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Hyperbolic sections in surface bundles

Kazuhiro Ichihara [a,∗,1], Kimihiko Motegi [b,2]

[a] *College of General Education, Osaka Sangyo University, 3-1-1 Nakagaito, Daito, Osaka 574-8530, Japan*
[b] *Department of Mathematics, Nihon University, 3-25-40 Sakurajosui, Setagaya-ku, Tokyo 156-8550, Japan*

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Dedicated to Takao Matumoto on the occasion of his 60th birthday

Abstract

We give conditions assuring that the given section in a surface bundle over the circle is hyperbolic in terms of the "projection" in the fiber surface according to the Nielsen–Thurston types of the monodromies.

MSC: primary 57M50; secondary 57N10, 37E30

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1. Introduction

As usual, we mean by a *knot* an embedding of the circle S^1 or its image in a 3-manifold. By the well-known Thurston's Uniformization Theorem [12,10] the exterior of a knot is canonically decomposed into geometric pieces. Thus the knots are classified into some classes with respect to the decompositions and the geometric structures admitted by the exteriors.

It is widely believed that the most interesting and richest class is that of *hyperbolic knots*; the knots with the complements admitting a complete hyperbolic metric of finite volume. See [4] for a survey. Thus it seems natural to ask: How to recognize whether the given knot is hyperbolic or not?

In the present paper, we focus on knots appearing as sections of surface bundles over the circle, and give an answer to the above question.

Let F be an orientable, closed surface with negative Euler characteristic and f an automorphism of F which fixes a specified point $x_0 \in F$. (For later convenience, we assume that the condition "$f(x_0) = x_0$" implies also that "$f(D_0) = D_0$" for some small diskal neighborhood D_0 of x_0.) Then the manifold M_f obtained from $F \times [0, 1]$ by identifying $(x, 0) \in F \times \{0\}$ with $(f(x), 1) \in F \times \{1\}$ is a fiber bundle over S^1 with fiber F. We call f a *monodromy*

∗ Corresponding author.
E-mail addresses: ichihara@las.osaka-sandai.ac.jp (K. Ichihara), motegi@math.chs.nihon-u.ac.jp (K. Motegi).
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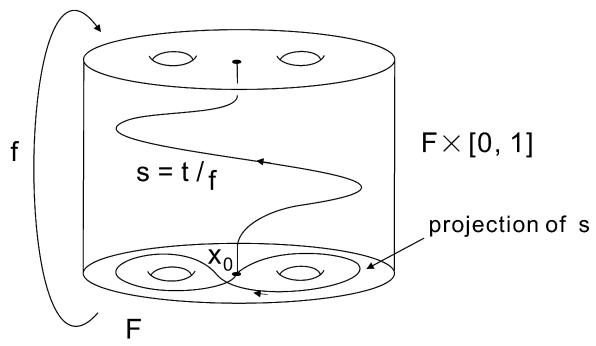

Fig. 1. Section and its projection.

of M_f. Let t be a monotone arc in $F \times [0,1]$ connecting $(x_0, 0)$ and $(x_0, 1)$; t is oriented from $(x_0, 0)$ to $(x_0, 1)$. Then t defines a section s in M_f. Projecting $t \subset F \times [0,1]$ into F, we obtain a closed oriented curve c based at x_0, which represents an element in $\pi_1(F, x_0)$, see Fig. 1. We call c a *projection* of the section s.

Conversely for a given oriented closed curve c based at x_0, we have a section s whose projection is c, uniquely up to level preserving isotopy. Henceforth we denote such a section by s_c. Furthermore, if $[c_1] = [c_2] \in \pi_1(F, x_0)$, then s_{c_1} and s_{c_2} are (level preservingly) isotopic in M_f.

We say that a section s_c is *hyperbolic* (respectively *atoroidal* or *Seifert fibered*) if its exterior is hyperbolic (respectively atoroidal or Seifert fibered). Precisely we can formulate the above question in the following:

Question 1. Can we detect the hyperbolicity of such sections by their projections?

In [6], Kra gave a necessary and sufficient condition for a given section being hyperbolic in terms of their projections in the case where f is the identity map.

Extending this result, in the present paper, we will give an answer to Question according to the Nielsen–Thurston types of monodromies. An automorphism f (i.e., an orientation preserving self diffeomorphism) of a compact, orientable surface with possibly non-empty boundary is said to be *periodic* if its some power is equal to the identity map. We say that f is *reducible* if there is an *essential* 1*-submanifold* $C \subset F$ (i.e., a union of pairwise disjoint simple closed curves such that each curve is homotopically non-trivial and not boundary-parallel, and that no two components are homotopic) so that $f(C)$ is isotopic to C; C is said to be *isotopically f-invariant*. Throughout this paper, we assume that an isotopically f-invariant essential 1-submanifold is *minimal* in the sense that there is no proper subset of C is isotopically invariant under f. It is known by [13,5,3] that if an automorphism is isotopic to neither a periodic automorphism nor a reducible automorphism, then it is isotopic to a *pseudo-Anosov* automorphism (i.e., an automorphism leaving singular foliations invariant) and vice versa; for the precise definition of a pseudo-Anosov automorphism, see [13], [5, Exposé 11, see also p. 286], [3]. Thus each automorphism is isotopic to an automorphism with (at least) one of the above three types which we refer to as *Nielsen–Thurston types*.

