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Data compression with homomorphism in covering information systems

Changzhong Wang [a,*], Degang Chen [b], Chong Wu [c], Qinhua Hu [d]

[a] Department of Mathematics, Bohai University, Jinzhou 121000, PR China
[b] Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, PR China
[c] School of Management, Harbin Institute of Technology, Harbin, Heilongjiang 150001, PR China
[d] Power Engineering, Harbin Institute of Technology, Harbin 150001, PR China

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ABSTRACT

In reality we are always faced with a large number of complex massive databases. In this work we introduce the notion of a homomorphism as a kind of tool to study data compression in covering information systems. The concepts of consistent functions related to covers are first defined. Then, by classical extension principle the concepts of covering mapping and inverse covering mapping are introduced and their properties are studied. Finally, the notions of homomorphisms of information systems based on covers are proposed, and it is proved that a complex massive covering information system can be compressed into a relatively small-scale information system and its attribute reduction is invariant under the condition of homomorphism, that is, attribute reductions in the original system and image system are equivalent to each other.

1. Introduction

In daily production practice, there are always complex and massive data in information systems. How to remove data redundancy while maintaining the same data structures, such as attribute reduction and decision rules, and how to uncover potential useful information are some important issues concerned. The theory of rough sets, proposed by Pawlak, is a useful tool for studying these problems [12]. It has been successfully applied in such artificial intelligence fields as machine learning, pattern recognition, decision analysis, process control, knowledge discovery in databases, and expert systems.

According to the theory of rough sets, a rough approximation space is actually a granular information world. As for an information system, it can be seen as a combination of some approximation spaces on the same universe [13]. The data compression in information systems is referred to two aspects of operations on data, one is to reduce stored and transferred data volume, the other is to reduce data dimension. Data volume reduction, in mathematics, can be explained as a many-to-one mapping between two information systems. A homomorphism is a special mapping between two information systems. Data dimension reduction can be seen as attribute reduction in information systems. Compression of massive complex data is a promising technique for storage and transmission of data.

Although in recent years many topics on information systems have been widely investigated with rough sets [2,5–9,11, 14–25], there are a few researches that focus on data compression in information systems. The notion of homomorphism as a kind of tool to study the relationship between two information systems was first proposed by Graymala-Busse [3]. Later, Graymala-Busse studied the conditions which make an information system to be selective under endomorphism of the system [4]. Li and Ma discussed some properties of redundancy and reduction of information systems under some

* Corresponding author. Tel.: +86 416 3400192; fax: +86 416 3400196.
E-mail address: changzhongwang@126.com (C. Wang).

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homomorphisms [8]. However, they did not study data compression using homomorphism. In [18], Wang et al. investigated some invariant properties of relation information systems under homomorphisms and proved that attribute reductions in the original system and image system are equivalent to each other under the condition of homomorphism. In [19], Wang defined the notions of homomorphisms in fuzzy information systems and studied equivalent attribute reductions in fuzzy information systems. In these two works, Wang pointed out that the notion of homomorphism between two information systems can be regarded as a kind of tool to study the compression of data volume in complex massive databases.

In some situations, information systems are based on covers rather than binary relations. We call cover-based information systems covering information systems. These kinds of information systems are quite different from relation information systems [18]. Thus the methods to find homomorphisms between two relation information systems [18] are not suitable for dealing with covering information systems. How to search for homomorphisms and study data compression in covering information systems are our new problems.

Covering rough sets [24], as a generalization of classical rough sets, have powerful prospects in applications [1,9,10,22,25]. In this paper, we introduce covering rough sets as a basic tool to study homomorphisms between covering information systems and data compression with homomorphisms in covering information systems. By classical extension principle we develop a method for defining a cover on a universe according to a cover on another universe. We then propose the concept of homomorphism between two information systems. Under the condition of homomorphism, we prove that a complex-massive covering information system can be compressed into a relatively small-scale information system and attribute reducts in the original system and image system are invariant.

