \( 1 \leq i \leq n \) let \( S_i \) be the \( k \)-span of the monomials \( c_{i1}^m \cdots c_{in}^m \), with \( a_{i1}, \ldots, a_{in} \geq 0 \). Multiplication \( S_1 \otimes \cdots \otimes S_n \to k[M(n)] \) is a \( B \)-module surjection so it is enough to prove that each \( S_i \) belongs to \( \text{Mod}_d(B) \). Now \( S_i \) decomposes as \( S_i = \bigoplus_{m=0}^{\infty} S_{i_1 \cdots i_n} \), where \( S_{i_1 \cdots i_n} \) is the \( k \)-span of the monomials \( c_{i_1}^m \cdots c_{i_n}^m \) with \( a_{i_1} + \cdots + a_{i_n} = m \). Hence it is enough to prove that \( S_{i_1 \cdots i_n} \) belongs to \( \text{Mod}_d(B) \), for \( m \geq 0 \). Let \( x_i = c_{i1} \), for \( 1 \leq i \leq n \), and order the monomials \( x^\lambda = x_{i1}^{\lambda_1} \cdots x_{in}^{\lambda_n} \) lexicographically, where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a sequence of non-negative integers with \( \lambda_1 + \cdots + \lambda_n = m \).

Putting \( c_\lambda = c_{i1}^{\lambda_1} \cdots c_{in}^{\lambda_n} \) (where \( c_{ij} \) denotes the restriction of \( c_{ij} \) to \( B \), for \( 1 \leq i, j \leq n \)) we obtain \( \tau(x^\lambda) - x^\lambda \otimes c_\lambda \in Y \otimes k[B] \), where \( \tau: S_{i_1 \cdots i_n} \to S_{i_1 \cdots i_n} \otimes k[B] \) is the structure map of the \( B \)-module \( S_{i_1 \cdots i_n} \) and \( Y \) is the \( k \)-span of monomials bigger than \( x^\lambda \). It follows that \( S_{i_1 \cdots i_n} \) has a composition series with factors of the form \( k_\mu \) with \( \mu \) as above. In particular \( S_{i_1 \cdots i_n} \) belongs to \( \text{Mod}_d(B) \) and hence \( \text{Mod}_d(B) = \text{Mod}(B) \). In particular \( \text{Mod}_d(B) \) contains every simple \( B \)-module and the Lemma, with \( B = B(a) \) is proved.
Now let $\alpha$ be arbitrary. An irreducible $B(a)$-module $L$ is a composition factor of the regular module $k[B(a)]$ and, since $k[B] \to k[B(a)]$ is surjective, is a $B(a)$-module composition factor of $k[B]$. Thus $L$ is isomorphic to $k_{\lambda(B,a)}$ for some $\lambda \in X$, as required.

We give $X$ the following partial order. For $\lambda, \mu \in X$ we decree $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive roots.

We write simply $c_{ij}$ for the restriction of $c_{ij}$ to $B$ (for $1 \leq i, j \leq n$) and simply $d$ for the restriction of $d$ to $B$. The elements $c_{11}, \ldots, c_{nn}$ commute with each other and $d$ is the product $c_{11}c_{22} \cdots c_{nn}$. It follows that $c_{ii}$ is a unit in $k[B]$, $1 \leq i \leq n$. It is easy to check that $c_{ij}c_{ij}^{-1}$ is a scalar multiple of $c_{ii}$, for all $1 \leq i \leq n$, $1 \leq r \leq s \leq n$ and it follows that $k[B]$ has a basis consisting of the monomials $c_{11}^{r_1}c_{21}^{r_2}\cdots c_{nn}^{r_n}$, where all $a_{ij} \in \mathbb{Z}$ and $a_{ij} \geq 0$ for $i \neq j$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in X$ we let $k[B]_{\alpha}$ denote the $k$-span of all monomials $c_{11}^{r_1}c_{21}^{r_2}\cdots c_{nn}^{r_n}$ with $a_{11} = \alpha_1$, $a_{21} + a_{22} = \alpha_2, \ldots, a_{n1} + a_{n2} + \cdots + a_{nn} = \alpha_n$ and put $c_{\alpha} = c_{11}^{\alpha_1}c_{22}^{\alpha_2}\cdots c_{nn}^{\alpha_n}$. Let $E(\alpha)$ denote the injective hull of $k_{\alpha}$.

**Lemma 2.7.** (i) $k[B]_{\alpha}$ is a $B$-submodule of $k[B]$, for $\alpha \in X$ and $k[B] = \bigoplus_{\alpha \in X} k[B]_{\alpha}$.

(ii) For $\alpha \in X$ the module $k[B]_{\alpha}$ is the injective hull of $k_{\alpha}$. All weights of $E(\alpha)$ are greater than or equal to $\alpha$ and $\dim E(\alpha)^a = 1$.

**Proof.** (i) This is an easy check. For $\alpha \in X$, the module $k[B]_{\alpha}$ is a direct summand of $k[B]$ and hence injective. Furthermore, $k[B]_{\alpha}$ contains the copy $k_{\alpha}$ of $k_{\alpha}$ and so contains a copy $E(\alpha)$, say, of $E(\alpha)$. Now if $E(\alpha)' \neq k[B]_{\alpha}$ we get $k[B]_{\alpha} = E(\alpha)' \oplus Y$ for some non-zero submodule $Y$. Now if $S$ is a simple submodule of $Y$ then $S \cong k_{\mu}$ for some $\mu \in X$. But then $k_{\mu}$ occurs with multiplicity at least 2 in the socle of $k[B]$ (either twice in $k[B]_{\lambda}$ if $\lambda = \mu$ or in both $k[B]_{\lambda}$ and $k[B]_{\mu}$ if $\lambda \neq \mu$). However, we have $k[B] \cong \text{Ind}_G^H k$ so that $\text{Hom}_G(k_{\lambda}, k[B]) \cong \text{Hom}_G(k_{\mu}, k) \cong k$, by Frobenius Reciprocity, so this is impossible. We leave it to the reader to check that all weights of $k[B]_{\alpha}/k_{\alpha}$ are greater than $\alpha$, completing the proof of (ii).

Let $a = (a_1, \ldots, a_m)$ be a composition of $n$. For $1 \leq i, j \leq n$ with $i \neq j$ we define $\alpha_{ij} = \epsilon_i - \epsilon_j$. We let $\Phi(a)$ be the set of $\alpha_{ij}$ such that either $1 \leq i, j \leq a_1$ or, for some $1 \leq r < n$, we have $a_1 + \cdots + a_r < i, j \leq a_1 + \cdots + a_r + a_{r+1}$. Let $\Phi^+(a)$ be the set of $\alpha_{ij} \in \Phi(a)$ with $i < j$ and let $\Phi^-(a)$ be the set of $\alpha_{ij} \in \Phi(a)$ with $i > j$. Thus $\Phi(a)$ is the disjoint union of $\Phi^+(a)$ and $\Phi^-(a)$. Note that we have a natural bijection $\Phi(a_1) \times \cdots \times \Phi(a_m) \to \Phi(a)$ inducing bijections $\Phi^+(a_1) \times \cdots \times \Phi^+(a_m) \to \Phi^+(a)$ and $\Phi^-(a_1) \times \cdots \times \Phi^-(a_m) \to \Phi^-(a)$. For $\lambda, \mu \in X$, we write $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive roots.
is a sum of elements of \( \Phi^+(a) \). We denote by \( E_a(\lambda) \) and \( B(a) \)-module injective hull of \( k_\lambda \), for \( \lambda \in X \).

**Lemma 2.8.** Let \( \lambda, \mu \in X \).

(i) The dimension of \( E_a(\lambda)^\mu \) is the number of \(|\Phi^+(a)|\)-tuples of nonnegative integers \( (r_\alpha)_{\alpha \in \Phi^+(a)} \) such that \( \mu - \lambda = \sum_{\alpha \in \Phi^+(a)} r_\alpha \alpha \).

(ii) If \( \text{Ext}^1_B(k_\mu, k_\lambda) \neq 0 \) then \( \lambda <_a \mu \). In particular, if \( \text{Ext}^1_B(k_\mu, k_\lambda) \neq 0 \) then \( \lambda < \mu \).

**Proof.** (i) For \( a = (n) \) this follows from Lemma 2.7(ii). In general we write \( \lambda(1) = (\lambda_1, \ldots, \lambda_m) \), \( \lambda(2) = (\lambda_{1+1}, \ldots, \lambda_{1+1}) \), \( \lambda(m) = (\lambda_{m+1}, \ldots, \lambda_m) \). I dentifying \( B(a) \) with \( B(a_1) \times \cdots \times B(a_m) \) via the natural isomorphism, we have \( k_\lambda \cong k_{\lambda(1)} \otimes \cdots \otimes k_{\lambda(m)} \) and \( E_a(\lambda) \cong E_{a_1}(\lambda(1)) \otimes \cdots \otimes E_{a_m}(\lambda(m)) \). The result follows.

(ii) Suppose that \( 0 \to k_\lambda \to V \to k_\mu \to 0 \) is a non-split \( B(a) \)-module extension. Let \( L \equiv k_\lambda \) be the \( B(a) \)-socle of \( E_a(\lambda) \). The embedding of \( k_\lambda \) into \( E_a(\lambda) \) extends, by injectivity, to an embedding of \( V \) into \( E_a(\lambda) \). Thus \( \mu \) is a weight of \( E_a(\lambda)/L \) and therefore \( \lambda <_a \mu \), by (i).

It is easy to prove (cf. [26, 8.1.1 Theorem]) the following. We leave the details to the reader.

**Lemma 2.9.** The natural map \( k[P(a)] \to k[B] \otimes k[B^+(a)] \) is injective.

One can now continue with the development, as in the case of algebraic groups [8, Sect. 1.5] (or the Manin quantization of \( \text{GL}_n \); see [26, Sect. 8.1, 8.2]) to obtain the following.

**Theorem 2.10.** Let \( a \) be a composition of \( n \).

(i) If \( V \in \text{Mod}(B) \) is finite dimensional then \( \text{Ind}^B_{a}V \) is finite dimensional.

(ii) Let \( X^+(a) \) denote the set of \( \lambda \in X \) such that \( \text{Ind}^B_{a}k_\lambda \neq 0 \). For \( \lambda \in X^+(a) \), the induced module \( \text{Ind}^B_{a}k_\lambda \) has simple socle \( L_a(\lambda) \), say. For \( \lambda \in X^+(a) \) we have \( \dim L_a(\lambda)^\lambda = 1 \) and we have \( \mu \leq_\lambda \mu \) for every weight \( \mu \) of \( L_a(\lambda) \). The collection \( \{L_a(\lambda): \lambda \in X^+(a)\} \) is a complete set of irreducible, inequivalent \( P(a) \)-modules and a complete set of inequivalent, irreducible \( G(a) \)-modules.

We write \( X^+(n) \) as \( X^+ \) for short. The set \( X^+(a) \) is described explicitly in Lemma 3.2.

For a subgroup \( H \) of a quantum group \( G \), we write \( k[G/H] \) for the set of \( f \in k[G] \) such that \( (\pi \otimes \text{id})\delta_\xi(f) = 1 \otimes f \), where \( \pi: k[G] \to k[H] \) is restriction. Note that \( k[G/H] \) is a left submodule of \( k[G] \) isomorphic to \( \text{Ind}_{1H}^{G}k \).
Lemma 2.11. Restriction \( k[P(a)] \to k[B] \) induces an isomorphism \( k[P(a)/G(a)] \to k[B/B(a)] \).