Note that the manifold M_f is a hyperbolic 3-manifold, a Seifert fiber space, or a toroidal 3-manifold if and only if the monodromy map $f: F \to F$ is isotopic to a pseudo-Anosov automorphism, a periodic automorphism or a reducible automorphism [14,11,9].

In Section 2, we will state our results, and give their proofs in Section 3. In Section 4, we will translate the results in Section 2 into terminologies of surface automorphisms.

2. Hyperbolic sections and their projections

2.1. Hyperbolic surface bundles—pseudo-Anosov monodromies

Theorem 1. *Let M_f be a hyperbolic surface bundle over S^1, i.e., the monodromy f is isotopic to a pseudo-Anosov automorphism. Then a section s_c is hyperbolic for any curve c.*

Note that M_f is hyperbolic if and only if the monodromy is irreducible and not isotopic to a periodic map. Since a section s_c is hyperbolic if and only if its exterior is atoroidal (i.e., contains no essential tori) and not Seifert fibered [12,10], Theorem 1 follows from the following lemmas.

Lemma 2. *Suppose that the monodromy f is irreducible. Then for a tubular neighborhood $N(s)$ of a section s, $M_f - \operatorname{int} N(s)$ is atoroidal.*

Proof. Suppose for a contradiction that $M_f - \operatorname{int} N(s)$ contains an essential torus T. Then we have a family of disjoint (monotone) annuli $A_1, \ldots, A_n$ in $F \times [0,1]$. Let $C \subset F$ be an essential 1-submanifold $p((F \times \{0\}) \cap \bigcup_{i=1}^n A_i)$, for the natural projection $p : F \times [0,1] \to F$. Then $f(C) = p((F \times \{1\}) \cap \bigcup_{i=1}^n A_i)$. The annuli give an isotopy between C and $f(C)$, thus f would not be irreducible. □

Lemma 3. *Suppose that the monodromy f is not isotopic to a periodic automorphism. Then for any section s, $M_f - \operatorname{int} N(s)$ is not Seifert fibered.*

Proof. Suppose for a contradiction that $M_f - \operatorname{int} N(s_c)$ is Seifert fibered.

The claim below shows that we extend the Seifert fibration to that of M_f so that s_c is a Seifert fiber. Thus f would be isotopic to a periodic automorphism [9], contradicting the assumption. □

Claim 4. *Let M be an irreducible 3-manifold and k a knot in M. If $M - \operatorname{int} N(k)$ is Seifert fibered, then M admits a Seifert fibration in which k is a fiber.*

Proof. If the meridian of k is not a fiber in the Seifert fibration of $M - \operatorname{int} N(k)$, then we can extend it to a Seifert fibration of M so that k is a fiber in the Seifert fibration. Thus in the following we assume that the meridian of k is a fiber.

Let B be the base orbifold of $M - \operatorname{int} N(k)$. If B is a disk or a disk with one singular point, then $M - \operatorname{int} N(k)$ is homeomorphic to $S^1 \times D^2$. Then by choosing a suitable Seifert fibration of $M - \operatorname{int} N(k)$ so that the meridian of k is not a regular fiber, we can reduce to the first situation. Otherwise, there exists an essential arc properly embedded in the orbifold B, i.e., a properly embedded arc which does not cut off a disk without cone points. This implies that M is reducible. This contradicts the assumption. □

2.2. Seifert fibered surface bundles—periodic monodromies

This case was studied in [8].

First we assume that the monodromy $f : F \to F$ is irreducible.

Theorem 5. [8] *Let F be a closed, orientable surface of genus $\geqslant 2$ and f an irreducible, periodic automorphism of period p with $f(x_0) = x_0$. Let s_c be a section in M_f containing $(x_0, 0) = (x_0, 1)$ whose projection is c. Then the following three conditions are equivalent.*

(1) *s_c is hyperbolic.*
(2) *$[c] f_*([c]) \cdots f_*^{p-1}([c]) \neq 1 \in \pi_1(F, x_0)$, where f_* denotes the induced isomorphism of $\pi_1(F, x_0)$.*
(3) *$[c] \neq [\bar{\gamma} * (f \circ \gamma)]$ in $\pi_1(F, x_0)$ for any path γ from x_i to x_0, where x_i is a fixed point of f.*

Remark 6. If $\operatorname{Fix}(f) = \{x_0\}$, the last condition is simplified as: $[c] \neq \alpha^{-1} f_*(\alpha)$ for any $\alpha \in \pi_1(F, x_0)$.

Next we assume that the monodromy $f : F \to F$ is reducible.

An element $\alpha \in \pi_1(F, x_0)$ is said to be *filling* if any representative of α intersects every essential simple closed curve in F.

