MATHEMATICS

An Extension Theorem for Riesz Homomorphisms

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Dedicated to Professor Dr. A. C. Zaanen on the occasion of his 65th birthday

1. INTRODUCTION

The celebrated Hahn-Banach extension theorem for linear functionals has been generalized in many different directions. In particular, extension theorems for linear transformations rather than linear functionals have been developed. One of the earliest results concerning extensions of positive linear transformations was suggested by an extension theorem of L. V. Kantorovitch [5] for positive linear functionals. In its generalized form the theorem of Kantorovitch can be formulated as follows:

(1.1) THEOREM. (*L. V. Kantorovitch*). *Let K be a Riesz space and let L be a linear subspace of K majorizing K* (i.e., for each $f \in K$ there exists an element $0 \leqslant g \in L$ such that $|f| \leqslant g$). *Then every positive linear transformation T defined on L with values in a Dedekind complete Riesz space M can be extended to a positive linear transformation of K into M.*

It is not too hard to see that this result is a direct generalization of the classical Hahn-Banach theorem. Indeed, the Dedekind complete Riesz space M takes over the role of the real number system for the values of the linear transformation and the majorizing property of L replaces the role of the sublinear functional in the Hahn-Banach extension theorem.

For a proof of Kantorovitch's theorem we refer to [10] Theorem X.3.1.

The purpose of the present paper is to answer the following question. If in Kantorovitch's theorem L is a Riesz space and T is a Riesz homomorphism (= linear lattice homomorphism) of L into M does there exist among the positive linear extensions of T to K an extension which also preserves the lattice operations of K? In other words, can every Riesz homomorphism be extended to a Riesz homomorphism in the setting of Kantorovitch's theorem. The somewhat surprising answer to this question turns out to be affirmative. In fact, we shall show that the Riesz homomorphic extensions are precisely the extreme points of the convex set of all the positive linear extensions whose existence are guaranteed by the theorem of Kantorovitch.

The reader who is familiar with the theory of Boolean algebras will immediately observe that the extension theorem for Riesz homomorphisms generalizes the well-known Sikorski extension theorem for Boolean homomorphisms. For Sikorski's theorem we refer the reader to [8], [5], and [1] (Chapter V.9, Theorem 2 page 113).

The proof of the extension theorem for Riesz homomorphisms presented in this paper will not be along the lines followed in the proofs of the Hahn-Banach theorem and the Sikorski extension theorem. The reason for this is that the process of extending a Riesz homomorphism from a Riesz subspace to the Riesz subspace generated by it and an additional element presents unsurmountable problems. Instead we shall deduce the extension theorem from the intermediate result that every Dedekind complete Riesz space is a retract of the Riesz spaces it majorizes. This in turn will follow from a property of the Dedekind completion of an Archimedean Riesz space which is of independent interest and can be formulated in the following way. The Dedekind completion of an Archimedean Riesz space is a retract of the Dedekind complete Riesz spaces it majorizes.

For this reason the next section is devoted to proving the above result about the Dedekind completion of an Archimedean Riesz space. The extension theorem for Riesz homomorphisms will be derived in section 3 of the paper and in section 4 we characterize the Riesz homomorphic extensions as the extreme points of the convex set of all positive linear extensions. For terminology concerning Riesz spaces and Boolean algebras not explained in this paper we refer the reader to [1], [7] and [10].

2. A PROPERTY OF THE DEDEKIND COMPLETION OF AN ARCHIMEDEAN RIESZ SPACE

In this section we shall first prove a new result concerning the Dedekind completion of an Archimedean Riesz space referred to in the introduction and which in a sense is the key result in establishing the extension principle for Riesz homomorphisms.

We shall begin with a few lemmas which are of independent interest.

Let L be a Riesz subspace of a Riesz space K and let $I \subset K$ be an ideal satisfying $L \cap I = \{0\}$.