This paper is organized as follows. In Section 2, we review some basic notions related to covering rough sets. In Section 3, we define the concepts of consistent functions and study their main properties. In Section 4, we introduce the definitions of covering mappings and investigate their main properties. In Section 5, we introduce the concept of homomorphism between two information systems based on covers and study data compression under the condition of homomorphism. Section 6 presents conclusions.

2. Basic notions related to covering rough sets

In production practice, there always exist a large number of databases which are not suitable for being dealt with by classical rough sets. For example, in a database some objects have multiple attribute values for a given attribute. This kind of database is available when some objects have multi-selection of attribute values for a given attribute. An attribute in this kind of database may induce a cover on the universe rather than a partition. Some illustrative examples of this kind of databases are given in [1,25]. In this section, we review basic concepts about covering rough sets [1].

Definition 2.1. Let U be a universe of discourse, $\mathbf{C}$ a family of subsets of U. $\mathbf{C}$ is called a cover of U if no subset in $\mathbf{C}$ is empty and $\cup\mathbf{C} = U$.

It is clear that a partition of U is certainly a cover of U, so the concept of a cover is an extension of the concept of a partition.

Definition 2.2. Let $\mathbf{C} = \{C_1, C_2, \ldots, C_n\}$ be a cover of U. For every $x \in U$, let

$$C_x = \cap\{C_i : x \in C_i \wedge C_i \in \mathbf{C}\},$$
$$Cov(\mathbf{C}) = \{C_x : x \in U\}.$$

Then Cov(C) is also a cover on U, we call it the induced cover of $\mathbf{C}$.

For every $x \in U$, C_x is the minimal subset including x in $Cov(\mathbf{C})$, every element in $Cov(\mathbf{C})$ cannot be written as the union of other elements in $Cov(\mathbf{C})$. $Cov(\mathbf{C}) = \mathbf{C}$ if and only if $\mathbf{C}$ is a partition. For any $x, y \in U$, if $y \in C_x$ then $C_x \supseteq C_y$. So if $y \in C_x$ and $x \in C_y$, then $C_x = C_y$.

Definition 2.3. Let $\mathbf{\Delta} = \{\mathbf{C}_i: i = 1, \ldots, m\}$ be a family of covers of U. For every $x \in U$, let

$$\Delta_x = \cap\{C_{ix} : C_{ix} \in Cov(\mathbf{C}_i),\ i = 1, 2, \cdots, m\},$$
$$Cov(\mathbf{\Delta}) = \{\Delta_x : x \in U\}.$$

Then $Cov(\mathbf{\Delta})$ is also a cover on U, we call it the induced cover of $\mathbf{\Delta}$.

Clearly Δ_x is the intersection of all the covering elements including x in all covers in $\mathbf{\Delta}$, so for every $x \in U$, Δ_x is the minimal subset including x in $Cov(\mathbf{\Delta})$ and $Cov(\mathbf{\Delta})$ can be viewed as the intersection of covers in $\mathbf{\Delta}$. Every element in $Cov(\mathbf{\Delta})$ cannot be written as the union of other elements in $Cov(\mathbf{\Delta})$. If every cover in $\mathbf{\Delta}$ is a partition, then $Cov(\mathbf{\Delta})$ is also a partition and Δ_x is the equivalence class including x. For any x, $y \in U$, if $y \in \Delta_x$, then $\Delta_x \supseteq \Delta_y$. So if $y \in \Delta_x$ and $x \in \Delta_y$, then $\Delta_x = \Delta_y$.

3. Main properties of consistent functions related to covers

Let U and V be universes. The class of all covers on U (respectively, on V) will be denoted by $C(U)$ (respectively, by $C(V)$). In the subsequent discussion, a cover in $C(U)$ (respectively, in $C(V)$) is always denoted by $\mathbf{C}$ (respectively, by $\mathbf{E}$). In this section, we introduce the concepts of consistent functions related to covers and investigate their properties.