Theorem 7. [8] *Let F be a closed, orientable surface of genus $\geqslant 2$ and f a reducible, periodic automorphism of period p with $f(x_0) = x_0$. Let s_c be a section in M_f containing $(x_0, 0) = (x_0, 1)$ whose projection is c. Then the following two conditions are equivalent.*

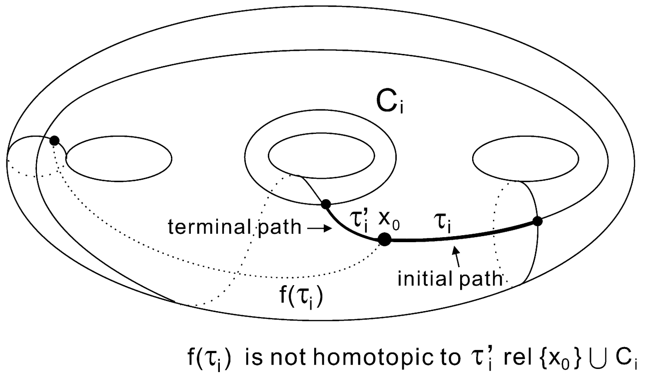

Fig. 2. Well-terminated curve.

(1) *s_c is hyperbolic.*
(2) $[c]f_*([c])\cdots f_*^{p-1}([c]) \in \pi_1(F, x_0)$ *is filling.*

2.3. Toroidal surface bundles—reducible monodromies

A reducible automorphism f is said to be *reduced* if there is a system of essential 1-submanifolds $C_1, \dots, C_n$ such that $f(C_i) = C_i$, $f(N(C_i)) = N(C_i)$ for some small tubular neighborhood $N(C_i)$ and any isotopically f-invariant essential 1-submanifold C can be isotoped into $N(C_i)$ for some i. We call $\{C_1, \dots, C_n\}$ a *reducing system of f* and denote it by $\mathcal{C}_f$. Here we assume that $\{C_1, \dots, C_n\}$ is a finite set.

Now let us introduce a notion of a simple closed curve based at x_0 being "well-terminated", which seems to be artificial. However, as we will see in Section 3, it is crucial to describe a condition which assures a section in a toroidal surface bundle being hyperbolic in terms of its projection.

Let c be a representative of $\alpha \in \pi_1(F, x_0)$ intersecting $C_1 \cup \cdots \cup C_n$ transversely, except for a component of C_j containing x_0 (if such a component exists). Then $c^{-1}(C_i)$ consists of finite points in $[0, 1]$ with minimum $t_{\min}$ and maximum $t_{\max}$. Put $\tau_i = c([0, t_{\min}])$, $\tau_i' = c([t_{\max}, 1])$; τ_i is a path from x_0 to the first intersection point $c(t_{\min}) \in C_i$, and τ_i' is a path from the last intersection point $c(t_{\max}) \in C_i$ to x_0. We call τ_i (respectively τ_i') an *ith initial path* (respectively *ith terminal path*) of c. We say that c is *well-terminated* if $f(\tau_i)$ is not homotopic to τ_i' keeping $\{x_0\} \cup C_i$ invariant for each i. See Fig. 2 for example; there C_i consists of three disjoint simple closed curves.

Let c be a *minimal* representative of $\alpha \in \pi_1(F, x_0)$ in the sense that $|c \cap (C_1 \cup \cdots \cup C_n)|$ is minimal among elements representing α. Here we put a convention: *If $x_0 \in C_i$ and c is tangent to C_i at x_0 and lies locally in one side of C_i, then we do not regard x_0 as an intersection point.* Then we say that α *essentially intersects* $\mathcal{C}_f$ if a minimal representative c of α is well-terminated and intersects some component of each C_i $(1 \leqslant i \leqslant n)$.

Now we can state our main result, whose proof will be given in the next section.

Theorem 8. *Let F be a closed, orientable surface of genus $\geqslant 2$ and f an automorphism of F with $f(x_0) = x_0$. Suppose that f is reduced by a reducing system $\mathcal{C}_f = \{C_1, \dots, C_n\}$ and not isotopic to a periodic automorphism. Then a section $s_c \subset M_f$ is hyperbolic if $[c] \in \pi_1(F, x_0)$ essentially intersects $\mathcal{C}_f$.*

If each member C_i of $\mathcal{C}_f$ is non-separating, i.e., $F - C_i$ is connected, then we have a simpler condition on the projection c for the corresponding section s_c being hyperbolic.

Corollary 9. *Let F and f be as above. Suppose that each member C_i of $\mathcal{C}_f$ is non-separating. Then a section $s_c \subset M_f$ is hyperbolic if $|c \cap C_i|$ is odd for each i.*

3. Proof of Theorem 8

This section is devoted to a proof of Theorem 8 and Corollary 9.

First let us recall from [2] the *geometric intersection number*.

Definition 10 *(Geometric intersection number).* Let c be a closed curve and X a codimension 1-submanifold in a manifold M; c intersects X transversely. Let x, y be points in $c \cap X$. We say that x and y are *equivalent* if we have a continuous map $\varphi:[0,1] \to M$ with $\varphi(0) = x, \varphi(1) = y$ and $\varphi([0,1]) \subset c$ such that φ can be homotoped $\varphi':[0,1] \to M$ so that $\varphi'([0,1]) \subset X$ relative $\{x, y\}$. We call a point $x \in c \cap X$ a *trivial intersection* if c is homotoped into X keeping x invariant. Then the *geometric intersection number* of c and X, denoted by $i(c, X)$, is defined to be the number of equivalence classes each of which consists of odd non-trivial intersection points.