(2.1) LEMMA. *The Riesz quotient space L/I is order dense in the Riesz quotient space K/I if and only if I is maximal with respect to the property $L \cap I = \{0\}$.*

PROOF. Assume first that the ideal $I \subset K$ is maximal w.r.t. the property $L \cap I = \{0\}$ and that $0 < f \in K$ satisfies $f \notin L$. Then the maximality property of I implies that there exists an element $0 < u \in I$ and an element $0 < g \in L$ such that $0 < g < f + u$. Hence, if p denotes the Riesz homomorphism of K onto K/I with kernel I, then $0 \leqslant p(g) \leqslant p(f) + p(u) = p(f)$. Since $g \notin L$ we may conclude that $p(g) > 0$, and so L/I in order dense in K/I.

Conversely, assume that L/I is order dense in K/I and I is not maximal w.r.t. the property $I \cap L = \{0\}$. The latter implies that there exists an element $0 < f \in K$ such that $f \notin I$ and the ideal I' generated by I and $\{f\}$ still satisfies $L \cap I' = \{0\}$. Let p and p' be the Riesz homomorphisms of K with kernels I and I' respectively. By hypothesis, L/I is order dense in K/I, and so there exists an element $0 < g \in L$ such that $0 < p(g) \leqslant p(f)$. Then $I \subset I'$ implies that also $0 \leqslant p'(g) \leqslant p'(f)$. But $f \in I'$ implies $p'(f) = 0$ and $0 < g \in L$ implies $p'(g) > 0$ and a contradiction is obtained finishing the proof.

For the sake of convenience we recall the following definition.

(2.2) DEFINITION. *A Riesz space L is said to majorize a Riesz space K whenever L in a Riesz subspace of K and for each $f \in K$ there exists an element $0 \leqslant g \in L$ such that $|f| \leqslant g$.*

(2.3) LEMMA. *If a Riesz space L majorizes an Archimedean Riesz space K and if I is an ideal in K maximal w.r.t. the property $L \cap I = \{0\}$, then the Riesz quotient space K/I is Archimedean.*

PROOF. By [7] Theorem 60.2, page 427, we have only to show that if $0 < f$, $g \in K$ and $(nf - g)^+ \in I$ for all $n = 1, 2, \ldots$, then $f \in I$. If $f \notin I$, then the maximality property of I implies that there is an element $0 < u \in L$ and an element $0 < v \in I$ such that $0 < u < f + v$. Hence, $(nu - g)^+ \leqslant (nf - g)^+ + nv \in I$ for all $n = 1, 2, \ldots$, implies that for all $n = 1, 2, \ldots$ we have $(nu - g)^+ \in L$. From the assumption that L majorizes K it follows that there is an element $0 < w \in L$ such that $0 < g < w$. Then $(nu - w)^+ \leqslant (nu - g)^+ \in I$ $(n = 1, 2, \ldots)$ implies that $(nu - w)^+ \in I$ for all $n = 1, 2, \ldots$. Since $(nu - w)^+ \in L$ $(n = 1, 2, \ldots)$ and $L \cap I = \{0\}$ we obtain finally that $nu \leqslant w$ for all $n = 1, 2, \ldots$. Since K is Archimedean we may conclude that $u = 0$, contradicting $u > 0$ finishing the proof.

We shall need the following definition of a retract of a Riesz space.

(2.4) DEFINITION. *A Riesz space L is called an order retract (Riesz retract) of a Riesz space K whenever there exists a positive linear transformation (Riesz homomorphism) T of K into L and a positive linear transformation (Riesz homomorphism) S of L into K such that the composition map $T \circ S$ is the identity on L.*

It is obvious that a Riesz retract is always an order retract.

Concerning retracts we shall need the following result whose counterpart in the theory of Boolean algebras is well-known (see [1], Chapter V.9, Lemma 1, p. 112). We abbreviate Dedekind complete by D-complete.