Definition 3.1. Let f: $U \to V$ a mapping from U to V, $\mathbf{C} = \{C_1, C_2, \ldots, C_n\}$ a cover of U and $Cov(\mathbf{C}) = \{C_x : x \in U\}$. Let $[x] = \{y \in U: f(y) = f(x)\}$. If $[x] \subseteq C_x$ for any $x \in U$, then f is referred to as a consistent function with respect to $\mathbf{C}$.

Theorem 3.1. *Let f: $U \to V$ be a mapping from U to V and $\mathbf{C} = \{C_1, C_2, \ldots, C_n\}$ a cover of U. If f is a consistent function with respect to $\mathbf{C}$, then $f(C_i \cap C_j) = f(C_i) \cap f(C_j)$, $\forall i, j \leqslant n$.*

Proof. At first, we are to prove that if $C_i \cap C_j = \emptyset$, then $f(C_i) \cap f(C_j) = \emptyset$.

Assume that $f(C_i) \cap f(C_j) \neq \emptyset$, let $u \in f(C_i) \cap f(C_j)$, then $u \in f(C_i)$ and $u \in f(C_j)$. Thus there exist $x \in C_i$ and $y \in C_j$ such that $f(x) = u$ and $f(y) = u$, which implies $[x] = [y]$. By the definitions of C_x and C_y, we have $C_x \subseteq C_i$ and $C_y \subseteq C_j$. Since f is a consistent function with respect to $\mathbf{C}$, it follows that $[x] \subseteq C_x$ and $[y] \subseteq C_y$. Hence $[x] \subseteq C_i$ and $[y] \subseteq C_j$, and so $[x] = [y] \subseteq C_i \cap C_j$, which is a contradiction. Therefore $f(C_i) \cap f(C_j) = \emptyset$.

Next, we are to prove that if $C_i \cap C_j \neq \emptyset$, then $f(C_i \cap C_j) = f(C_i) \cap f(C_j)$.

Since it is always true that $f(C_i \cap C_j) \subseteq f(C_i) \cap f(C_j)$, we only need to prove that $f(C_i \cap C_j) \supseteq f(C_i) \cap f(C_j)$.

Let u be an arbitrary element in $f(C_i) \cap f(C_j)$, i.e., $u \in f(C_i) \cap f(C_j)$, then $u \in f(C_i)$ and $u \in f(C_j)$. Thus there exist $x \in C_i$ and $y \in C_j$ such that $f(x) = u$ and $f(y) = u$, which implies $[x] = [y]$. By the definitions of C_x and C_y, we have $C_x \subseteq C_i$ and $C_y \subseteq C_j$. Since f is a consistent function with respect to $\mathbf{C}$, it follows that $[x] \subseteq C_x$ and $[y] \subseteq C_y$. Hence $[x] \subseteq C_i$ and $[y] \subseteq C_j$, and so $[x] = [y] \subseteq C_i \cap C_j$. This means that $f([x]) = f([y]) \subseteq f(C_i \cap C_j)$. Again, Since f is a consistent function with respect to $\mathbf{C}$, we have that $f([x]) = f([y]) = f(x) = f(y) = u$, which implies $u \in f(C_i \cap C_j)$. Hence $f(C_i \cap C_j) \supseteq f(C_i) \cap f(C_j)$. Therefore $f(C_i \cap C_j) = f(C_i) \cap f(C_j)$, $\forall i, j \leqslant n$. □

Theorem 3.2. *Let f: $U \to V$ be a mapping from U to V and $\mathbf{C} = \{C_1, C_2, \cdots, C_n\}$ a cover of U. If f is a consistent function with respect to $\mathbf{C}$, then $\forall C_i \in \mathbf{C}$, $f^{-1}(f(C_i)) = C_i$.*

Proof. Since $f^{-1}(f(C_i)) \supseteq C_i$ is obviously true, we only need to prove $f^{-1}(f(C_i)) \subseteq C_i$.