We remark the following elementary fact.

Lemma 11. *Assume that X is two-sided (i.e., bi-collared). If $c \cap X$ contains a trivial intersection point x, then there exists another point $y \in c \cap X$ equivalent to x.*

Proof. Since x is a trivial intersection point, by definition, we have a continuous map $\psi: S^1 \times [0,1] \to M$ such that $\psi(S^1 \times \{0\}) = c$, $\psi(S^1 \times \{1\}) \subset X$ and $\psi(\{a\} \times [0,1]) = \{x\}$. We may assume that $\psi|_{S^1 \times [0,1)}$ is transverse to X so that $\psi^{-1}(X)$ consists of some properly embedded arcs connecting two points in $S^1 \times \{0\}$ and some arcs each of which has one open end in $S^1 \times \{1\}$; $\{a\} \times [0,1] \subset \psi^{-1}(X)$.

Since X is two-sided and c is homotoped into X, the algebraic intersection number of c and X is zero. Thus the cardinality of $c \cap X$ is even. Hence $c \cap X$ has odd number of points other than x. This then implies that we have an arc γ which has one end $(b, 0) \in S^1 \times \{0\}$ and one open end in $S^1 \times \{1\}$. Then $\{a\} \times [0,1]$ and $\bar{\gamma}$ (the closure of γ in $S^1 \times [0,1]$) cobound a disk Δ. Thus $\psi|_\Delta$ shows that x and $y = \psi(b, 0)$ are equivalent. □

Now let us turn to the proof of Theorem 8.

Recall that the monodromy f is a reduced automorphism with a reducing system $\mathcal{C}_f = \{C_1, \ldots, C_n\}$ and that $[c]$ essentially intersects $\mathcal{C}_f$.

We assume hereafter that c is transverse to $C_1 \cup \cdots \cup C_n$ except for at x_0 and $|c \cap (C_1 \cup \cdots \cup C_n)|$ and $|c \cap \partial N(C_1 \cup \cdots \cup C_n)|$ is minimal among the representatives of the (relative) homotopy class $[c]$.

Since the monodromy f is not isotopic to a periodic automorphism, Lemma 3 shows that $M_f - \operatorname{int} N(s_c)$ is not Seifert fibered. So to prove Theorem 8, it is sufficient to that $M_f - \operatorname{int} N(s_c)$ is atoroidal.

Suppose for a contradiction that $M_f - \operatorname{int} N(s_c)$ contains an essential torus T.

Let t_c be a monotone arc in $F \times [0,1]$ used to define the section s_c. Since $F \times [0,1] - \operatorname{int} N(t_c)$ is level preservingly diffeomorphic to $(F - \operatorname{int} D_0) \times [0,1]$, which contains no essential tori, T cannot be isotoped to be disjoint from $F \times \{0\} - \operatorname{int} N(s_c)(= F \times \{1\} - \operatorname{int} N(s_c))$. Furthermore, since $F \times \{0\} - \operatorname{int} N(s_c)(= F \times \{1\} - \operatorname{int} N(s_c))$ is incompressible in $M_f - \operatorname{int} N(s_c)$, by an isotopy, we assume that the intersection $T \cap (F \times \{0\} - \operatorname{int} N(s_c))$ consists of non-empty circles each of which is essential in both T and $F \times \{0\}$, and that the number of components of $T \cap (F \times \{0\} - \operatorname{int} N(s_c))$ is minimal. Cut open M_f (respectively $M_f - \operatorname{int} N(s_c)$) along $F \times \{0\}$ (respectively $F \times \{0\} - \operatorname{int} N(s_c)$) to obtain $F \times [0,1]$ (respectively $F \times [0,1] - \operatorname{int} N(t_c)$). Then the torus T cut into incompressible annuli $A_1, \ldots, A_m \subset F \times [0,1] - \operatorname{int} N(t_c) \cong (F - \operatorname{int} D_0) \times [0,1]$. If both components of ∂A_j is contained in $F \times \{0\} - \operatorname{int} N(t_c)$ or $F \times \{1\} - \operatorname{int} N(t_c)$ for some A_j, then by [15, Corollary 3.2] A_j is boundary parallel, contradicting the minimality of the number of components of $T \cap (F \times \{0\} - \operatorname{int} N(s_c))$. Hence for each A_i $(i = 1, \ldots, m)$, one boundary component is in $F \times \{0\} - \operatorname{int} N(t_c)$ and the other is in $F \times \{1\} - \operatorname{int} N(t_c)$. If A_j is compressible in $F \times [0,1]$ for some A_j, then Claim 12 below shows that A_j is parallel to the frontier of $N(t_c)$. This then implies that $m = 1$ and the original torus T would be boundary-parallel, a contradiction. It follows that each A_i $(i = 1, \ldots, m)$ is incompressible in $F \times [0,1]$.