Proof. Certainly restriction \( k[P(a)] \to k[B] \) takes \( k[P(a)/G(a)] \) into \( k[B/B(a)] \), giving rise to a \( k \)-algebra map \( \psi: k[P(a)/G(a)] \to k[B/B(a)] \). Say we have \( k[P(a)] \cong \text{Ind}_{G(a)}^{P(a)}k[G(a)] \). Regarding \( k[G(a)] \) as a \( P(a) \)-module via inflation we therefore get \( k[P(a)] \cong k[G(a)] \otimes \text{Ind}_{G(a)}^{P(a)}k \) from the tensor identity and by exhibiting the isomorphism explicitly, we find that the multiplication map \( k[G(a)] \otimes k[P(a)/G(a)] \to k[P(a)] \) is an isomorphism. Similarly, we obtain that multiplication \( k[B(a)] \otimes k[B/B(a)] \to k[B] \) is an isomorphism. Now restriction \( k[P(a)] \to k[B] \) is surjective and takes \( k[G(a)] \) onto \( k[B(a)] \). Thus we have that multiplication \( k[B(a)] \otimes k[B/B(a)] \to k[B] \) is an isomorphism and that multiplication \( k[B(a)] \otimes Y \to k[B] \) is an isomorphism, where \( Y = \psi(k[P(a)/G(a)]) \). But \( Y \leq k[B/B(a)] \) so we must have \( Y = k[B/B(a)] \). Thus \( \psi \) is surjective. It is injective by Lemma 2.9, and hence an isomorphism.

Lemma 2.12. For \( V \in \text{Mod}(B(a)) \) we have \( R^i \text{Ind}_{B(a)}^P(k(V) \otimes \text{Ind}_{B(a)}^Gk[V]) \approx \text{Ind}_{B(a)}^G \circ R^i \text{Ind}_{B(a)}^Gk[V] \) for all \( i \geq 0 \).

Proof. We shall omit \( \text{Ind}_{B(a)}^B \) and \( \text{Ind}_{B(a)}^G \). We first consider the case \( i = 0 \). We have, from the definition, \( \text{Ind}_{B(a)}^Gk(V) \leq \text{Ind}_{B(a)}^P(k(V)) \). By embedding \( V \) in a direct sum of copies of \( k[B(a)] \), it suffices to show that \( \text{Ind}_{B(a)}^P(k[B(a)] \cong k[G(a)] \). Let \( Q = \text{Ind}_{B(a)}^P(k[B(a)] \cong k[G(a)] \). We have

\[
\text{Ind}_{B(a)}^P(k[B(a)] \cong k[P(a)/G(a)] \\
\cong \text{Ind}_{B(a)}^P(k[B(a)] \otimes k[P(a)/G(a)]) \\
\cong \text{Ind}_{B(a)}^P(k[B(a)]) \cong k(P(a)).
\]

Hence, for \( S \in \text{Mod}(P(a)) \), we have

\[
\dim S = \dim(S \otimes \text{Ind}_{B(a)}^P(k[B(a)] \otimes \text{Ind}_{B(a)}^P(k)P(a)) \\
= \dim(S \otimes \text{Ind}_{B(a)}^P(k[B(a)] \overset{G(a)}{\rightarrow} \text{Ind}_{B(a)}^P(k[B(a)]))^{G(a)}.
\]

Suppose, for a contradiction, that \( Q \neq 0 \) and choose a finite dimensional \( G(a) \)-submodule \( L \). We get a short exact sequence of \( G(a) \)-modules

\[
0 \to L^* \otimes k[G(a)] \to L^* \otimes \text{Ind}_{B(a)}^P(k[B(a)] \to L^* \otimes Q \to 0,
\]

giving an exact sequence of fixed points

\[
0 \to \left( L^* \otimes k[G(a)] \right)^{G(a)} \to \left( L^* \otimes \text{Ind}_{B(a)}^P(k[B(a)]) \right)^{G(a)} \\
\to (L^* \otimes Q)^{G(a)} \to 0.
\]
But we have \( \dim(L^* \otimes k[G(a)])^{G(a)} = \dim(L^* \otimes \text{Ind}_{a}^{B(a)} k[B(a)])^{G(a)} = \dim L \) and \((L^* \otimes Q)^{G(a)} \neq 0\), a contradiction. Hence \( k[G(a)] = \text{Ind}_{a}^{B(a)} k[B(a)] \) and therefore \( \text{Ind}_{B(a)}^{L(a)} V = \text{Ind}_{B(a)}^{G(a)} V \), for any \( B(a) \)-module \( V \).

Let \( 0 \rightarrow V \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \) be a resolution of \( V \) by injective \( B(a) \)-modules. We claim that each \( I_i \) is \( \text{Ind}_{B(a)}^{L(a)} \)-acyclic. Since every injective \( B(a) \)-module is a direct summand of a direct sum of copies of \( k[B(a)] \), it is enough to show that \( k[B(a)] \) is \( \text{Ind}_{B(a)}^{L(a)} \)-acyclic. However, we have \( k[B(a)] \otimes \text{Ind}_{B(a)}^{L(a)} k \equiv k[B] \) so \( k[B(a)] \) is \( \text{Ind}_{B(a)}^{L(a)} \)-acyclic by Corollary 1.4. Thus \( R^i \text{Ind}_{B(a)}^{L(a)} V \) is the homology of \( 0 \rightarrow \text{Ind}_{B(a)}^{L(a)} I_0 \rightarrow \text{Ind}_{B(a)}^{L(a)} I_1 \rightarrow \cdots \).

However, we have a commutative diagram

\[
\begin{array}{c}
0 \rightarrow \text{Ind}_{B(a)}^{L(a)} I_0 \rightarrow \text{Ind}_{B(a)}^{L(a)} I_1 \rightarrow \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \rightarrow \text{Ind}_{B(a)}^{L(a)} I_0 \rightarrow \text{Ind}_{B(a)}^{L(a)} I_1 \rightarrow 
\end{array}
\]

where the vertical maps are isomorphisms. Thus we obtain \( R^i \text{Ind}_{B(a)}^{L(a)} V \equiv R^i \text{Ind}_{B(a)}^{L(a)} V \) for all \( i \geq 0 \), as required.

3. VANISHING THEOREMS

We write \( \text{Sym}(S) \) for the group of permutations of a set \( S \). We write \( \text{Sym}(n) \) for \( \text{Sym}[1, n] \) and let \( s_i \in \text{Sym}(n) \) denote the transposition \((i, i + 1)\), for \( 1 \leq i < n \). For \( a = (a_1, \ldots, a_m) \), a composition of \( n \), we write \( \text{Sym}(a) \) for the subgroup \( \text{Sym}[1, a_1] \times \text{Sym}[a_1 + 1, a_1 + a_2] \times \cdots \times \text{Sym}[a_1 + \cdots + a_{m-1} + 1, n] \) of \( \text{Sym}(n) \). Note that \( \text{Sym}(n) \) acts on \( X \) by \( w \cdot \lambda = (\lambda_{w(1)}, \ldots, \lambda_{w(n)}) \), for \( w \in \text{Sym}(n) \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in X \). We also introduce the “dot” action \( w \cdot \lambda = w(\lambda + \rho) - \rho \), for \( w \in \text{Sym}(n) \), \( \lambda \in X \), where \( \rho = (n, n - 1, \ldots, 1) \). The natural action of \( \text{Sym}(n) \) on \( X \) induces an action on \( \mathbb{Z} X \), given by \( w(\sum_{\lambda \in X} a_{\lambda} e^{\lambda}) = \sum_{\lambda \in X} a_{\lambda} e^{w(\lambda)} \).

We consider now the “rank one” case. Let \( 1 \leq i < n \) and let \( a = (1, 1, \ldots, 1, 2, 1, \ldots, 1) \), with the 2 in the \( i \)th position. We let \( Y_j \leq k[c_{ii}, c_{i,i+1}] \) be the \( j \)th homogeneous component, \( j \geq 0 \). Then \( k[c_{ii}, c_{i,i+1}] = \bigoplus_{j \geq 0} Y_j \) is a \( G(a) \)-module decomposition.

**Lemma 3.1.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in X \).

(i) If \( \lambda_i - \lambda_{i+1} = m \geq 0 \) then \( \text{Ind}_{B(a)}^{L(a)} k_{\lambda} \equiv k_{\mu} \otimes Y_m \), where \( \mu = (\lambda_1, \ldots, \lambda_1, \lambda_i - m, \lambda_{i+1}, \ldots, \lambda_n) \), and \( R^1 \text{Ind}_{B(a)}^{L(a)} k_{\lambda} = 0 \).

(ii) If \( \lambda_i - \lambda_{i+1} < 0 \) then \( \text{Ind}_{B(a)}^{L(a)} k_{\lambda} = 0 \) and if \( \lambda_i - \lambda_{i+1} = -1 \) then \( R^1 \text{Ind}_{B(a)}^{L(a)} k_{\lambda} = 0 \).

(iii) \( R^j \text{Ind}_{B(a)}^{L(a)} = 0 \) for all \( j > 1 \).
(iv) \( \text{ch } R^1 \text{Ind}_{B(a)}^{G(a)} k_{s, \lambda} = \text{ch } \text{Ind}_{B(a)}^{G(a)} k_{\lambda}. \)

(v) The character of any finite dimensional \( G(a) \)-module is \( s_j \)-invariant.

(vi) For any composition \( b = (b_1, \ldots, b_m) \) of \( n \) the character of any finite dimensional \( G(b) \)-module is \( \text{Sym}(b) \)-invariant. In particular the character of any finite dimensional \( G(a) \)-module is \( \text{Sym}(n) \)-invariant.

Proof. The calculation in the classical case given in [8, Sect. 12.2] easily adapts to the case considered here and gives parts (i), (ii), (iii).

(iv) We define the Euler character \( \phi(V) \) of a finite dimensional \( B(a) \)-module module \( V \) by \( \phi(V) = \text{ch } \text{Ind}_{B(a)}^{G(a)} V = \text{ch } R^1 \text{Ind}_{B(a)}^{G(a)} V. \) Then \( \phi \) is additive on short exact sequences and hence induces a map on \( \text{Gr}(B(a)). \) Hence, by Lemma 2.6(ii), \( \phi \) induces a map \( \psi : \mathbb{Z} X \rightarrow \mathbb{Z} X \) satisfying \( \psi(\text{ch } V) = \phi(V). \) From the tensor identity we get \( \psi(xy) = x \psi(y), \) for \( x, y \in \mathbb{Z} X, \) if \( x \) is the character of a \( G(a) \)-module. Also, we have \( \psi(1) = \phi(k) = 1, \) by (i). Hence we get \( \psi(x) = x \) whenever \( x \) is the difference of characters of \( G(a) \)-modules.