Let $x \in f^{-1}(f(C_i))$, then $f(x) \in f(C_i)$. Thus there exists $y \in C_i$ such that $f(x) = f(y)$. By the definition of C_y, we have $C_y \in C_i$. Since f is a consistent function with respect to $\mathbf{C}$, it follows that $[x] = [y] \subseteq C_y$. This implies $[x] = [y] \subseteq C_i$. Hence $x \in [x] \subseteq C_i$. So $f^{-1}(f(C_i)) \subseteq C_i$. Therefore $f^{-1}(f(C_i)) = C_i$. □

Definition 3.2. Let U be a finite universe and $\mathbf{C}_1$, $\mathbf{C}_2 \in C(U)$, where $\mathbf{C}_1 = \{C_{11}, C_{12}, \ldots, C_{1m}\}$, $\mathbf{C}_2 = \{C_{21}, C_{22}, \ldots, C_{2n}\}$. Let

$$\mathbf{C}_1 \cap \mathbf{C}_2 = \{C_{1x} \cap C_{2x} : C_{ix} \in Cov(\mathbf{C}_i),\ i = 1, 2,\ x \in U\}.$$

Then $\mathbf{C}_1 \cap \mathbf{C}_2$ is called the intersection of $\mathbf{C}_1$ and $\mathbf{C}_2$.

From Definition 3.2, for every $x \in U$, $C_{1x} \cap C_{2x}$ is the minimal subset including x in $\mathbf{C}_1 \cap \mathbf{C}_2$. Clearly $\mathbf{C}_1 \cap \mathbf{C}_2$ is a cover of U.

Theorem 3.3. *Let f: $U \to V$ and $\mathbf{C}_1$, $\mathbf{C}_2 \in C(U)$. If f is consistent with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, then f is consistent with respect to $\mathbf{C}_1 \cap \mathbf{C}_2$.*

Proof. For any $x \in U$, let C_x be the minimal subset including x in $\mathbf{C}_1 \cap \mathbf{C}_2$, C_{1x} the minimal subset including x in $Cov(\mathbf{C}_1)$ and C_{2x} the minimal subset including x in $Cov(\mathbf{C}_2)$. From Definition 3.2, we have $C_{1x} \cap C_{2x} = C_x$.

Since f is consistent with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, it follows that $[x] \subseteq C_{1x}$ and $[x] \subseteq C_{2x}$ for any $x \in U$. Thus we get $[x] \subseteq C_{1x} \cap C_{2x} = C_x$. Then we know that f is consistent with respect to $\mathbf{C}_1 \cap \mathbf{C}_2$ from Definition 3.1. □

4. Main properties of covering mappings

In this section, we extend the concepts of mappings between classical sets to the mappings between two power sets. Based on the idea of the classical extension principle, we introduce the concepts of covering mappings as follows.

Definition 4.1. Let f: $U \to V$, $x| \to f(x)$ be a surjection. f can induce a mapping from $C(U)$ to $C(V)$ and a mapping from $C(V)$ to $C(U)$, that is,

$$\hat{f} : C(U) \to C(V), \quad \mathbf{C}| \to f(\mathbf{C}) \in C(V), \quad \forall \mathbf{C} \in C(U);$$

$$\hat{f}(\mathbf{C}) \triangleq \{f(C_i) : C_i \in \mathbf{C}\}.$$

$$\hat{f}^{-1} : C(V) \to C(U), \quad \mathbf{E}| \to f^{-1}(\mathbf{E}) \in C(U), \quad \forall \mathbf{E} \in C(V);$$

$$\hat{f}^{-1}(\mathbf{E}) \triangleq \{f^{-1}(E_i) : E_i \in \mathbf{E}\}.$$

Then $\hat{f}$ and $\hat{f}^{-1}$ are called covering mapping and inverse covering mapping induced by f, respectively; $\hat{f}(\mathbf{C})$ and $\hat{f}^{-1}(\mathbf{E})$ are called the image of $\mathbf{C}$ and the inverse image of $\mathbf{E}$, respectively. In the subsequent discussion, we simply denote $\hat{f}$ and $\hat{f}^{-1}$ by f and f^{-1}, respectively.