Claim 12. *Let F be a closed orientable surface and t a monotone arc in $F \times [0,1]$ connecting $(x_0, 0)$ and $(x_0, 1)$ for some point $x_0 \in F$. Let A be an incompressible annulus in $E(t) = F \times [0,1] - \operatorname{int} N(t)$ with one boundary component in $F \times \{0\} - \operatorname{int} N(t)$ and the other in $F \times \{1\} - \operatorname{int} N(t)$. If A is compressible in $F \times [0,1]$, then A is parallel to the frontier of $N(t)$ in $F \times [0,1]$.*

Proof. Let c_i be the simple closed curve $\partial A \cap (F \times \{i\})$ for $i = 0, 1$. Suppose that A is compressible in $F \times [0,1]$. Then by the incompressibility of $F \times \{i\}$ in $F \times [0,1]$, c_i bounds a disk D_i in $F \times \{i\}$ for $i = 0, 1$. The union of $A \cup D_0 \cup D_1$ gives a 2-sphere S embedded in $F \times [0,1]$. By the irreducibility of $F \times [0,1]$, S bounds a 3-ball B. Note that each D_i intersects t at exactly one point for $i = 0, 1$, since A is incompressible in $E(t)$. Since t is monotone, t is unknotted in B and hence A is parallel to the frontier of $N(t)$ in $F \times [0, 1]$. □

By an isotopy of $F \times [0, 1]$ which is the identity on $F \times \{0, 1\}$, we may assume that each A_j is monotone in the sense that there is no local minima and maxima.

Since A_j is incompressible in $F \times [0, 1]$, each component of $C = (\bigcup_{j=1}^{m} A_j) \cap (F \times \{0\})$ is homotopically non-trivial in F. The monotone annuli A_i give an isotopy from $C = p((\bigcup_{j=1}^{m} A_j) \cap (F \times \{0\}))$ on F and $f(C) = p((\bigcup_{j=1}^{m} A_j) \cap (F \times \{1\}))$ on F for a natural projection $p : F \times [0, 1] \to F$; i.e., $f(C)$ is isotopic to C on F.

Claim 13. *The curves C are isotopic to some $C_i \in \mathcal{C}_f$ or $\partial N(C_i)$.*

Proof. If no two components of C are homotopic, then C is an isotopically f-invariant, essential 1-submanifold. Hence C is isotoped into $N(C_i)$ for some $C_i \in \mathcal{C}_f$; actually C is isotoped to C_i.

Suppose that C has two mutually homotopic curves. We may assume, after changing indices if necessary, that $\{c_1 \dots, c_k\}$ ($k \geqslant 2$) be a maximal family of parallel curves and that these curves appear in this order; no other curve is parallel to a member of this family. Suppose that $k \geqslant 3$. Then since F is not a torus, we can take a unique annulus $F_{c_1,\dots,c_k} \subset F$ so that its boundary consists of c_1 and c_k and it contains $c_2, \dots, c_{k-1}$ in its interior. Note that each maximal family of parallel curves has such an annulus in F and that $f(F_{c_1,\dots,c_k})$ is an annulus cobounded by $f(c_1)$ and $f(c_k)$. Then there are two possibilities: $f(c_1) = p(A_1 \cap (F \times \{1\}))$, $f(c_k) = p(A_k \cap (F \times \{1\}))$, or $f(c_1) = p(A_k \cap (F \times \{1\}))$, $f(c_k) = p(A_1 \cap (F \times \{1\}))$. In either case, considering how to join $A_1, A_2, \dots, A_k$ ($k \geqslant 3$) at their boundaries, we see that A_1 and A_2 cannot be contained in the same torus. This contradicts that $A_1 \cup A_2 \cup \dots \cup A_m$ gives a single torus in M_f. Hence at most two curves can be parallel each other. Furthermore, we observe that C consists of $\frac{m}{2}$ pairs of mutually parallel curves; in particular m is even. Let $c'_1, \dots, c'_{\frac{m}{2}}$ be closed curves each of which lies between each parallel pair. Then we see that $C' = c'_1 \cup \dots \cup c'_{\frac{m}{2}}$ is an isotopically f-invariant, essential 1-submanifold. Hence C' is isotoped into $N(C_i)$ for some $C_i \in \mathcal{C}_f$, and in fact, C' can be isotoped to C_i. This then implies that C is also isotoped into $N(C_i)$; precisely, C can be isotoped to $\partial N(C_i)$. □

In the following, let $c_{i,1} \cup \dots \cup c_{i,m}$ denote C_i or $\partial N(C_i)$.

Now let us consider two families of annuli; one is $A_1 \cup \dots \cup A_m$, and, the other is $(c_{i,1} \times [0, 1]) \cup \dots \cup (c_{i,m} \times [0, 1])$. From the former one, the essential torus T is obtained by identifying $(A_1 \cup \dots \cup A_m) \cap (F \times \{0\})$ and $(A_1 \cup \dots \cup A_m) \cap (F \times \{1\})$ by f, and, from the latter one, also an essential torus T_i is obtained by identifying $(c_{i,1} \cup \dots \cup c_{i,m}) \times \{0\}$ and $(c_{i,1} \cup \dots \cup c_{i,m}) \times \{1\}$ by f.