Let \( \mu = (\mu_1, \ldots, \mu_n) \in X. \) If \( \mu_i - \mu_{i+1} = m > 0 \) then, putting \( \alpha = \epsilon_i - \epsilon_{i+1} \) from (i) we get \( \text{ch } \text{Ind}_{B(a)}^{G(a)} k_{\mu} = e^\mu + e^{\mu - a} + \cdots + e^{\mu - ma} \) and \( \text{ch } \text{Ind}_{B(a)}^{G(a)} k_{\lambda} = e^\lambda - e^{\lambda - a} + \cdots + e^{\lambda - ma} \). We therefore get \( \psi(e^\mu + e^{\mu - ma}) = e^\mu + e^{\mu - ma} \), which is \( \psi(e^\mu + e^{\lambda, \mu}) = e^\mu + e^{\lambda, \mu} \) for all \( \mu \in X. \)

Let \( \lambda_i - \lambda_{i+1} = m \) so that \( \mu = s_i \cdot \lambda = \lambda - (m + 1)\alpha, \) where \( \alpha = \epsilon_i - \epsilon_{i+1}. \) If \( m \leq -1 \) then \( R^1 \text{Ind}_{B(a)}^{G(a)} k_{s, \lambda} = \text{Ind}_{B(a)}^{G(a)} k_{\lambda} = 0, \) by (i), (ii). Thus we may assume \( m \geq 0. \) Now \( s_i(\lambda + \alpha) = \lambda - (m + 1)\alpha \) so that \( \psi(e^{\lambda + a} + e^{\lambda - (m + 1)a}) = e^{\lambda + a} + e^{\lambda - (m + 1)a}. \) However, \( \psi(e^{\lambda + a}) = \text{ch } \text{Ind}_{B(a)}^{G(a)} k_{\lambda + a} = e^{\lambda + a} + e^{\lambda - (m + 1)a} \) so we get \( \psi(e^{\lambda - (m + 1)a}) = -(e^{\lambda} + \cdots + e^{\lambda - ma}), \) i.e. \( \psi(e^{\lambda, \alpha}) = - \text{ch } \text{Ind}_{B(a)}^{G(a)} k_{\lambda} \), by (i). Now \( \psi(e^{\lambda, \alpha}) = \text{ch } \text{Ind}_{B(a)}^{G(a)} k_{s, \lambda} - \text{ch } R^1 \text{Ind}_{B(a)}^{G(a)} k_{s, \lambda} \) and \( \text{Ind}_{B(a)}^{G(a)} k_{s, \lambda} = 0 \) so we get \( \text{ch } R^1 \text{Ind}_{B(a)}^{G(a)} k_{s, \lambda} = \text{ch } \text{Ind}_{B(a)}^{G(a)} k_{\lambda}, \) as required.

(v) It follows from (i), (ii), (iii), and (iv) that \( ch R^1 \text{Ind}_{B(a)}^{G(a)} k_{\lambda} \) is \( s_j \)-invariant for every \( \lambda \in X \) and \( j \geq 0. \) Hence, by additivity of the Euler character, we have that \( \phi(\lambda) \) is \( s_j \)-invariant for every finite dimensional \( B(a) \)-module \( V \) (where \( \phi \) is as in the proof of (iii)). Now if \( V \) is a \( G(a) \)-module then \( R^1 \text{Ind}_{B(a)}^{G(a)} V \cong V \otimes R^1 \text{Ind}_{B(a)}^{G(a)} k \) for \( j \geq 0, \) by the tensor identity, and this is \( \text{ch } V \) for \( j = 0 \) and \( 0 \) for \( j = 1, \) by (i). Thus \( \text{ch } V = \phi(V) \) and hence \( s_j \)-invariant.

(vi) Let \( V \) be a finite dimensional \( G(b) \)-module. We have \( G(b) \geq G(a) \) where \( a' = (1, \ldots, 1, 2, 1, \ldots, 1) \) has 2 in the \( j \)th position and either \( j \leq b_1 \) or \( b_1 + \cdots + b_j < b_1 + \cdots + b_{j+1} \) for some \( 1 \leq r < m. \) Thus \( V \) is a \( G(a) \)-module by restriction and hence \( \text{ch } V = \phi(V) \) is \( s_j \)-invariant. But \( \text{Sym}(b) \) is generated by all such \( s_j \) so \( \text{ch } V = \text{Sym}(b) \)-invariant.
We can now describe the sets $X^+ (a)$ explicitly.

**Lemma 3.2.** (i) We have $X^+ (n) = \{ (\lambda_1, \ldots, \lambda_n) \in X : \lambda_1 \geq \cdots \geq \lambda_n \}$ and for any composition $a = (a_1, \ldots, a_m)$ of $n$ we have $X^+ (a) = X^+ (a_1) \times \cdots \times X^+ (a_m)$.

(ii) If $\lambda_i - \lambda_{i+1} = -1$ for some $1 \leq i < n$ then $R^1 \mathrm{Ind}^G_k \kappa_\lambda = 0$ for all $j \geq 0$.

**Proof.** (i) For $\lambda \in X^+ (n)$, one gets a module $L(\lambda)$ with highest weight $\lambda$ as a composition factor of a tensor product of exterior powers of the natural representation, as in [5, 2.3.4, 3.3.2]. Conversely, suppose $\lambda = (\lambda_1, \ldots, \lambda_n) \in X^+ (n)$ and choose $1 \leq i < n$ such that $\lambda_i - \lambda_{i+1} < 0$. Let $a = (1, \ldots, 1, 2, 1, \ldots, 1)$, with $2$ in the $i$th position. We have $\mathrm{Ind}^G_k (\mu, \mu_j) = 0$, by Lemma 2.10 and Lemma 3.1. Hence $\mathrm{Ind}^G_k \kappa_\lambda = 0$, by transitivity of induction. However, if there is a simple module $L$, say, of highest weight $\lambda$ then we have a non-zero $B$-module homomorphism $L \rightarrow k$, and hence, by Frobenius reciprocity, a non-zero $G$-homomorphism $L \rightarrow \mathrm{Ind}^G_k \kappa_\lambda$. Hence there is no simple module of highest weight $\lambda$ and we have the required description of $X^+ (n)$. Thus, for $1 \leq j \leq m$ we have a complete set of inequivalent $G(\mu_j)$-modules $(L_\mu (\mu_j), \mu_j \in X^+ (a_j))$. We identify $G(\mu_j)$ with $G(\mu_j) \times \cdots \times G(\mu_m)$ via the natural isomorphism. Thus we have a complete set of inequivalent irreducible $G(\mu_j)$-modules $(L_\mu (\mu_j), \mu_j \in X^+ (a_j) ; 1 \leq j \leq m)$. This gives the required description of $X^+ (a)$.

(ii) The first part of the above argument also gives (ii).

In preparation for Kempf's vanishing theorem, we now discuss some maps occurring in a certain Koszul resolution. We suppose, for the moment, that $G$ is an arbitrary quantum group over $k$. For $k$-spaces $M, N$ we identify a bilinear map $b : M \times N \rightarrow k$ with its $k$-linear extension $M \otimes N \rightarrow k$. Let $M$ be a finite dimensional right $G$-module and let $N$ be a finite dimensional left $G$-module. We call a bilinear map $b : M \times N \rightarrow k$ a $G$-module pairing if it is non-degenerate and $(\mathrm{id}_{G(\mu_j)} \otimes b) \circ (\tau_{\lambda_j} \otimes \mathrm{id}) = (b \otimes \mathrm{id}_{G(\mu_j)}) \circ (\mathrm{id}_{\mu_j} \otimes \tau_{\lambda_j})$. Suppose that $b : M \times N \rightarrow k$ is a $G$-module pairing. We note that if $M_1 \leq M$ and $N_1 \leq N$ are $G$-submodules then $M_1^\perp = \{ n \in N : b(m_1, n) = 0 \} $ is a $G$-submodule of $M$ and $N_1^\perp = \{ m \in M : b(m, n_1) = 0 \} $ is a $G$-submodule of $M$.

Note also that if $b_1 : M_1 \times N_1 \rightarrow k$ and $b_2 : M_2 \times N_2 \rightarrow k$ are $G$-module pairings then the bilinear form $b : M_1 \otimes M_2 \times N_1 \otimes N_2 \rightarrow k$, satisfying $b(m_1 \otimes m_2, n_1 \otimes n_2) = b_1 (m_1, n_1) b_2 (m_2, n_2)$ (for $m_1 \in M_1$, $n_1 \in N_1$, $i = 1, 2$) is a $G$-module pairing.

We now take $G$ to be the quantum group $G(n)$ once more. We have the natural right $G$-module $V$ with basis $v_1, \ldots, v_n$ and structure map $\tau_V$, satisfying $\tau_V (v_i) = \sum c_{ij} v_j$ (for $1 \leq i \leq n$) and the natural left $G$-module $E$ with $k$-basis $e_1, \ldots, e_n$ and the structure map $\tau_E$ satisfying $\tau_E (e_i) = \sum c_{ji} e_j$. We have a complete set of inequivalent irreducible $G$-modules $(L_\mu (\mu_j), \mu_j \in X^+ (a_j))$. We identify $G(\mu_j)$ with $G(\mu_j) \times \cdots \times G(\mu_m)$ via the natural isomorphism. Thus we have a complete set of inequivalent irreducible $G(\mu_j)$-modules $(L_\mu (\mu_j), \mu_j \in X^+ (a_j) ; 1 \leq j \leq m)$. This gives the required description of $X^+ (a)$. We have $\mathrm{Ind}^G_k \kappa_\lambda = 0$, by transitivity of induction. However, if there is a simple module $L$, say, of highest weight $\lambda$ then we have a non-zero $B$-module homomorphism $L \rightarrow k$, and hence, by Frobenius reciprocity, a non-zero $G$-homomorphism $L \rightarrow \mathrm{Ind}^G_k \kappa_\lambda$. Hence there is no simple module of highest weight $\lambda$ and we have the required description of $X^+ (n)$. Thus, for $1 \leq j \leq m$ we have a complete set of inequivalent $G(\mu_j)$-modules $(L_\mu (\mu_j), \mu_j \in X^+ (a_j))$. We identify $G(\mu_j)$ with $G(\mu_j) \times \cdots \times G(\mu_m)$ via the natural isomorphism. Thus we have a complete set of inequivalent irreducible $G(\mu_j)$-modules $(L_\mu (\mu_j), \mu_j \in X^+ (a_j) ; 1 \leq j \leq m)$. This gives the required description of $X^+ (a)$.
for \(1 \leq i \leq n\). Further, we have the natural \(G\)-module pairing\
\[ V \times E \text{ taking } (v_i, e) \text{ to } \delta_{ij} \text{ for } 1 \leq i, j \leq n. \]

Let \(r\) be a positive integer. We write \(I(n, r)\) for the set of maps\
\[ [1, r] \rightarrow [1, n]. \]
For \(i = (i_1, \ldots, i_r)\), \(j = (j_1, \ldots, j_r) \in I(n, r)\) we put \(c_{ij} = c_{i_1 j_1} \cdots c_{i_r j_r}\). For \(i = (i_1, \ldots, i_r) \in I(n, r)\) we write \(v_i = v_{i_1} \otimes \cdots \otimes v_{i_r} \in V^{*r}\)
and \(e_i = e_{i_1} \otimes \cdots \otimes e_{i_r} \in E^{*r}\). We have the product \(G\)-module pairing \(b:\)
\[ V^{*r} \times E^{*r} \rightarrow k \text{ satisfying } b(v_i, e_j) = \delta_{ij}, \text{ for } i, j \in I(n, r). \]
In [5, Sect. 2.1], we defined the \(G\)-module exterior powers \(\wedge^r V\) and \(\wedge^r E\). The natural map \(V^{*r} \rightarrow \wedge^r V\) has kernel \(M\), spanned by the elements \(v_i\), for \(i = (i_1, \ldots, i_r) \in I(n, r)\) having a repeated entry, together with the elements \(v_i - \text{sgn}(\pi)v_{i_{\pi}}\) for \(i \in I(n, r), \pi \in \text{Sym}(r)\). It follows that \(M^+ = N_i\), say, is spanned by the elements \(\sum_{\pi \in \text{Sym}(r)} \text{sgn}(\pi)e_{i_{\pi}}, i \in I(n, r)\).