In the following, we investigate the basic properties of covering mappings.

Theorem 4.1. *Let $\mathbf{C} \in C(U)$. If f is a consistent function with respect to $\mathbf{C}$, then $f^{-1}(f(\mathbf{C})) = \mathbf{C}$.*

Proof. Let C_i be arbitrary subset in $\mathbf{C}$, i.e., $C_i \in \mathbf{C}$. Since f is a consistent function with respect to $\mathbf{C}$, it follows from Theorem 3.2 that $f^{-1}(f(C_i)) = C_i$ for any $C_i \in \mathbf{C}$. Hence $f^{-1}(f(\mathbf{C})) = \mathbf{C}$. □

Corollary 4.1. *Let $f: U \to V$, $\mathbf{\Delta} = \{\mathbf{C}_i: i = 1, \ldots, m\}$. If $\forall \mathbf{C}_i \in \mathbf{\Delta}$, f is a consistent function with respect to $\mathbf{C}_i$ on U, then $f^{-1}(f(\cap\mathbf{\Delta})) = \cap\mathbf{\Delta}$.*

Proof. It follows immediately from Theorems 3.3 and 4.1. □

Theorem 4.2. *Let $f: U \to V$ and $\mathbf{C}_1, \mathbf{C}_2 \in C(U)$. If f is a consistent function with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, then $f(\mathbf{C}_1 \cap \mathbf{C}_2) = f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$.*

Proof. For any $x \in U$, let C_x be the minimal element containing x in $\mathbf{C}_1 \cap \mathbf{C}_2$, C_{1x} the minimal element containing x in $Cov(\mathbf{C}_1)$, C_{2x} the minimal element containing x in $Cov(\mathbf{C}_2)$. From Definition 3.2, we get $C_x = C_{1x} \cap C_{2x}$. Since f is a consistent function with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, it follows from Theorem 3.3 that f is a consistent function with respect to $\mathbf{C}_1 \cap \mathbf{C}_2$. By Theorem 3.1 we have $f(C_x) = f(C_{1x}) \cap f(C_{2x})$.

In the following we prove that $f(C_{1x}) \cap f(C_{2x})$ is the minimal subset containing $f(x)$ in $f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$.

By the reflexivity of C_x, we have $x \in C_x$. Thus $f(x) \in f(C_x)$. Hence $f(x) \in f(C_{1x}) \cap f(C_{2x})$.

Assume that $f(C_{1x}) \cap f(C_{2x})$ is not the minimal subset containing $f(x)$ in $f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$. Then there is at least $y \in U$ such that C_{iy} ($i = 1$ or 2) satisfies

$$f(x) \in f(C_{iy}) \quad \text{and} \quad f(C_{1x}) \cap f(C_{2x}) \cap f(C_{iy}) \subset f(C_{1x}) \cap f(C_{2x}).$$

This implies $f(x) \in f(C_{1x}) \cap f(C_{2x}) \cap f(C_{iy})$. Thus there exist $u \in C_{1x}$, $v \in C_{2x}$ and $w \in C_{iy}$ such that $f(u) = f(x)$, $f(v) = f(x)$ and $f(w) = f(x)$. Since f is a consistent function with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively, it follows from Theorem 3.3 that f is consistent with respect to $\mathbf{C}_1 \cap \mathbf{C}_2$, and so $[x] = [u] = [v] = [w]$ and $[x] \subseteq C_{1x} \cap C_{2x} \cap C_{iy}$ which implies $x \in C_{iy}$. By the definition of C_x, we have $C_x \subseteq C_{iy}$. Thus $C_{1x} \cap C_{2x} \subseteq C_{iy}$. Hence

$$f(C_{1x}) \cap f(C_{2x}) = f(C_{1x} \cap C_{2x}) \subseteq f(C_{iy})$$

and

$$f(C_{1x}) \cap f(C_{2x}) \cap f(C_{iy}) = f(C_{1x}) \cap f(C_{2x}).$$

This is a contradiction. It follows that $f(C_{1x}) \cap f(C_{2x})$ is the minimal subset containing $f(x)$ in $f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$.