Claim 14. *There is a level preserving isotopy $\Phi_s : F \times [0, 1] \to F \times [0, 1]$ satisfying the following property*:

(1) $\Phi_0 = \mathrm{id}$,
(2) $\Phi_1(C, 0) = (C_i, 0)$, $\Phi_1(f(C), 1) = (C_i, 1)$, *and*
(3) *f-equivariant condition*: $f \circ \Phi_s|_{F \times \{0\}} = \Phi_s|_{F \times \{1\}} \circ f$, *where we identify $F \times \{t\}$ with F in a natural way.*

Proof. We give a proof in the case where C is isotopic to C_i; the same argument works for the case where C is isotopic to $\partial N(C_i)$.

Since C is isotopic to C_i in F, we have an isotopy ψ_t ($0 \leqslant t \leqslant 1$) of F such that $\psi_0 = \mathrm{id}$ and $\psi_1(C) = C_i$. Define $\Phi_s : F \times [0, 1] \to F \times [0, 1]$ as

$$\Phi_s(x, t) = \begin{cases} (\psi_{s(1-2t)}(x), t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ (f \circ \psi_{s(2t-1)} \circ f^{-1}(x), t) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then it is straightforward to check that Φ_s is the required isotopy of $F \times [0, 1]$. □

Let us denote essential annuli $\Phi_1(A_j)$ by A'_j ($j = 1, \dots, m$). Then $(\bigcup_{j=1}^{m} A'_j) \cap (F \times \{0\}) = (c_{i,1} \cup \dots \cup c_{i,m}) \times \{0\}$ and $(\bigcup_{j=1}^{m} A'_j) \cap (F \times \{1\}) = (c_{i,1} \cup \dots \cup c_{i,m}) \times \{1\}$.

Claim 15. *$A'_j \cap (F \times \{0\}) = c_{i,j} \times \{0\}$, $A'_j \cap (F \times \{1\}) = c_{i,j} \times \{1\}$ (after changing indices if necessary) and the union of annuli $A'_1 \cup \cdots \cup A'_m$ can be isotoped to the union of vertical annuli $(c_{i,1} \times [0,1]) \cup \cdots \cup (c_{i,m} \times [0,1])$ by an isotopy which is the identity on $F \times \{0,1\}$.*

Proof. This is the assertion of [7, Lemma 2.1]. □

It follows from Claims 14 and 15 that the torus T is isotopic to T_i in the surface bundle M_f.

Claim 16. *$i(s_c, T_i) > 0$.*

Proof. Recall that $c \cap C_i \neq \emptyset$; this also implies that $c \cap \partial N(C_i) \neq \emptyset$. Then we have $s_c \cap T_i \neq \emptyset$.

Suppose to the contrary that $i(s_c, T_i) = 0$. Since T_i is two-sided in M_f, from Lemma 11, we see that there are two points x and y in $s_c \cap T_i$ which are equivalent. It follows that we have a continuous map $\varphi : [0,1] \times [0,1] \to M_f$ such that $\alpha = \varphi([0,1] \times \{0\}) \subset s_c$, $\beta = \varphi([0,1] \times \{1\}) \subset T_i$, $\varphi(\{0\} \times [0,1]) = x$ and $\varphi(\{1\} \times [0,1]) = y$. We may assume that $\varphi|_{(0,1)\times[0,1)}$ transverses to T_i. Then $(\varphi|_{(0,1)\times[0,1)})^{-1}(T_i)$ consists of properly embedded arcs, each of which is parallel into $[0,1] \times \{0\}$, and some arcs with one open end in $[0,1] \times \{1\}$. Note that each of these arcs cuts off a disks from $[0,1] \times [0,1]$. Let us choose an innermost one, say Δ, which gives an innermost equivalent pair. We rewrite this pair by x, y again and we have a continuous map φ as above for this new pair x, y. At this stage $(\varphi|_{(0,1)\times\{0\}})^{-1}(T_i) = \emptyset$.

Now we have two possibilities:

(1) $x_0 \notin \alpha = \varphi([0,1] \times \{0\}) \subset s_c$, or
(2) $x_0 \in \alpha = \varphi([0,1] \times \{0\})$.

(1) In this case α is entirely in $F \times [0,1]$. Since α and β are homotopic fixing their endpoints x, y, the union $\alpha \cup \beta$ bounds a singular disk. Hence we can assume, by an homotopy if necessary, that β is also entirely in $F \times [0,1]$. For otherwise, $\pi(\alpha \cup \beta)$ wraps in S^1 direction, where π is the bundle projection $M_f \to S^1$, and so, it never bounds a singular disk. Thus $p \circ \varphi$ gives a homotopy from $p(\alpha) \subset c$ to $p(\beta) \subset C_i$ fixing $p(x)$ and $p(y)$ for a natural projection $p : F \times [0,1] \to F$. This implies that we can eliminate $p(x)$ and $p(y)$ from $c \cap C_i$ fixing x_0, without creating new intersection points with $C_1 \cup \cdots \cup C_n$, contradicting the minimality of c among $[c] \in \pi_1(F, x_0)$.