(2.5) THEOREM. *If a Riesz space L is an order retract of a D-complete Riesz space K, then L is D-complete.*

PROOF. Let $A \subset L$ be non-empty and bounded above by an element $f \in L$. Then, with the notation introduced in (2.4), we have that $S(A) \subset K$ is non-empty and bounded above by $Sf \in K$. Since K is D-complete $\sup S(A) = g$ exists in K. We shall now prove that $\sup A$ exists in L and is equal to Tg. To this end, observe that $TS(A) = A$ implies that Tg is an upper bound of A. If u is an upperbound of A and $u \leqslant Tg$, then Su is an upperbound of $S(A)$, and so, by the definition of g, we have $g \leqslant Su$. Hence, $Tg \leqslant TSu = u$, and the proof is complete.

The reader may observe that other Riesz space properties such as the Archimedean property and the countable sup property are preserved under retracts.

In the proof of the main theorem we shall use the following result which follows from Theorem 32.6, [7], page 195.

(2.6) LEMMA. *If a Riesz space L majorizes an Archimedean Riesz space K and L is order dense in K, then for each $f \in K$ we have*

$$\sup (g : g \in L \text{ and } g \leqslant f) = f = \inf (g' : g' \in L \text{ and } f \leqslant g'),$$

where the sup *and* inf *exist in K.*

We are now in a position to prove the main theorem of this section.

(2.7) THEOREM. *Let L be a Riesz space majorizing a D-complete Riesz space K and let I be an ideal in K maximal w.r.t. the property $L \cap I = \{0\}$. Then the quotient Riesz space K/I is D-complete and Riesz isomorphic to the D-completion of L.*

PROOF. Let p be the Riesz homomorphism of K onto K/I with kernel I. Then $L \cap I = \{0\}$ implies that the restriction of p to L is a Riesz isomorphism of L onto $p(L) = L/I$. Let q be the inverse mapping of p. Then q is a Riesz isomorphism of L/I onto $L \subset K$. Since K is D-complete and L/I majorizes K/I it follows from the extension theorem (1.1) of Kantorovitch that q can be extended to a positive linear transformation T of

K/I into K. From the definition of q it follows that the composition map $p \circ q$ is the identity on L/I. We shall now show that also the composition map $p \circ T$ is the identity on K/I. To this end, observe that the maximality property of I implies, using (2.1) and (2.3), that L/I majorizes and is order dense in the Archimedean Riesz space K/I. Now $f \in K/I$ and $g, g' \in L/I$ and $g \leqslant f \leqslant g'$ implies that $q(g) \leqslant Tf \leqslant q(g')$, and so, by applying p, we obtain $g = p(q(g)) \leqslant p(Tf) \leqslant p(q(g')) = g'$ for arbitrary $g, g' \in L/I$ satisfying $g \leqslant f \leqslant g'$. Hence, by (2.6), $f = p(Tf)$ for all $f \in K/I$. This shows that T is a Riesz homomorphism and that K/I is a Riesz retract of K. K being D-complete implies, using Theorem 2.5, that K/I is D-complete. Since L is Riesz isomorphic to L/I and L/I majorizes K/I and is order dense in the D-complete Riesz space K/I it follows finally that K/I is Riesz isomorphic to the Dedekind completion of L. This completes the proof of the theorem.

The proof of Theorem 2.7 shows a little more, namely that K/I is a Riesz retract of K. This observation leads immediately to the following corollary.

(2.8) COROLLARY. *The D-completion of an Archimedean Riesz space is Riesz isomorphic to a Riesz retract of every D-complete Riesz space it majorizes.*

By specialization in (2.7) we obtain the following result due to A. I. Veksler [9], who derived the result in quite a different way.

(2.9) THEOREM. *Let L be a Riesz subspace of a D-complete Riesz space K. Then there exists a Riesz subspace L' of K containing L as a Riesz subspace and a Riesz isomorphism of L' onto the D-completion of L which leaves every element of L fixed.*

PROOF. Let K' be the ideal in K generated by L. With the notation used in the proof of (2.7) it is easy to see that the Riesz subspace $L' = T(K'/I)$ of K satisfies all the required properties.