For \(i = (i_1, \ldots, i_r) \in I(n, r)\) we write \(\hat{e}_i\) or \(e_{i_1 \cdots i_r}\) for \(e_{i_1} \wedge \cdots \wedge e_{i_r} \in \wedge^r E\). The length of a permutation \(\pi \in \text{Sym}(r)\) will be denoted \(l(\pi)\).

**Lemma 3.3.** Let \(r\) be a positive integer.

(i) The \(k\)-linear map \(\phi: \wedge^r E \rightarrow E^{*r}\) satisfying \(\phi(\hat{e}_i) = \sum_{\pi \in \text{Sym}(r)} \text{sgn}(\pi)e_{i_{\pi}}, i \in (i_1, \ldots, i_r)\) with \(i_1 > \cdots > i_r\), is a \(G\)-module map.

(ii) the \(k\)-linear map \(\psi: \wedge^r E \otimes E \otimes \wedge^{r-1} E\) satisfying \(\psi(\hat{e}_i) = \sum_{\pi = (1 \cdots 1) \pi} (\sigma^{-1})^{n_{\pi}} \epsilon_{i_{\pi}} \otimes e_{i_1 \cdots i_{r-1}}\), (where \(i_{\pi}\) indicates that \(i_{\pi}\) is omitted), for \(i = (i_1, \ldots, i_r) \in I(n, r)\) with \(i_1 > \cdots > i_r\), is a \(G\)-module map.

**Proof.** (i) We write \(I_q(n, r)\) for the set of \(i = (i_1, \ldots, i_r) \in I(n, r)\) such that \(i_1 > \cdots > i_r\). We first work over the field \(K = \mathbb{Q}(r)\), where \(r\) is an indeterminate. We write \(G_K\) for the version of \(G\) over \(K\), write \(E_K\) for the natural \(G_K\)-module, \(N_{r,K}\) for the corresponding \(G_K\)-submodule of \(E_K^{*r}\) and write \(\phi_K\) for the corresponding map \(\wedge^r E_K \rightarrow E_K^{*r}\). However, we write simply \(c_{ij}\) for the \((i, j)\) coefficient function over \(K\) (and \(k\)), we write \(e_i\) for a basis element of \(E_K^{*r}\) and \(\hat{e}_i\) for its image in \(\wedge^r E_K\) (for \(i \in [1, n]\) or \(I(n, r)\) and \(j \in [1, r]\)) and hope that no confusion will arise.

First notice that the characters of \(N_{r,K}\) and \(\wedge^r E_K\) are equal and, since \(\wedge^r E_K\) is irreducible, \(N_{r,K}\) is isomorphic to \(\wedge^r E\). Thus we have a \(G_K\)-module isomorphism \(\theta: \wedge^r E_K \rightarrow N_{r,K}\). Each weight space of \(\wedge^r E_K\) is at most one dimensional so that for each \(i \in I_q(n, r)\) there is a scalar \(a_i \in K\) such that \(\theta(\hat{e}_i) = a_i \phi(\hat{e}_i)\). Let \(\eta: E_K^{*r} \rightarrow \wedge^r E_K\) be the natural map. For \(i \in I_q(n, r)\) we have \(\eta \circ \theta(\hat{e}_i) = a_i \sum_{\pi \in \text{Sym}(r)} \text{sgn}(\pi) \epsilon_{i_{\pi}} = a_i (\sum_{\pi \in \text{Sym}(r)} (-1)^{l(\pi)} \epsilon_{i_{\pi}})\) (using the defining relations for \(\wedge^r E_K\); see [5, Sect. 2.1]). Now \(\wedge^r E_K\) is absolutely simple so that \(\eta \circ \theta\) is a multiple of the identity and from the fact that \(\sum_{\pi \in \text{Sym}(r)} (-t)^{l(\pi)} \epsilon_{i_{\pi}} = 0\) we conclude \(a_i = a_{i}\) for all \(i, j \in I_q(n, r)\). Thus \(\phi_K\) is a scalar multiple of \(\theta\) and therefore \(\phi_K\) is a \(G_K\)-module map. Putting \(L = \wedge^r E_K, M = E^{*r}_K\) we have \(\pi_M \circ \phi_K = (\phi_K \otimes \text{id}_{K[G_K]} \circ T_L\). Applying this to \(\hat{e}_i, i \in I_q(n, r)\) we deduce the equa-
\[
\sum_{\pi \in \text{Sym}(r)} \text{sgn}(\pi) c_{i\pi, i} = 0
\]
for any \( (j_1, \ldots, j_r) \in I(n, r) \) which has \( j_a = j_b \) for some \( a \neq b \); (1)

and

\[
\sum_{\pi \in \text{Sym}(r)} (-t)^{l(\pi)} \text{sgn}(\pi') c_{j\pi, i} = \sum_{\pi \in \text{Sym}(r)} \text{sgn}(\pi) c_{j\pi', i\pi}
\]
for all \( j \in I_0(n, r), \pi' \in \text{Sym}(r) \). (2)

However, the equations are identities in \( A_{Z[t]}(n) \) and we have a natural
isomorphism \( k \otimes_{Z[t]} A_{Z[t]}(n) \to A(n) \) induced by the ring homomorphism
\( Z[t] \to k \) taking \( t \) to \( q \), by [5, 2.4.1]. Therefore the equations
obtained from (1), (2) by replacing \( t \) by \( q \) hold in \( A(n) \). These are exactly
the conditions for \( \psi: \wedge^r E \to E^{\otimes r} \) to be a \( G \)-module homomorphism.

(ii) The proof is similar to that of (i). The condition for \( \psi \) to be a
\( G \)-module map is that certain identities in the \( c_{ij} \) hold. By base change
these identities hold for arbitrary \( k, q \) provided that they hold in the
special case \( K = \mathbb{Q}(t), q = t \). In that case \((\text{id}_E \otimes \xi) \circ \phi \) is a non-zero
multiple of \( \psi \), where \( \xi: E^{\otimes (r-1)} \to \wedge^{r-1} E \) is the natural map, hence \( \psi \) is a
\( G \)-map by (i). We leave the details to the reader.

A much simpler proof can be obtained by realizing \( \phi: \wedge^r E \to E^{\otimes r} \)
(resp. \( \psi: \wedge^r E \to E \otimes \wedge^{r-1} E \)) as the adjoint of the natural map \( V^{\otimes r} \to \wedge^r V \)
(resp. \( V \otimes \wedge^{r-1} V \to \wedge^r V \)) via suitable \( G \)-module pairings, see [15,
Section 1.2].

We are now ready to prove the central result of this section.

**Theorem 3.4 (Kempf's Vanishing Theorem).** Let \( a \) be a composition of
\( n \) and let \( \lambda \in X \). If \( \lambda + \rho \in X^+(a) \) then \( R^i \text{Ind}_{B_1}^{G_1} k_{\lambda} = 0 \) for all \( i > 0 \). In
particular, if \( \lambda + \rho \in X^- \) then \( R^i \text{Ind}_{B_1}^{G_1} k_{\lambda} = 0 \) for all \( i > 0 \).

**Proof.** We argue by induction on \( n \). Contemplation of the case \( n = 1 \) is
left to the reader. Assume now that \( n > 1 \) and the result holds for all
compositions of \( n' \) for \( n' < n \). Suppose first that \( a = (a_1, \ldots, a_m) \) with
\( m > 1 \) (and \( a_m > 0 \)). We have \( R^i \text{Ind}^{P(a)}_{B_1} k_{\lambda} = R^i \text{Ind}^{G(a)}_{B_1} k_{\lambda} \), by Lemma 2.12.
Let \( a' = (a_1, \ldots, a_{m-1}) \). We indentify \( B(a) \) with \( B(a') \times B(a_m) \) and \( G(a) \)
with \( G(a') \times G(a_m) \). We have \( R^i \text{Ind}^{G(a')}_{B_1} k_{\lambda} \cong H^i(B(a), k_{\lambda} \otimes k(G(a))) \), for
\( i \geq 0 \), by Lemma 1.3(iii). Now we have a spectral sequence with \( E_2 \)-term
\( H^i(B(a'), H^j(B(a_m), k_{\lambda} \otimes k(G(a)))) \) converging to \( H^*(B(a), k_{\lambda} \otimes k(G(a))) \), by Proposition 1.6(i). We have \( k_{\lambda} = k_{\mu} \otimes k_{\nu} \) for elements
\[ \mu \in X(n - a_m), \nu \in X(a_m). \] Furthermore, we have \( k[G(a)] \cong k[G(a_m)] \otimes k[G(a')] \), so we get \( H^i(B(a), H^i(B(a,m), k_a \otimes k[G(a)])) \cong H^i(B(a), H^i(B(a,m), k_a \otimes k[G(a)])) \). By the inductive hypothesis, we have \( H^i(B(a_m), k_a \otimes k[G(a_m)]) \cong R^i\text{Ind}^G_{B(a_m)} k_a \) for \( j \neq 0 \). The spectral sequence therefore degenerates and we get \( R^i\text{Ind}^G_{B(a)} k_a \cong H^i(B(a), k_a \otimes k[G(a)]) \). This is \( H^0(B(a), k_a \otimes k[G(a)]) \) by Proposition 1.6(ii), (iii), Proposition 1.3(iii), and hence 0 for \( i > 0 \) by the inductive hypothesis.

Now suppose \( a = (n - 1,1) \). We have \( R^i\text{Ind}^G_{B(a)} k_a \cong R^i\text{Ind}^G_{B(a,m)} k_a \), by Lemma 2.12, and this is 0, for \( i > 0 \), by the above.

We let \( S = k[c_{12}, \ldots, c_{nn}] \cong k[G] \) and let \( V \) be the \( k \)-span of \( c_{12}, \ldots, c_{nn} \). It is easy to check that the set of elements \( c_{nn} e_{nn} c_{m,n-1} \ldots c_{11} \), with \( a_{nn} \ldots a_{11} \in N_0 \), forms a \( k \)-basis of \( k[M] \). On the other hand, \( \text{Ind}^G_{B(a)} k_a \) and \( k[G,a] \) with \( a_{ii} = 0 \) for \( 1 \leq i \leq n \), form a \( k \)-basis of \( k[G] \). Hence the elements \( c_{12}, c_{13}, \ldots, c_{nn} \) form a \( k(G) \)-regular sequence.