(2) We divide this case into two subcases:

(i) $x_0 \neq x, y$, see Fig. 3.
(ii) $x_0 = x$ or y.

(i) In this case, $t_0 = (\varphi|_{[0,1]\times\{0\}})^{-1}(x_0)$ is in $(0,1) \times \{0\}$. Since α and β are homotopic fixing their endpoints x, y, we may homotope β fixing x, y so that $\beta \cap (F \times \{0\})$ consists of a single point. We may assume further that φ transverses $F \times \{0\}$. Then $\varphi^{-1}(F \times \{0\})$ contains an arc γ_0 connecting $(t_0, 0)$ and $(t'_0, 1)$ for some t'_0. By incompressibility of $F \times \{0\}$ in M_f, we may assume by a further homotopy that actually $\varphi^{-1}(F \times \{0\}) = \gamma_0$. This γ_0 divides $[0,1] \times [0,1]$ into two rectangles D_1 and D_2. Denote $\varphi(\gamma_0)$ by γ (respectively $f(\gamma)$) when we regard it in $F \times \{0\}$ (respectively $F \times \{1\}$).

The composition $(p \circ \varphi)|_{D_2} : D_2 \to F$ gives a homotopy from γ to an ith initial path $\tau_i = p \circ \varphi([t_0, 1] \times \{0\})$ keeping $\{x_0\} \cup C_i$ invariant, and $(p \circ \varphi)|_{D_1} : D_1 \to F$ gives a homotopy from $f(\gamma)$ to an ith terminal path $\tau'_i = p \circ \varphi([0, t_0] \times \{0\})$ keeping $\{x_0\} \cup C_i$ invariant. Thus $f(\tau_i)$ is homotopic to $f(\gamma)$, which is homotopic to τ'_i keeping $\{x_0\} \cup C_i$ invariant. Hence c is not well-terminated, a contradiction.

(ii) Putting $x = x_0$ (or $y = x_0$), we can apply the argument in (1) so that we have a homotopy from $p(\alpha) \subset c$ to $p(\beta) \subset C_i$ fixing $p(x)$ and $p(y)$. This again implies that we can eliminate $p(x)$ and $p(y)$ from $c \cap C_i$ fixing x_0, contradicting the minimality of c among $[c] \in \pi_1(F, x_0)$. Here we adopt our counting rule: If the curve c is *tangent* to C_i at x_0 and lies locally in one side of C_i, then we *do not* regard it as their intersection point. □

Since T is isotopic to T_i in M_f, the invariance of geometric intersection numbers under homotopy [2, Lemme 3.1] assures that $i(s_c, T) = i(s_c, T_i)$, which is not zero by Claim 16. On the other hand, since $T \subset M_f - \operatorname{int} N(s_c)$,

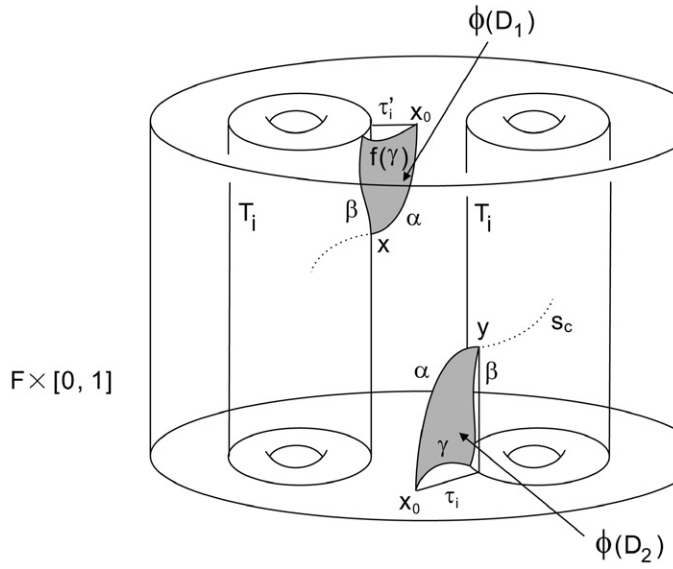

Fig. 3.

$s_c \cap T = \emptyset$, in particular, $i(s_c, T) = 0$, a contradiction. It follows that $M_f - \operatorname{int} N(s_c)$ is atoroidal and the proof of Theorem 8 is completed.

Proof of Corollary 9. If we show that $i(s_c, T_i) > 0$ for each i, then the argument in the proof of Theorem 8 implies that s_c is hyperbolic in M_f. Suppose for a contradiction that $i(s_c, T_i) = 0$ for some i. Then it turn out that there is a trivial intersection z in $s_c \cap T_i$, or there is an equivalent pair x, y in $s_c \cap T_i$. In the former case, by definition, s_c can be homotoped into T_i. Thus the algebraic intersection number of s_c and T_i is zero. This then implies that the algebraic intersection number of c and C_i is also zero, contradicting the assumption of Corollary 9. If the latter case happens, then since $i(s_c, T_i) = 0$, each equivalence class of an intersection point consists of even non-trivial intersection points. Therefore $|s_c \cap T_i|$ is even, and hence $|c \cap C_i|$ is also even, contradicting the assumption. □

4. Applications to surface-automorphisms

In this final section, we restate Theorems 1, 5, 7 and 8 in terminologies of surface-automorphisms.