(2.10) REMARKS. (i) The proof of (2.9) shows a little more than Veksler's in that L' is also a Riesz retract of the D-complete Riesz space K'/I.

(ii) The purpose of this remark is to indicate that in the theory of Boolean algebras there are results analogous to (2.1) and (2.9). Indeed, the following theorem about Boolean algebras holds.

(2.10) THEOREM. *Let $\mathscr{B}_0$ be a subalgebra of a Boolean algebra $\mathscr{B}$ and let I be an ideal in $\mathscr{B}$. Then the quotient algebra $\mathscr{B}_0/I$ is order dense in the quotient algebra $\mathscr{B}/I$ if and only if I is maximal w.r.t. the property $\mathscr{B}_0 \cap I = \{0\}$; and in this case, if $\mathscr{B}$ is in addition complete, then $\mathscr{B}/I$ is isomorphic to the minimal completion of $\mathscr{B}_0$.*

The proof of (2.10) can be given along the same lines as the proofs of (2.1) and (2.9) except that at the point in the proof of (2.9) the extension theorem of Kantorovitch is used we may now use Sikorski's extension theorem.

The theorem in the theory of Boolean algebras which corresponds to Veksler's theorem is a trivial consequence of Sikorski's extension theorem.

If we denote by $\mathscr{S}(\mathscr{B})$ the Stone representation space of $\mathscr{B}$ and by $O(I)$ the open subset of $\mathscr{S}(\mathscr{B})$ determined by the ideal I, then it is well-known that the closed set $F_I = \mathscr{S}(\mathscr{B}) \backslash O_I$ is homeomorphic with the Stone space $\mathscr{S}'\mathscr{B}/I)$ of $\mathscr{B}/I$. If now $\mathscr{B}$ is complete, then $\mathscr{S}(\mathscr{B})$ is extremely disconnected, and if I is maximal w.r.t. $\mathscr{B}_0 \cap I = \{0\}$ for some sub-algebra $\mathscr{B}_0$ of $\mathscr{B}$, then the closed subset F_I of $\mathscr{S}(\mathscr{B})$ is also extremely disconnected. It is not known to the authors whether every closed subset of an extremely disconnected Stone space which is extremely disconnected in the induced topology can be obtained in this manner.

The first-named author wishes to thank Professor Ph. Dwinger for many valuable discussions concerning the Boolean algebra results of this section, and he wishes to mention that Professor Dwinger found the same proof of (2.10).

3. AN EXTENSION THEOREM FOR RIESZ HOMOMORPHISMS

The main purpose of this section is to prove the following extension theorem for Riesz homomorphisms which generalizes Sikorski's extension theorem for Boolean homomorphisms.

(3.1) THEOREM. *Let L be a Riesz space majorizing a Riesz space K and let T be a Riesz homomorphism of L into a D-complete Riesz space M. Then T can be extended to a Riesz homomorphism of K into M.*

The proof of (3.1) will be divided into several steps.

First we observe that an immediate consequence of (3.1) is the following result. Every D-complete Riesz space is a Riesz retract of every Riesz space it majorizes. We shall now show that this consequence of (3.1) follows from Theorem 2.7 and its Corollary 2.8.

(3.2) THEOREM. *Every D-complete Riesz space is a Riesz retract of every Riesz space it majorizes.*

PROOF. Let the Riesz space L be D-complete and majorizing the Riesz space K and let K' be the D-completion of K by cuts. Observe that L still majorizes K'. If we can show that L is a Riesz retract of K' then it is obviously also a Riesz retract of $K \subset K'$. Hence, there is no loss in generality to assume that K is D-complete. Then, by using the notation introduced in the proof of (2.7), we observe that since L/I is now D-complete it coincides with K/I. Hence, in this case $p(L) = K/I$ and $T = q$,

and so the composition map $q \circ p$ is the required Riesz retract of K onto L, completing the proof.