For a positive integer \( r \), we write \( I_0(n;r) \) for the set of \( i = (i_1, \ldots, i_n) \in I(n,r) \) with \( i_1 > \cdots > i_r > 1 \). We let \( U_i \) be the \( k \)-vector space on basis \( u_i, i \in I_0(n;r) \). We also write \( u_i \) as \( u_{i_1 \cdots i_r} \) for \( i = (i_1, \ldots, i_r) \in I_0(n;r) \). By [23, Theorem 43], we have the Koszul resolution

\[
0 \to k[G] \otimes U_{n-1} \to k[G] \otimes U_{n-2} \to \cdots \to k[G] \otimes U_1 \to k[G] \to k[P] \to 0.
\]

The maps \( \theta_i: k[G] \otimes U_i \to k[G] \otimes U_{i-1}, 1 < r < n \) and \( \theta_1: k[G] \otimes U_1 \to k[G] \) in the above are described as follows. For \( 1 < r < n, f \in k[G] \) and \( i = (i_1, \ldots, i_n) \in I(n,r), \) we have \( \theta_i(f \otimes u_i) = \Sigma_{a=1}^r (-1)^{r-1} f c_{i_a} \otimes u_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_r} \). For \( f \in k[G] \) and \( 2 \leq i \leq n \) we have \( \theta_i(f \otimes u_i) = f c_{1i} \). We write \( \Lambda U \) for the subspace of \( \Lambda E \) spanned by \( e_2, \ldots, e_n \). Let \( 1 \leq r < n \). We write \( \Lambda^r U \) for the subspace of \( \Lambda E \) spanned by \( \hat{e}_r, i \in I_0(n;r) \). We regard \( \Lambda E \) as a \( P \)-module by restriction: let \( \tau \) be the structure map. For \( i = (i_1, \ldots, i_r) \in I_0(n;r) \) we have \( \tau(\hat{e}_r) = \Sigma_{j \in I(n,r)} \hat{e}_{j} \otimes \hat{c}_{ij}, \) where \( \hat{c}_{ij} = k[G] \rightarrow k[P] \) is the natural map. If \( j = (j_1, \ldots, j_r) \in I(n,r) \) then \( j_a = 1 \), for some \( 1 \leq a \leq r \), then \( \hat{c}_{ij} = 0 \) and hence \( \hat{c}_{ij} = 0 \). It follows that \( \tau(\hat{e}_r) \in \Lambda^r U \otimes k[P] \) and so \( \Lambda^r U \) is a \( P \)-submodule of \( \Lambda E \). We make \( U_i \) into a \( P \)-module by transport of structure via the \( k \)-isomorphism \( \eta: U_i \to \Lambda^r U \) given by \( \eta(u_i) = \hat{e}_r, i \in I_0(n;r) \). We claim that each \( \theta_i \) is a \( P \)-module map. It is easy to see that the linear map \( \zeta: U_i \to k[G] \) given by \( \zeta(e_i) = c_{1i} \), for \( 2 \leq i \leq n \), is a \( P \)-module map. Now \( \theta_1 = m \circ (\text{id}_{k[G]} \otimes \zeta) \), where \( m: k[G] \otimes k[G] \to k[G] \) is multiplication. Hence \( \theta_1 \) is a \( P \)-module map. Now suppose \( 1 < r < n \). Then \( \theta_r = m \circ (\zeta \otimes \text{id}_{U_i}) \circ \xi_r, \) where \( \xi_r: U_i \to U_1 \otimes U_{n-1} \) is the \( k \)-linear map such that \( \xi_r(u_i) = \Sigma_{a=1}^r (-1)^{r-1} u_{i_a} \otimes u_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_r}, \)
Thus it suffices to prove that $\xi_r: U_r \to U_{r-1} \otimes U_{r-1}$ is a $P$-module map. But the $G$-module map $\psi: \Lambda^r E \to E \otimes \Lambda^{r-1} E$, of Lemma 3.3(ii), takes $\Lambda^r U$ into $U \otimes \Lambda^{r-1} U$ and so restricts to a $P$-module map $\psi': \Lambda^r U \to U \otimes \Lambda^{r-1} U$. But $\xi_r: U_r \to U_{r-1} \otimes U_{r-1}$ is the map obtained from $\psi': \Lambda^r U \to U \otimes \Lambda^{r-1} U$ by transport of structure and hence $\xi_r$ is $P$-module map. Thus each $\theta_r$ is a $P$-module map. Hence we have an exact sequence of $P$-modules

$$0 \to k[G] \otimes \Lambda^{n-1} U \to \cdots \to k[G] \otimes U \to k[G] \to k[P] \to 0 \ (\dagger)$$

Note that if $\lambda + \rho \in X^+$ but $\lambda = (\lambda_1, \ldots, \lambda_n) \not\in X^+$ then $\lambda_i - \lambda_{i+1} = -1$ for some $1 \leq j < n$ so we get $R^i \text{Ind}_B^G k_\lambda = 0$ for all $i \geq 0$ by Lemma 3.2(ii). For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X^+$ we define $f(\lambda) = n\lambda_1 - (\lambda_1 + \cdots + \lambda_n)$. Suppose that $\lambda \in X^+$ and that, for all $\mu \in X^+$ with $f(\mu) < f(\lambda)$, we have $R^i \text{Ind}_B^G k_\mu = 0$ for all $i > 0$. Tensoring $(\dagger)$ by $k_\lambda$, we obtain the exact sequence

$$0 \to k[G] \otimes \Lambda^{n-1} U \otimes k_\lambda \to k[G] \otimes \Lambda^{n-2} U \otimes k_\lambda \to \cdots$$

$$\to k[G] \otimes U \otimes k_\lambda \to k[G] \otimes k_\lambda \to k[P] \otimes k_\lambda \to 0.$$

Let $1 \leq r \leq n - 1$. As a $B$-module, $\Lambda^r U \otimes k_\lambda$ has a filtration with sections $k_{\lambda_1 + \epsilon_{i_1} + \cdots + \epsilon_{i_r}}$ with $1 \leq i_1 < \cdots < i_r \leq n$. Note that $\lambda + \epsilon_{i_1} + \cdots + \epsilon_{i_r} + \rho$ belongs to $X^+$ and that if $\lambda + \epsilon_{i_1} + \cdots + \epsilon_{i_r} \in X^+$ then $f(\lambda + \epsilon_{i_1} + \cdots + \epsilon_{i_r}) < f(\lambda)$. Thus by the inductive hypothesis and the case already considered, we have $R^i \text{Ind}_B^G k_{\lambda + \epsilon_{i_1} + \cdots + \epsilon_{i_r}} = 0$, for all $i > 0$. By the long exact sequence, we get $R^i \text{Ind}_B^G (\Lambda^r U \otimes k_\lambda) = 0$ and hence by the tensor identity we get $R^i \text{Ind}_B^G (k[G] \otimes \Lambda^r U \otimes k_\lambda) = 0$ for all $i > 0$.

Let $Z_r$ denote the image of the map $k[G] \otimes \Lambda^r U \otimes k_\lambda \to k[G] \otimes \Lambda^{r-1} U \otimes k_\lambda$, for $1 \leq r \leq n - 2$. We have a short exact sequence

$$0 \to k[G] \otimes \Lambda^{n-1} U \otimes k_\lambda \to k[G] \otimes \Lambda^{n-2} U \otimes k_\lambda \to Z_{n-2} \to 0$$

and exact sequences

$$0 \to Z_r \to k[G] \otimes \Lambda^{r-1} U \otimes k_\lambda \to Z_{r-1} \to 0$$

for $n - 2 \geq r \geq 2$. Thus we obtain $R^i \text{Ind}_B^G Z_{n-2} = 0$, for $i > 0$, by the long exact sequence and $R^i \text{Ind}_B^G Z_r = 0$ for $1 \leq r \leq n - 2$ and all $i > 0$ by the long exact sequence and downward induction on $r$. From the exact sequence $0 \to Z_1 \to k[G] \otimes k_\lambda \to k[P] \otimes k_\lambda \to 0$ and the tensor identity, we obtain

$$k[G] \otimes R^i \text{Ind}_B^G k_\lambda = R^i \text{Ind}_B^G (k[P] \otimes k_\lambda)$$
for $i > 0$. Now, by Proposition 1.2, we have a spectral sequence with $E_2$-term $R^i \text{Ind}^P_b R^j \text{Ind}^P_{Gl}(k[P] \otimes k_\lambda)$ converging to $R^i \text{Ind}^P_{Gl}(k[P] \otimes k_\lambda)$. By the tensor identity we have $R^i \text{Ind}^P_{Gl}(k[P] \otimes k_\lambda) \cong k[P] \otimes R^i \text{Ind}^P_{Gl} k_\lambda$, which is an injective $P$-module by Proposition 1.3(iii) and Remark 2.2. So the spectral sequence degenerates and we get $R^i \text{Ind}^P_{Gl}(k[P] \otimes k_\lambda) \cong \text{Ind}^P_{Gl}(k[P] \otimes R^i \text{Ind}^P_{Gl} k_\lambda)$. But we have already shown that $R^i \text{Ind}^P_{Gl} k_\lambda = 0$, for $i > 0$, and the proof is complete.

**Proposition 3.5.** Let $a, b$ be compositions of $n$ and suppose that $P(a) \leq P(b)$. If $V$ is a finite dimensional $P(a)$-module then $R^i \text{Ind}^P_{Gl}(b)V$ is finite dimensional for all $i \geq 0$ and $R^i \text{Ind}^P_{Gl}(b)V = 0$ for all $i > 0$. In particular, if $V$ is any finite dimensional $B$-module then $R^i \text{Ind}^P_{Gl} V$ is finite dimensional for all $i \geq 0$ and $R^i \text{Ind}^P_{Gl} V = 0$ for all $i > 0$.

**Proof.** By Proposition 1.2, we have a spectral sequence with $E_2$-term $R^i \text{Ind}^P_{Gl}(a) \otimes R^j \text{Ind}^P_{Gl}(b)V$ converging to $R^i \text{Ind}^P_{Gl}(b)V$. Furthermore, we have $R^i \text{Ind}^P_{Gl}(a) \otimes V \otimes R^j \text{Ind}^P_{Gl}(b)k_\lambda = 0$, which is $V$ for $t = 0$ and 0 for $t > 0$, by the tensor identity, Lemma 2.10 and the Kempf's vanishing theorem. Thus the spectral sequence degenerates and we get $R^i \text{Ind}^P_{Gl}(a)V \cong R^i \text{Ind}^P_{Gl}(b)V$, for all $i \geq 0$. We may therefore assume that $P(a) = B$. By the long exact sequence we may assume that $V$ is simple, i.e., $V \cong k_\mu$ for some $\mu \in X$.