Let F be a closed, orientable surface of genus $\geqslant 2$ and f an automorphism of F with $f(x_0) = x_0$ for some point x_0. (Recall that the condition "$f(x_0) = x_0$" implies $f(D_0) = D_0$ for some small diskal neighborhood D_0 of x_0.) For another automorphism f' of F isotopic to f and satisfying $f'(x_0) = x_0$, by isotoping f' to f, we obtain a closed curve c based at x_0 which traces x_0 under the isotopy. We call c the *sliding curve* of f' and write $f' = f_c$. Let f'_1 and f'_2 be automorphisms isotopic to f such that their sliding curves represent the same element of $\pi_1(F, x_0)$. Then following Birman [1, Chapter 4], there is an isotopy between them keeping x_0 invariant.

Consider the mapping torus $(F \times [0,1])/\{(x,0) = (f_c(x), 1)\}$ with the gluing map f_c. Since f_c is isotopic to f on F, we can apply a level ($t \in [0,1]$) preserving isotopy to $F \times [0,1]$ so that the gluing map becomes f. This level preserving isotopy deforms the vertical segment $\{x_0\} \times [0,1]$ to a monotone arc t whose projection is c; t gives a section s in M_f by identifying its endpoints. Let us denote $F - \operatorname{int} D_0$ by $\hat{F}$. Then $(\hat{F} \times [0,1])/\{(x,0) = (f_c|_{\hat{F}}(x), 1)\} \cong M_{f_c} - \operatorname{int} N(s_0) \cong M_f - \operatorname{int} N(s)$. It follows from [14,11] that s_c is hyperbolic if and only if the monodromy map $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.

Thus Theorems 1, 5, 7 and 8 can be restated as:

Corollary 17. *If f is isotopic to a pseudo-Anosov automorphism, then the restriction $f_c|_{\hat{F}}$ is also isotopic to a pseudo-Anosov automorphism for any curve c.*

Corollary 18. [8] *Suppose that f is irreducible and periodic with period p. Then the following three conditions are equivalent.*

(1) *The restriction $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.*
(2) $[c]f_*([c])\cdots f_*^{p-1}([c]) \neq 1 \in \pi_1(F, x_0)$.
(3) $[c] \neq [\bar{\gamma} * (f \circ \gamma)]$ *in $\pi_1(F, x_0)$ for any path γ from x_i to x_0, where x_i is a fixed point of f.*

Corollary 19. [8] *Suppose that f is reducible and periodic with period p. Then the following two conditions are equivalent.*

(1) *The restriction $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.*
(2) $[c]f_*([c])\cdots f_*^{p-1}([c]) \in \pi_1(F, x_0)$ *is filling.*

Corollary 20. *Suppose that f is reduced by a reducing system $\mathcal{C}_f = \{C_1, \ldots, C_n\}$ and not isotopic to a periodic automorphism. Then the restriction $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism if $[c] \in \pi_1(F, x_0)$ essentially intersects $\mathcal{C}_f$.*

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References

[1] J.S. Birman, Braids, Links, and Mapping Class Groups, Annals of Mathematics Studies, vol. 82, Princeton University Press, Princeton, NJ, 1974.
[2] F. Bonahon, Bouts des varieties hyperboliques de dimension 3, Ann. of Math. (2) 124 (1) (1986) 71–158.
[3] A.J. Casson, S.A. Bleiler, Automorphisms of Surfaces after Nielsen and Thurston, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988.
[4] P.J. Callahan, A.W. Reid, Hyperbolic structures on knot complements. Knot theory and its applications, Chaos Solitons Fractals 9 (4–5) (1998) 705–738.
[5] A. Fathi, F. Laudenbach, V. Poenaru, Travaux de Thurston sur les surfaces, Asterisque 66–67 (1991/1979).
[6] I. Kra, On the Nielsen–Thurston–Bers type of some self-maps of Riemann surfaces, Acta Mathematica 146 (1981) 231–270.
[7] K. Ichihara, K. Motegi, Braids and Nielsen–Thurston types of automorphisms of punctured surfaces, Tokyo J. Math. 28 (2005) 527–538.
[8] K. Ichihara, K. Motegi, H.-J. Song, Longitudinal Seifert fibered surgeries on hyperbolic knots, Preprint.
[9] W. Jaco, Lectures on Three-Manifold Topology, Conf. Board of Math. Sci., vol. 43, Amer. Math. Soc., Providence, RI, 1980.
[10] J. Morgan, H. Bass (Eds.), The Smith Conjecture, Academic Press, San Diego, 1984.
[11] J.-P. Otal, Le theoreme d'hyperbolisation pour les varietes fibrees de dimension 3, Asterisque 235 (1996).
[12] W.P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982) 357–381.
[13] W.P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. 19 (2) (1988) 417–431.
[14] W.P. Thurston, Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle, math.GT/9801045.
[15] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. 87 (1968) 56–88.