(3.3) REMARK. In the theory of Boolean algebras the theorem which corresponds to (3.2) states that every complete Boolean algebra is a retract of every Boolean algebra it contains. This immediate consequence of Sikorski's theorem was shown in [5] to be logically equivalent to Sikorski's extension theorem.

In order to be able to derive (3.1) from (3.2) we need two more intermediate results. The first one is a corollary of (3.2).

(3.4) COROLLARY. *Let L be a D-complete Riesz space majorizing a Riesz space K and let T be a Riesz homomorphism of L into a D-complete Riesz space M. Then T can be extended to a Riesz homomorphism of K into M.*

PROOF. By (3.2), there exists a Riesz retract p of K onto L. Then the composition map $T \circ p$ is a Riesz homomorphism of K into M which extends T, which finishes the proof.

The reader should note that for (3.4) to hold M need not be D-complete.

The next result deals with the extension property of order continuous homomorphisms.

(3.5) THEOREM. *Let L be a Riesz space majorizing a Riesz space K and let T be an order continuous Riesz homomorphism of L into a D-complete Riesz space M. Then T can be extended to a Riesz homomorphism of K into M.*

PROOF. Since T is order continuous T extends uniquely by order limits to a Riesz homomorphism, which we shall again denote by T, of the D-completion L' of L into M. Let K' be the D-completion of K. Then Veksler's theorem (2.9) implies that there exists a Riesz homomorphism p of L' onto a Riesz subspace L_0 of K' such that $L \subset L_0$ and p leaves L invariant. It is obvious that L_0 majorizes K', and so, by (3.4), the Riesz homomorphism $T \circ p^{-1}$ of L_0 into M can be extended to a Riesz homomorphism T' of K' into M. Since L is invariant under p it follows that T' extends T, and the proof is finished.

PROOF OF THEOREM 3.1. Let J be the ideal in K generated by the kernel in L of T and let p be the canonical Riesz homomorphism of K onto K/J. For every $f \in L$ we set $T_0(p(f)) = T(f)$. Then it is easy to see that T_0 defines a Riesz isomorphism of $p(L) = L/J$ onto the Riesz subspace $X = T(L)$ of M. Let X' be the D-completion of X. Then T_0 is an order continuous Riesz isomorphism of L/J into X' ([7], Theorem 18.13, p. 104). Since L/J majorizes K/J it follows from (3.5) that T_0 can be extended to a

Riesz homomorphism T_0 of K/J into X'. From $X \subset M$ and M is D-complete it follows from Veksler's theorem (2.9) that there exists a Riesz isomorphism q of X' into a Riesz subspace Y of M such that $X \subset Y$ and q leaves X invariant. Then for each $f \in K$ we set $T_1(f) = q(T_0'(p(f)))$. Then T_1 is a Riesz homomorphism of K into M such that for all $f \in L$ we have $T_1(f) = q(T_0'(p(f))) = q(Tf) = Tf$, i.e., T_1 extends T, which finishes the proof.

(3.6) REMARK. Using the Stone representation theorem for Boolean algebras or using Carathéodory's theory of place functions (see [2]) it can readily be shown that the extension theorem (3.5) for Riesz homomorphisms implies Sikorski's extension theorem for Boolean homomorphisms.

4. RIESZ HOMOMORPHIC EXTENSIONS AND EXTREME POINTS

Assume that the Riesz space L majorizes the Riesz space K and assume that T_0 is a positive linear transformation of L into a D-complete Riesz space M. By $\mathscr{L}_b(K, M)$ we shall denote the D-complete Riesz space of all order bounded linear transformations of K into M. Then the subset $E(T_0)$ of $\mathscr{L}_b^+(K, M)$ of all the positive linear transformations which extend T_0 is a convex set, which by Kantorovitch's theorem is non-empty. If T_0 is a Riesz homomorphism, then the extension theorem (3.1) implies that the convex set $E(T_0)$ contains certain Riesz homomorphisms of K into M. It is a natural question to ask to characterize those elements of $E(T_0)$ which are the Riesz homomorphic extensions of T_0? The complete answer is contained in the following theorem.