Let $\lambda$ be the element of $X^+(b)$ which is $\text{Sym}(b)$-conjugate to $\mu$ and let $\tau$ be the $\leq b$ minimal element of the $\text{Sym}(b)$-orbit of $\lambda$. Let $Z = \{ \nu \in X : \tau \leq w \nu \leq b \lambda \}$ for all $w \in \text{Sym}(b)$). We prove by descending induction that $R^i \text{Ind}^P_{Gl} k_\nu$ is finite dimensional for all $i$ and is 0 for all $i > 0$. Note that $\text{Ind}^P_{Gl}(b)k_\nu$ is finite dimensional by Theorem 2.10(i). If $\nu \in X^+(b)$ (in particular if $\nu = \lambda$) then $R^i \text{Ind}^P_{Gl}(b)k_\nu = 0$ for all $i > 0$ by Kempf's vanishing theorem. So we assume that $\nu = (\nu_1, \ldots, \nu_s)$ does not belong to $X^+(b)$. Thus we have $\nu_i - \nu_{i+1} = -m < 0$ for some $j$ satisfying $j < b_1$ or $b_1 + \cdots + b_r + 1 \leq j < b_1 + \cdots + b_r + b_{r+1}$ with $1 \leq r < m$. Let $c = (1, \ldots, 1, 2, 1, \ldots, 1)$, with 2 in the $j$th position, and $\phi = \nu + me_j - (m - 1)e_{j+1}$, and $U = \text{Ind}^P_{Gl}(c)k_\phi \otimes k_{-e_j}$. Then Lemma 3.1 and an easy spectral sequence argument give that $R^i \text{Ind}^P_{Gl}(b)U = 0$ for all $i \geq 0$. Moreover, identifying $k_\nu$ with the $B$-socle of $U$, we have a short exact sequence $0 \rightarrow k_\nu \rightarrow U \rightarrow U/k_\nu \rightarrow 0$. Hence, we get $R^i \text{Ind}^P_{Gl}(b)k_\nu = R^{i-1} \text{Ind}^P_{Gl}(U/k_\nu)$. It is easy to check that all weights of $U/k_\nu$ belong to $Z$ and are greater than $\nu$. The result therefore follows from the long exact sequence and the inductive hypothesis.

One now obtains Weyl's character formula, as in [8, (2.2.6)],. Thus we have the following formula, valid in the field of fractions of the integral group ring of $\mathbb{R} \otimes \mathbb{Z} X$. 

| Q U A N T U M  G L \_n | 257 |
THEOREM 3.6 (Weyl’s Character Formula). For \( \lambda \in X^+ \) we have
\[
\chi(\lambda) = \frac{\text{ch Ind}^G_{B} k_{\lambda}}{\text{ch Ind}^G_{B} k_{\lambda}} = \sum_{w \in \text{Sym}(\lambda)} \text{sgn}(w) e^{\pi(k + r)} / \sum_{w \in \text{Sym}(\lambda)} \text{sgn}(w) e^{\pi r}.
\]

We put \( \chi(\lambda) = \text{ch Ind}^G_{B} k_{\lambda} \), for \( \lambda \in X^+ \).

Remark 3.7. If \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \) then \( \text{ch Ind}^G_{B} k_{\lambda} \) is Schur’s symmetric function \( s_{\lambda} \); see [21, I, (3.1)]. In particular, for \( \lambda = (r, 0, \ldots, 0) \) the character of \( \text{Ind}^G_{B} k_{\lambda} \) is the character of the symmetric power \( S^r(E) \) of the natural module considered in [5, 2.1.9]. There is an obvious isomorphism from \( S^r(E) \) to the submodule \( V \) of \( k[G] \) consisting of the \( k \)-span of the elements \( c_{i_1}^1 c_{i_2}^2 \cdots c_{i_r}^r \) with \( i_1 + \cdots + i_r = r \). It follows that we have equality \( \text{Ind}^G_{B} k_{\lambda} = V \) and therefore \( \text{Ind}^G_{B} k_{(r, 0, \ldots, 0)} \equiv S^r(E) \). For a description of the submodule structure of \( S^r(E) \), see [30].

Similarly, for \( 1 \leq r \leq n \), we have \( \text{Ind}^G_{B} k_{\omega_r} = \Lambda^r(E) \), where \( \omega_r = (1, \ldots, 1, 0, \ldots, 0) \), and \( \Lambda^r(E) \) is the \( r \)th exterior power of \( E \), as in [5, 2.1.4].

We shall prove a more precise version of part of Proposition 3.5 known as “Grothendieck vanishing”; i.e., \( R^i \text{Ind}^G_{B} = 0 \) for \( i > (q) \). The form of the proof (unfortunately) depends on whether or not \( q \) is a root of unity. In case \( q \) is not a root of unity this is deduced from the stronger result which is known as Bott’s theorem in the analogous situation of reductive groups in characteristic 0. The proof of Bott’s Theorem given here is essentially Demazure’s “very simple proof of Bott’s Theorem” [3], but with the direction of induction reversed to allow for the fact that we already have Kempf’s vanishing theorem but do not yet have vanishing of \( R^i \text{Ind}^G_{B} \) for \( i > (q) \). In the case in which \( q \) is a root of unity we obtain “Grothendieck vanishing” as in [26] (but without restrictions on \( q \)) by viewing \( G \) as a covering group of classical \( GL_n \).

LEMMA 3.8. Suppose that \( q \) is not a root of unity. Let \( 1 \leq r \leq n \) and let \( a = (1, \ldots, 1, 2, 1, \ldots, 1) \), where 2 appears in the \( r \)th position.

(i) Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in X \) and suppose that \( \lambda_r = \lambda_{r+1} \geq 0 \). Then \( \text{Ind}_{B(a)}^{G(a)} k_{\lambda} \) is a uniserial module and irreducible as a \( B(a) \)-module.

(ii) Let \( \mu \in X \). The \( B(a) \)-module injective hull \( E_a(\mu) \) of \( k_{\mu} \) is uniserial. For each \( m \in \mathbb{N} \) there exists a unique (up to isomorphism) \( B(a) \)-module which has socle \( k_{\mu} \) and dimension \( m \).

Proof. (i) Let \( \lambda_r = \lambda_{r+1} = m \). We have \( \text{Ind}_{B(a)}^{G(a)} k_{\lambda} \equiv k_{a} \otimes Y_m \), where \( Y_m \) and \( \mu \) are as in Lemma 3.1. Hence it suffice to prove that \( Y_m \) is a uniserial \( B(a) \)-module. Let \( V = Y_m \). We write simple \( c_{ij} \) for the restriction of \( c_{ij} \in k[G] \) to \( G(a) \) and \( \overline{c}_{ij} \) for the restriction of \( c_{ij} \in k[G] \) to \( B(a) \) (for \( 1 \leq i, j \leq n \)). Let \( v_0 = e_{r, r+1}, v_1 = e_{r, r+1} e_{r+1, r+2}, \ldots, v_m = e_{r, r+1} \) and let \( V_j \) de-
note the \( k \)-span of \( v_0, \ldots, v_j \) for \( 0 \leq j \leq m \). Then \( 0 < V_0 < V_1 < \cdots < V_m = V \) is a \( B(a) \)-module composition series of \( V \). Let \( W \) be a non-zero \( B(a) \)-submodule of \( V \). We must show that \( W \) is equal to \( V_j \) for some \( j \). Let \( T = T(n) \). Then \( W \) is a \( T \)-submodule of \( V \). Now \( V = \bigoplus_{j=0}^{m} k v_j \) is the weight space decomposition of \( V \), in particular every non-zero weight space has dimension 1. Thus we have \( W = \bigoplus_{j \in J} k v_j \) where \( J \) is a non-empty subset of \( \{0, 1, \ldots, m\} \). Let \( l \) be maximal such that \( v_l \in W \). We claim that \( v_j \) belongs to the \( B(a) \)-submodule generated by \( v_l \), for every \( j \leq l \), and hence \( W = V_l \). Let \( \tau : k[G(a)] \rightarrow k[G(a)] \otimes k[B(a)] \) be the structure map of \( k[G(a)] \) regarded as a \( B(a) \)-module and regard \( V \) as a submodule of \( k[G(a)] \). We have

\[
\tau(v_l) = (c_{ii} \otimes c_{ii} + c_{i,i+1} \otimes c_{i+1,i})^{l} (c_{i,i+1} \otimes c_{i+1,i+1})^{m-l}.
\]

Putting \( x = c_{ii} \otimes c_{ii}, y = c_{i,i+1} \otimes c_{i+1,i} \), we have \( xy = qxy \) so that \( (x + y)^l = \sum_{j=0}^{l} q^j y^{l-j} \). Here \( [s] = 1 + q + \cdots + q^{s-1} \), for a non-negative integer \( s \) and

\[
\begin{bmatrix} s \\ t \end{bmatrix} = \frac{s(s-1) \cdots (s-t+1)}{[t][t-1] \cdots [1]}
\]

is the \( q \)-binomial coefficient, for non-negative integers \( s, t \) with \( s \geq t \).

Thus we get

\[
\tau(v_l) = \sum_{j=0}^{l} \binom{l}{j} v_j \otimes c_{i,i+1}^{l-j} c_{i+1,i}^{m-l}.
\]

The elements \( c_{i,i+1}^{l-j} c_{i+1,i}^{m-l} (\text{for } j = 0, \ldots, l) \) are linearly independent and it follows from \([16, (1.2b)(ii)]\) that \([v_j] \) belongs to the \( B(a) \)-submodule generated by \( v_l \). But \([v_j] \neq 0 \) since \( q \) is not a root of unity and hence \( v_j \) belongs to the submodule of \( W \) generated by \( v_l \), for every \( j \leq l \), and hence \( W = V_l \), as required.

Let \( L \) be the \( G(a) \)-module socle of \( \text{Ind}_{B(a)}^{G(a)} k \). Then \( L \) is an irreducible \( G(a) \)-module of highest weight \( \lambda \). Suppose for a contradiction that \( L \neq \text{Ind}_{B(a)}^{G(a)} k \). Let \( N \) be the sum of all weight spaces of \( \text{Ind}_{B(a)}^{G(a)} k \) which are less than \( \lambda \). Then \( N \) is a \( B(a) \)-submodule. But \( L \neq N \) and \( N \neq L \), contradicting uniseriality. One can also get irreducibility from \([30, 3.1 \text{ Theorem}]\).

(ii) It follows from Lemma 2.8(i) that \( E_{\mu} k \) is infinite dimensional and hence we can find a submodule of dimension \( m \) for each \( m \in \mathbb{N} \). Thus there is a \( B(a) \)-module with socle \( k_{\mu} \), for each \( m \in \mathbb{N} \). We now show
that \( E_s(\mu) \) is uniserial. Now \( E_s(\mu) \cong k_\mu \otimes E_s(0) \) (since \( k_\mu \otimes E_s(0) \) is injective and has socle \( k_\mu \)) so we may assume \( \mu = 0 \). Let \( V_1, V_2 \) be submodules of \( E_s(0) \). We must show that \( V_1 \leq V_2 \) or \( V_2 \leq V_1 \). If \( V_1 \nleq V_2 \) then there is a finite dimensional submodule \( U_1 \) of \( V_1 \) with \( U_1 \nsubseteq V_2 \). Also, there exists a finite dimensional submodule \( U_2 \) of \( V_2 \) with \( U_2 \nsubseteq V_1 \). Thus \( U_1 \nleq U_2 \) and \( U_2 \nleq U_1 \). We may therefore assume that \( V_1 \) and \( V_2 \) are finite dimensional. By Lemma 2.8(i), we may choose \( l \) so large that every weight of \( V_1 \) or \( V_2 \) has the form \( j\alpha \) for some \( 0 \leq j \leq l \), where \( \alpha = \alpha_{r,r+1} = \epsilon_r - \epsilon_{r+1} \). Now \( Y = \text{Ind}^{G(\mu)}_{B(\mu)} k_{\epsilon_r} \otimes k_{-\epsilon_{r+1}} \) has simple socle \( k = k_0 \) and weights \( 0, \alpha, \ldots, l\alpha \), by Lemma 3.1(i). Hence \( Y \) embeds in \( E_s(0) \) with image \( Z \), say. Each weight space of \( E_s(0) \) has dimension 0 or 1, by Lemma 2.6(i), so that \( E_s(0)^A = Z^+ \), for \( \lambda = j\alpha \), \( 0 \leq j \leq l \). Hence \( Z \) contains \( V_1 \) and \( V_2 \). But \( \text{Ind}^{G(\mu)}_{B(\mu)} k_{\epsilon_r} \), and hence \( Y = (\text{Ind}^{G(\mu)}_{B(\mu)} k_{\epsilon_r}) \otimes k_{-\epsilon_{r+1}} \), is uniserial (i). Thus \( Z \cong Y \) is uniserial and therefore \( V_1 \leq V_2 \) or \( V_2 \leq V_1 \). Hence \( E_s(\mu) \) is uniserial. Now if \( R_1, R_2 \) are \( B(\alpha) \)-modules of dimension \( m \) with socle \( k_\mu \), then these modules embed in \( E_s(\mu) \) with images \( R_1', R_2' \). By uniseriality we have \( R_1' \leq R_2' \) or \( R_2' \leq R_1' \) and by dimensions therefore \( R_1' = R_2' = R \). Thus \( R_1' \cong R \cong R_2' \), as required.