Since $f(C_{1x}) \cap f(C_{2x}) = f(C_{1x} \cap C_{2x}) = f(C_x)$, we have that for any $x \in U$, if $C_x \in \mathbf{C}_1 \cap \mathbf{C}_2$, then $f(C_x) \in f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$. Therefore $f(\mathbf{C}_1 \cap \mathbf{C}_2) \subseteq f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$.

Conversely, for any $x \in U$, by the above proof, $f(C_{1x}) \cap f(C_{2x})$ is the minimal subset containing $f(x)$ in $f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$ and we know $f(C_{1x}) \cap f(C_{2x}) = f(C_x) \in f(\mathbf{C}_1 \cap \mathbf{C}_2)$. Thus $f(C_{1x}) \cap f(C_{2x}) \in f(\mathbf{C}_1 \cap \mathbf{C}_2)$. Hence $f(\mathbf{C}_1 \cap \mathbf{C}_2) \supseteq f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$. Therefore $f(\mathbf{C}_1 \cap \mathbf{C}_2) = f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$. □

Remark. In general, a covering mapping f does not keep invariant the intersection of covers.

We immediately get the following corollary from Theorems 3.3 and 4.2.

Corollary 4.2. *Let $f: U \to V$ and $\mathbf{\Delta} = \{\mathbf{C}_i: i = 1, \ldots, m\}$ a family of covers of U. If $\forall \mathbf{C}_i \in \mathbf{\Delta}$, f is consistent respect to $\mathbf{C}_i$ of U, then $f(\bigcap_{i=1}^{n} \mathbf{C}_i) = \bigcap_{i=1}^{n} f(\mathbf{C}_i)$.*

5. Attribute reduction based on homomorphisms

In this section, we introduce the notion of a homomorphism as a kind of tool to study data compression in covering information systems. By compression with homomorphism we can get the relatively smaller image system that has the same reducts as a given original database. Let us start with introducing the notions of covering information systems.

Definition 5.1. Let U and V be finite universes, f: $U \to V$ a surjection from U to V, and $\mathbf{\Delta} = \{\mathbf{C}_i: i = 1, \ldots, m\}$ a family of covers on U. Then the pair $(U, \mathbf{\Delta})$ is referred to as a covering information system and the pair $(V, f(\mathbf{\Delta}))$ is referred to as a f-induced covering information system of $(U, \mathbf{\Delta})$.

Now by Theorem 4.2 and Corollary 4.2, we introduce the following concepts.

Definition 5.2. Let $(U, \mathbf{\Delta})$ be a covering information system and $(V, f(\mathbf{\Delta}))$ a f-induced covering information system of $(U, \mathbf{\Delta})$. If $\forall \mathbf{C}_i \in \mathbf{\Delta}$, f is consistent with respect to $\mathbf{C}_i$, then f is called a homomorphism from $(U, \mathbf{\Delta})$ to $(V, f(\mathbf{\Delta}))$.