(4.1) THEOREM. *If T_0 is a Riesz homomorphism of L into M, then an element $T \in E(T_0)$ is a Riesz homomorphism if and only if T is an extreme point of the convex set $E(T_0)$.*

PROOF. Assume first that $T \in E(T_0)$ is a Riesz homomorphism and that T is not an extreme point of $E(T_0)$. Then there exist elements $T_1, T_2 \in E(T_0)$ and a constant $0 < \lambda < 1$ such that $T_1 \neq T_2$ and $T = \lambda T_1 + (1-\lambda)T_2$. The latter implies that $\lambda T_1 \leqslant T$ and $(1-\lambda)T_2 < T$, and so, by a theorem of Kutateladze [4] (see also [6], Theorems 4.2 and 4.3 for a generalization and direct proof), there exist orthomorphisms π_1, π_2 of M into M such that $\lambda T_1 = \pi_1 T$ and $(1-\lambda)T_2 = \pi_2 T$. Since T_1, T_2 are extensions of T_0 it follows that for all $f \in L$ we have $\pi_1(T_0(f)) = \lambda \cdot T_1(f) = \lambda \cdot T_0(f)$ and $\pi_2(T_0(f))) = (1-\lambda)T_2(f) = (1-\lambda)T_2(f) = (1-\lambda)T_0(f)$. If we denote by I_m the identity orthomorphism of M onto M, then we may conclude now that $\pi_1 = \lambda I_m$ and $\pi_2 = (1-\lambda)I_m$ on $T(L)$. Then, by an important property of orthomorphisms (see [6], Theorem 1.6), it follows that $\pi_1 = \lambda I_m$ and $\pi_2 = (1-\lambda)I_m$ on the band $\{T(L)\}^{dd}$ generated by $T(L)$ in M. From the hypothesis that L majorizes K it follows that $\{T(L)\}^{dd} = \{T(K)\}^{dd}$, and so for all $f \in K$ we have $\pi_1(T(f)) = \lambda \cdot T(f) = \lambda \cdot T_1(f)$. Hence, $\lambda \neq 0$ implies

that $T = T_1$ and similarly we obtain that $T = T_2$, contradicting $T_1 \neq T_2$, which shows that T is an extreme point of $E(T_0)$.

Conversely assume that $T \in E(T_0)$ is an extreme point. According to the theorem of Kutateladze referred to above T is Riesz homomorphism if for every $S \in \mathscr{L}_b^+(K, M)$ satisfying $0 \leqslant S \leqslant T$ there exists an orthomorphism π of M into M such that $S = \pi T$. Now the restriction of T to L is the Riesz homomorphism T_0. Then, by Theorem 4.2 of [6], $0 \leqslant S \leqslant T_0$ on L implies that there exists an orthomorphism π of M into M such that $0 \leqslant \pi \leqslant I_m$ and $S = \pi T_0 = \pi T$ on L. Then the positive linear transformation $S_1 = \pi T$ of K into M coincides with S on L. From this fact we may conclude that $T \pm (S - S_1) \in E(T_0)$. Furthermore, from the definition of S_1 it follows immediately that $T \pm (S - S_1) = T \pm (S - \pi T) \geqslant 0$. Hence, T being an extreme point of $E(T_0)$ implies finally that $S = T_1 = \pi T$, and so, by Kutateladze's theorem, T is a Riesz homomorphism. This completes the proof of the theorem.

(4.2) REMARK. It is not without interest to observe that the Riesz homomorphic extensions of a Riesz homomorphism are all incomparable with respect to the order relation between operators.