**Theorem 3.9 (Bott’s Theorem).** Suppose that \( q \) is not a root of unity. For \( w \in \text{Sym}(n) \) and \( \lambda \in \mathcal{X} \) with \( \lambda + \rho \in \mathcal{X}^+ \) we have

\[
R^i \text{Ind}^G_B k_{w, \lambda} = \begin{cases} \text{Ind}^G_B k_\lambda, & \text{if } i = l(w); \\ 0, & \text{otherwise.} \end{cases}
\]

**Proof.** First suppose that \( \lambda \notin \mathcal{X}^+ \). We shall show that \( R^i \text{Ind}^G_B k_{w, \lambda} = 0 \) for all \( i \), by induction on \( i \). We have \( \text{Ind}^G_B k_{w, \lambda} = 0 \) by Lemma 3.2. Now suppose \( i > 0 \) and let \( \mu = w \cdot \lambda \). Since \( \mu \notin \mathcal{X}^+ \) we have \( \mu_j - \mu_{j+1} = -m < 0 \), for some \( j \), by Lemma 3.2(i). Let \( P = P(a) \), where \( a = (1, \ldots, 1, 2, 1, \ldots, 1) \), with 2 in the \( j \)-th position. We get \( R^i \text{Ind}^G_B k_{\mu} \cong R^{i-1} \text{Ind}^G_B k_{\mu} \), by Lemma 3.1. But \( R^i \text{Ind}^G_B k_{\mu} \) and \( \text{Ind}^G_B k_{s, \mu} \) have the same character and \( \text{Ind}^G_B k_{s, \mu} \) is irreducible, as a \( P \)-module, by Lemma 3.1(i), hence these modules are isomorphic. We therefore get \( R^i \text{Ind}^G_B k_{\mu} = R^{i-1} \text{Ind}^G_B \text{Ind}^G_B k_{s, \mu} \), and this is \( R^{i-1} \text{Ind}^G_B k_{s, \mu} = R^{i-1} \text{Ind}^G_B k_{s, \mu} \), by Lemma 3.1. Thus we get \( R^i \text{Ind}^G_B k_{\mu} = 0 \) by the inductive hypothesis.

Now suppose that \( \lambda \in \mathcal{X}^+ \). Then the \( \text{Sym}(n) \)-orbit of \( \lambda + \rho \) is regular. Note that for \( w = 1 \) we get \( R^i \text{Ind}^G_B k_{\lambda} = 0 \) for all \( i \neq 0 = l(1) \) by the vanishing theorem. Now suppose that \( l(w) > 0 \) and that the result holds for all \( w' \in \text{Sym}(n) \) with \( l(w') < l(w) \). Then \( \mu = w \cdot \lambda \notin \mathcal{X}^+ \) so we have \( \text{Ind}^G_B k_{\mu} = 0 \). Let \( 1 \leq j < n \) be such that \( \mu_j - \mu_{j+1} = -m < 0 \). As above we have \( R^i \text{Ind}^G_B k_{\mu} = R^{i-1} \text{Ind}^G_B k_{s, \mu} = R^{i-1} \text{Ind}^G_B k_{s, \mu} \). Furthermore, we have \( l(s_jw) = l(w) - 1 \) and the assertion for \( w \cdot \lambda \) follows from the inductive assumption.
**Proposition 3.10 (Grothendieck Vanishing).** We have $R^i \text{Ind}_{B}^{G} = 0$ for all $i > (\frac{n}{2})$.

**Proof.** First suppose that $q$ is not a root of unity. The functors $R^i \text{Ind}_{B}^{G}$ commute with direct limits so it suffices to show that $R^i \text{Ind}_{B}^{G} V = 0$ for every finite dimensional $B$-module $V$. By the long exact sequence, it suffices to show that $R^i \text{Ind}_{B}^{G} k_{\mu} = 0$ for every $\mu \in X$. Writing $\mu = w \cdot \lambda$ for some $\lambda + \rho \in X^+$ we see that this follows from Bott's theorem and the fact that $l(w) \leq (\frac{n}{2})$ for all $w \in \text{Sym}(n)$.

Now suppose that $q$ is a primitive $l$th root of unity, for some integer $l$. Let $k[\widehat{G}]$ be the subalgebra of $k[G]$ generated by $c_{xy}^i, d^{-i}$, for all $1 \leq x, y \leq n$, and $d^{-i}$. Then $k[\widehat{G}]$ is central in $k[G]$, by [5, 1.3.2 Corollary]. We denote by $G_1$ the corresponding subgroup (see Section 1). Then we get from Corollary 2.4 that $k[G]$ is free over $k[\widehat{G}]$ on $(c_{xy}^i : 0 \leq a_{xy} < l$ for all $1 \leq x, y \leq n)$. We also get that $k[G_1]$ has $l$-dimension $l^n$, in particular $G_1$ is finite. Let $\overline{B}$ be the quantum group whose coordinate algebra $k[\overline{B}]$ is the image of $k[\widehat{G}]$ under the restriction map $k[G] \to k[B]$, and let $B_1$ be the corresponding subgroup of $B$. Then Corollary 2.4 gives that $k[B]$ is free over $k[\overline{B}]$ of rank $l(\frac{n}{2})$, which is the dimension of $k[B_1]$.

Let $V$ be a $B$-module. For $i \geq 0$ we have $R^i \text{Ind}_{B}^{G} \cong H^i(B, V \otimes k[G])$, by Proposition 1.3(iii). We also have a spectral sequence with $E^2$-term $H^i(\overline{B}, H^j(B_1, V \otimes k[G]))$ converging to $H^*(B, V \otimes k[G])$. Since $B_1$ is finite $k[G]_{B_1}$ is injective, by the Theorem of [14]. (One could also argue that $k[G]_{B_1}$ is injective by Proposition 1.5(iii) and the restriction of an injective $G_1$ to $B_1$ is injective by the freeness theorem of Nichols and Zoeller [24], so that $k[G]_{B_1} = (k[G]_{G_1})_{B_1}$ is injective.) Therefore the spectral sequence degenerates and we get $H^i(B, V \otimes k[G]) \cong H^i(\overline{B}, H^j(B, V \otimes k[G]))$. Now let $S$ be a finite dimensional submodule of $k[G]$ such that the restriction map $S \to k[G_1]$ is surjective. This map splits over $G_1$ since $k[G_1]$ is injective and hence projective (by the self injectivity of finite dimensional Hopf algebras [20]). Thus $k[G_1]$ embeds in $S$, as a $G_1$-module. Applying $\text{Ind}_{G_1}^{G}$ to inclusion an injection $k[G_1] \to S$ gives a $G$-module embedding $k[G] \to S \otimes \text{Ind}_{G_1}^{G} k = S \otimes k[\overline{G}]$, which is $G$-split, as $k[G]$ is injective. Thus we have an embedding of vector space $H^i(\overline{B}, H^j(B_1, V \otimes k[G])) \cong H^i(\overline{B}, H^j(B_1, S \otimes V) \otimes k[\overline{G}])$. Since $k[\overline{G}]$ is trivial as a $G_1$-module, and hence $B_1$-module, we get $H^i(\overline{B}, H^j(B_1, S \otimes V) \otimes k[\overline{G}]) \cong H^i(\overline{B}, Z \otimes k[\overline{G}])$, for some $\overline{B}$-module $Z$. However, we also have $H^i(B, Z \otimes k[\overline{G}]) \cong R^i \text{Ind}_{B}^{G} Z$, by Proposition 1.3(iii). But $i > (\frac{n}{2}) = \dim \overline{G}/\overline{B}$ so that $R^i \text{Ind}_{B}^{G} Z$ vanishes, by the classical form of Grothendieck vanishing for algebraic groups [19, 1, 5.12 Prop. b]). Hence $R^i \text{Ind}_{B}^{G} V = 0$, as required.
We now connect the representation theory of quantum $GL_n$ with the $q$-Schur algebras. A great deal of information on the Schur algebras and their $q$-analogues (treated from the homological point of view relevant here) can be found in the recent book by Martin [22].

For $\lambda \in X^+$ we put $\nabla(\lambda) = \text{Ind}^G_B k_{\lambda}$. We put $\Delta(\lambda) = \nabla(\lambda^*)^*$, where $\lambda^* = -w_0 \lambda$ and $w_0$ is the longest element of $\text{Sym}(n)$. The module $\nabla(\lambda)$ has simple socle $L(\lambda)$ and $\Delta(-w_0 \lambda) = \nabla(\lambda)^*$ has simple head $L(-w_0 \lambda)$. Thus $Z = \Delta(-w_0 \lambda)^*$ has simple socle $L(-w_0 \lambda)^* \cong L(\lambda)$. Now $\text{ch} Z = \chi(\lambda)$ so $Z$ has highest weight $\lambda$ giving rise to a non-zero $B$-homomorphism $Z \to k$. We obtain, via Frobenius reciprocity, a $G$-homomorphism $\theta: Z \to \text{Ind}^G_B k_{\lambda} = \nabla(\lambda)$, which is non-zero on $\lambda$ weight spaces. Thus $\theta$ is injective on the $G$-socle of $Z$ and hence a monomorphism. But $\dim Z = \dim \nabla(\lambda)$ so that $\theta$ is an isomorphism. Thus we have $\Delta(\lambda)^* \cong \nabla(\lambda)$. Replacing $\lambda$ by $\lambda^*$ we get the first part of the following. The argument also yields $\Delta(\lambda^*)^* \cong \nabla(\lambda^*)$.

(1) For $\lambda \in X^+$ we have $\Delta(\lambda)^* \cong \nabla(\lambda^*)$ and $\nabla(\lambda)^* \cong \Delta(\lambda^*)$.

The following is deduced from Kempf’s Vanishing Theorem as in the reductive group case (due to Cline, Parshall, Scott and van der Kallen; see [19, II, 4.13 Proposition])

(2) For $\lambda, \mu \in X^+$ and $i \geq 0$ we have

$$\text{Ext}^i_G(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k, & \text{if } i = 0 \text{ and } \mu = -w_0 \lambda; \\ 0, & \text{otherwise}. \end{cases}$$

Let $V \in \text{Mod}(G)$. We say that an ascending $G$-module filtration $0 = V_0 \leq V_1 \leq \cdots$ (with $V = \bigcup_{i=0}^\infty V_i$) is good if each successive quotient $V_i/V_{i-1}$ is either 0 or isomorphic to $\nabla(\lambda_i)$ for some $\lambda_i \in X^+$. More generally, for a composition $a$ of $n$ and $V \in \text{Mod}(P(a))$ we say an ascending filtration $0 = V_0 \leq V_1 \leq \cdots$ with $V = \bigcup_{i=0}^\infty V_i$ is good if, for each $i > 0$, $V_i/V_{i-1}$ is either 0 or isomorphic to $\text{Ind}^P_B k_{\lambda_i}$, for some $\lambda_i \in X^+$. It is easy to prove the following as in [8, Sect. 4.2] (see also [31] for the first part).

(3) (i) For $\lambda, \mu \in X^+$ the $G$-module $\nabla(\lambda) \otimes \nabla(\mu)$ has a good filtration.

(ii) For $\lambda \in X^+$ and a composition $a = (a_1, \ldots, a_m)$ of $n$ the restriction $\nabla(\lambda)_{P(a)}$ has a good filtration.
The multiplicity of $\nabla(\lambda)$ in a $G$-module $V$ which has a good filtration is independent of the choice filtration, we denote this number $(V : \nabla(\lambda))$. We have:

(4) If $V \in \text{Mod}(G)$ has a good filtration then $(V : \nabla(\lambda)) = \dim(V \otimes \nabla(\lambda))^G$.

A subset $\pi$ of $X^+$ is called saturated if $\lambda \in \pi$ and $\mu \leq \lambda$ with $\mu \in X^+$ implies $\mu \in \pi$. Let $\pi$ be a finite saturated subset of $X^+$. If $V$ is a $G$-module we denote by $O_\mu(V)$ the largest $G$-submodule of $V$ such that all composition factors of $V$ belong to the set $\{L(\lambda) : \lambda \in \pi\}$. We have the Generalized Schur coalgebra $A(\pi) = O_\mu(k[G])$, a finite dimensional subcoalgebra of $k[G]$. We call the dual algebra $S(\pi) = A(\pi)^*$ a generalized Schur algebra. One gets, as in the "classical case" [9, 2.2c, 2.2d] that $S(\pi)$ has dimension $\sum_{\lambda \in \pi} (\dim \nabla(\lambda))^2$. Any $S(\pi)$-module is naturally a $G$-module and for $S(\pi)$-modules $V_1, V_2$ we have:

(5) $\text{Ext}^i_{S(\pi)}(V_1, V_2) \cong \text{Ext}^i_{A(\pi)}(V_1, V_2)$ for all $i \geq 0$.

Also, as in the "classical case" one has (cf. [9, 2.2h]):

(6) $\{L(\lambda) : \lambda \in \pi\}$ is a full set of simple $S(\pi)$-modules. For $\lambda \in \pi$, let $I(\lambda)$ be the injective hull of $L(\lambda)$, as an $S(\pi)$-module. Then $I(\lambda)$ has a good filtration, with sections in $\nabla(\mu)$, $\mu \in \pi$, and the filtration multiplicity $(I(\lambda) : \nabla(\mu))$ (for $\mu \in \pi$) is equal to the composition multiplicity $[\nabla(\mu) : L(\lambda)]$.

Thus the category of $S(\pi)$-modules is a high weight category, equivalently $S(\pi)$ is a quasi-hereditary algebra in the sense of Cline et al. [2]. As in the classical case [10, 1.3], taking $\pi = \Lambda^+(n, r)$, we get $S(n, r)$, the Schur algebra, i.e., the algebra dual $A(n, r)^*$ of the coalgebra $A(n, r)$.

Remark. The proof given in [26, 11.5.2] that $S(n, r)$ is quasi-hereditary is valid when $q$ is not a root of unity or $\pm q$ is an odd root of unity. The proof given there (essentially the argument given by Parshall [25] for ordinary Schur algebras) is rather different from the above. The key point is that the module $\Delta(\lambda)$ (for $\lambda \in \Lambda^+(n, r)$) is the universal module of high weight $\lambda$ and has dimension independent of $q$ and $k$. (This is closely related to the universal property for Weyl modules for algebraic groups, first proved by Humphreys; see [18, Sect. 1].) These properties follow also in our setup, from (1) and Weyl's Character Formula, Theorem 3.6. The Parshall–Wang argument is therefore valid in the present context. This argument is also given [22, pp. 200–202], but without justification of the "universal property of Weyl modules."

We prove now a general result on quasihereditary algebras. Let $S$ be a finite dimensional $k$-algebra. We assume that $S$ is Schurian, that is
Endₜₜₑ(赳) = k, for every simple ₜₜₑ-module 赳. Let {赳(ₜₜₑ): ₜₜₑ ∈ ₜₜₑ} be a full set of simple ₜₜₑ-modules and let ≤ be a partial ordering on ₜₜₑ with respect to which ₜₜₑ is quaquereditary in the sense of Cline et al. [2]. For each ₜₜₑ ∈ ₜₜₑ let Δ(ₜₜₑ) and ℱ(ₜₜₑ) be respectively the standard and costandard modules labelled by ₜₜₑ. By a partial tilting module, we mean a finite dimensional ₜₜₑ-module ₜₜₑ such that ₜₜₑ has a filtration whose sections are all isomorphic to standard modules and a filtration whose sections are all isomorphic to costandard modules. By a result of Ringel [27], for each ₜₜₑ there exists an indecomposable partial tilting module ₜₜₑ such that in a costandard filtration of ₜₜₑ the module ℱ(ₜₜₑ) occurs exactly once as a section and all other sections have the form ℱ(ₜₜₑ) for some ₜₜₑ < ₜₜₑ. Furthermore, every partial tilting module is a direct sum of copies of ₜₜₑ, ₜₜₑ ∈ ₜₜₑ.

For a finite dimensional ₜₜₑ-module ₐ, we denote by [ₐ] the corresponding element of the Grothendieck group of finite dimensional ₜₜₑ-modules. If ₜₜₑ is a quaquereditary algebras with ₜₜₑ ₜₜₑ for all ₜₜₑ, then ₜₜₑ is semisimple.

Proof. Let ₐₜₜₑ denote the ₜₜₑ-module injective hull of ₜₜₑ(ₜₜₑ), for ₜₜₑ ∈ ₜₜₑ. We prove by induction that ₜₜₑ(ₜₜₑ) = ₐₜₜₑ(ₜₜₑ) = ₜₜₑ(ₜₜₑ) = Δ(ₜₜₑ), for all ₜₜₑ ∈ ₜₜₑ. We suppose ₜₜₑ ∈ ₜₜₑ and that this property holds for all λ < ₜₜₑ. Suppose, for a contradiction, that (ₜₜₑ(ₜₜₑ): ℱ(ₜₜₑ)) ≠ 0 for some λ < ₜₜₑ and let ₜₜₑ be minimal with this property. Then ℱ(ₜₜₑ) embeds in ₜₜₑ(ₜₜₑ), and by injectivity of ℱ(ₜₜₑ) we have that ℱ(ₜₜₑ) is a direct summand and, by indecomposability, ₜₜₑ(ₜₜₑ) = ℱ(ₜₜₑ). This is impossible, as (ₜₜₑ(ₜₜₑ): ℱ(ₜₜₑ)) = 1 and so ₜₜₑ(ₜₜₑ) = ℱ(ₜₜₑ). The partial tilting module ₜₜₑ(ₜₜₑ) also has a standard filtration and we must have ₜₜₑ(ₜₜₑ) = Δ(ₜₜₑ), by the hypothesis. However, ℱ(ₜₜₑ) has simple socle ₜₜₑ(ₜₜₑ) and Δ(ₜₜₑ) has simple head ₜₜₑ(ₜₜₑ) so we get ₜₜₑ(ₜₜₑ) ≅ ℱ(ₜₜₑ) ≅ Δ(ₜₜₑ) ≅ ₜₜₑ(ₜₜₑ). Now ₜₜₑ(ₜₜₑ) ≅ ₐₜₜₑ(ₜₜₑ), for some ₜₜₑ ∈ ₜₜₑ and ₜₜₑ(ₜₜₑ) ≅ ℱ(ₜₜₑ) has a simple socle ₜₜₑ(ₜₜₑ) so ₜₜₑ = ₜₜₑ and hence ₜₜₑ(ₜₜₑ) is injective. Thus we have ₜₜₑ(ₜₜₑ) = ₐₜₜₑ(ₜₜₑ) = ₜₜₑ(ₜₜₑ) = Δ(ₜₜₑ), for all ₜₜₑ ∈ ₜₜₑ. Thus every simple module is injective and therefore ₜₜₑ is semisimple.

The following result is obtained in [5, (3.3.2)] and [26, (11.4.4)] with the aid of Hecke algebras:

(8) Mod(G) is semisimple if and only if q is not a root of unity or k has characteristic 0 and q = 1.

Proof. Suppose that q ≠ 1 and q is an lth root of unity. We have ℱ(lₑₙ) = S'(ᴱ). Now ℱ(lₑₙ) has a simple socle and the k-span of the lth
powers $e_1^j, \ldots, e_n^j$ of the canonical basis elements of $E$ forms a proper submodule. Hence $\text{Mod}(G)$ is not semisimple. If $q = 1$ and $k$ has characteristic $p > 0$ the $k$-span of $e_1^j, \ldots, e_n^j$ is a proper submodule of $\nabla(p e_i)$ and so $\text{Mod}(G)$ is not semisimple.

Now suppose that $q$ is not a root of unity or $q = 1$ and $k$ has characteristic $0$. For $j \geq 0$, the symmetric power $\nabla(j e_i) = S^j(E)$ is simple, by [30, 3.1 Theorem]. Hence $S^j(E) = L(j e_i) = \Delta(j e_i)$. Thus $S^j(E) \otimes \cdots \otimes S^j(E)$ is a partial tilting module, for all $j_1, \ldots, j_m \geq 0$. However, every injective $S(n, r)$-module is a direct summand on $S^j(E) \otimes \cdots \otimes S^j(E)$, for some $j_1, \ldots, j_m \geq 0$ with $j_1 + \cdots + j_m = r$ (by the argument of [17, Sect. 4.8]; see also [11, 3.4 Lemma]). Thus the injective modules coincide with the partial tilting modules and so, by (7), the algebra $S(n, r)$ is semisimple for all $r \geq 0$. Thus every $\mathcal{A}(n, r)$-comodule is semisimple and, by [17, Remark following 2.2c], every $M(n)$-module is semisimple. Given a finite dimensional $G$-module $V$ the module $V \otimes D^r$ is naturally an $M(n)$-module, for $r > 0$, where $D$ is the determinant module, i.e., the one dimensional $G$-module of weight $(1, \ldots, 1)$. It follows that every finite dimensional $G$-module is semisimple and therefore every $G$-module is semisimple.

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