(4.3) ADDENDUM. In the proof of Theorem 3.1 the ideal J in K should be chosen in such a way that the kernel $I = \{f : T(f) = 0\} = \{f : T(|f|) = 0\}$, of T satisfies $I = J \cap L$ and the quotient space K/J is Archimedean. For this to be true it is in general not enough to take for J the ideal in K generated by the kernel I of T as was done in the proof of Theorem 3.1. That such an ideal J exists with the above properties is a consequence of the fact that I, being the kernel of a Riesz homomorphism, is closed in the relative uniform topology of L, i.e., L/I is Archimedean, and the following generalization of Lemma 2.3.

(2.3)′ LEMMA. *If the Riesz subspace L of a Riesz space K majorizes K and I is an ideal in L such that the quotient space L/I is Archimedean, then every ideal J in K maximal w.r.t. the property $J \cap L = I$ has the property that the quotient space K/J is Archimedean.*

We shall not give the proof as it is completely analogous to the proof of Lemma 2.3.

Professor Ph. Dwinger kindly informed the authors that the answer to the question raised in the last paragraph of section 2 is no. This can be shown by means of the following counterexample. Let $\mathscr{B}'$ be the complete atomic Boolean algebra of all subsets of an ordinal $\alpha > \aleph_0$. Let $\mathscr{J}(2^\alpha)$ be the free Boolean algebra on 2^α free generators. Then $\mathscr{J}(2^\alpha)$ satisfies the countable chain condition (see [1], Thm. 24, p. 93), and hence the normal completion $\mathscr{B}$ of $\mathscr{J}(2^\alpha)$ also satisfies the countable chain condition. Since $\mathscr{J}(2^\alpha)$ is free there exists a Boolean homomorphism h of $\mathscr{J}(2^\alpha)$ onto $\mathscr{B}'$. By Sikorski's extension theorem [8] h can be extended

to a homomorphism of $\mathscr{B}$ onto $\mathscr{B}'$. Since $\mathscr{B}'$ does not satisfy the countable chain condition $(\alpha > \aleph_0)$ and $\mathscr{B}$ does, it follows that $\mathscr{B}'$ is not a retract of $\mathscr{B}$. But $\mathscr{B}'$ is isomorphic to a complete quotient space of $\mathscr{B}$. This shows that not necessarily every complete quotient space of a complete Boolean algebra is of the type introduced at the end of section 2.

REFERENCES

1. Balbes, R. and Ph. Dwinger – Distributive Lattices, Univ. of Missouri Press., 1974.
2. Carathéodory, C. – Algebraic Theory of Measure and Integration, Chelsea Publ. Co., New York, 1963.
3. Kantorovitch, L. V. – Concerning the problem of moments for a finite interval, Dokl. Ak. Nauk SSSR, 14, 531–536 (1937).
4. Kutateladze, S. S. – Support sets for sublinear operators, Dokl. Ak. Nauk. SSSR, 230 (1976) and Soviet Math., 17, 1428–1431 (1976).
5. Luxemburg, W. A. J. – A remark on Sikorski's extension theorem for homomorphisms in the theory of Boolean algebras, Fund. Math., 55, 239–247 (1964).
6. Luxemburg, W. A. J. and A. R. Schep – A Radon-Nikodym type theorem for positive operators and a dual, Ind. Math., 81, 357–375 (1978).
7. Luxemburg, W. A. J. and A. C. Zaanen – Riesz Spaces I, North-Holland Mathemathical Library, Amsterdam 1972.
8. Sikorski, R. – A theorem on extensions of homomorphisms, Ann. Soc. Pol. Math., 21, 332–335 (1948).
9. Veksler, A. I. – A new construction of the Dedekind completion of vector lattices and l-groups with division, Sibirsk Math. Ž., 10, 1206–1213 (1969) and Siber. Math. Journ. 10, 891–896, (1969).
10. Vulikh, B. Z. – Introduction to the Theory of Partially Ordered Spaces, Wolters-Noordhoff, Groningen, 1967.