Definition 5.3. Let $(U, \mathbf{\Delta})$ be a covering information system and $\mathbf{C}_i \in \mathbf{\Delta}$. Then $\mathbf{C}_i$ is called superfluous in $\mathbf{\Delta}$ if $\cap\{\mathbf{\Delta} - \{\mathbf{C}_i\}\} = \cap\mathbf{\Delta}$; otherwise, $\mathbf{C}_i$ is called indispensable in $\mathbf{\Delta}$. The collection of all indispensable elements in $\mathbf{\Delta}$ is called the core of $\mathbf{\Delta}$, denoted as $Core(\mathbf{\Delta})$. Let $\mathbf{P} \subseteq \mathbf{\Delta}$, then $\mathbf{P}$ is referred to as a reduct of $\mathbf{\Delta}$ if $\mathbf{P}$ satisfies the following conditions:

(1) $\cap\mathbf{P} = \cap\mathbf{\Delta}$;
(2) $\forall \mathbf{C}_i \in \mathbf{P}$, $\cap\mathbf{P} \neq \cap(\mathbf{P} - \mathbf{C}_i)$.

Theorem 5.1. *Let $(U, \mathbf{\Delta})$ be a covering information system, $(V, f(\mathbf{\Delta}))$ a f-induced covering information system of $(U, \mathbf{\Delta})$ and $\mathbf{P} \subseteq \mathbf{\Delta}$. If f is a homomorphism from $(U, \mathbf{\Delta})$ to $(V, f(\mathbf{\Delta}))$, Then $\mathbf{P}$ is a reduct of $\mathbf{\Delta}$ if and only if $f(\mathbf{P})$ is a reduct of $f(\mathbf{\Delta})$.*

Proof. It follows immediately from Definitions 5.2, 5.3 and Corollaries 4.1, 4.2. □

By Theorem 5.1, we immediately get the following corollary.

Corollary 5.1. *Let $(V, f(\mathbf{\Delta}))$ be a $f-$ induced covering information system of $(U, \mathbf{\Delta})$, $\mathbf{C}_i \in \mathbf{\Delta}$ and $\mathbf{P} \subseteq \mathbf{\Delta}$. If f is a homomorphism from $(U, \mathbf{\Delta})$ to $(V, f(\mathbf{\Delta}))$, then*

(1) *$\mathbf{C}_i$ is indispensable in $\mathbf{\Delta}$ if and only if $f(R)$ is indispensable in $f(\mathbf{\Delta})$.*
(2) *$\mathbf{P}$ is superfluous in $\mathbf{\Delta}$ if and only if $f(\mathbf{P})$ is superfluous in $f(\mathbf{\Delta})$.*
(3) *The image of the core of $\mathbf{\Delta}$ is the core of the image of $\mathbf{\Delta}$ and the inverse image of the core of $f(\mathbf{\Delta})$ is the core of the original image. That is, $f(Core(\mathbf{\Delta})) = Core(f(\mathbf{\Delta}))$ and $f^{-1}(Core(f(\mathbf{\Delta}))) = Core(\mathbf{\Delta})$.*

Remark. Theorem 5.1 tells us the fact that for a given covering information system that has great size, if we can find a many-to-one homomorphism of the given information system, then the system can be compressed into a relatively small-scale one and the attribute reductions of the original system and its image system are equivalent to each other. Therefore, we can quickly reduce the given covering information system by reducing its smaller image system. The kind of data compression method not only improves the efficiency of attribute reduction algorithm, but also saves costs in manpower and time.

The following example is employed to illustrate that under the condition of homomorphism, a complex-massive covering information system can be compressed into a relatively small-scale information system and the reducts in the original system are the same as ones in the image system.

Example 5.1. Now we consider a house evaluation problem. Suppose U = $\{x_1, \ldots, x_{15}\}$ is a set of nine houses, E = {price; structure; surrounding} is a set of attributes, the values of "price" are {high; middle; low}, the values of "structure" are {reasonable; ordinary; unreasonable}, and the values of "surrounding" are {quiet; noisy; quite noisy}. We have four specialists to evaluate the attributes of these houses, they are {A, B, C, D}, then it is possible that their evaluation results for the same attribute are not the same as one another. The evaluation results are listed below.

For attribute "price"

$$\text{A : high} = \{x_1, x_2, x_4, x_{10}, x_{15}\}, \quad \text{middle} = \{x_6, x_8, x_9, x_{13}, x_{14}\}, \quad \text{low} = \{x_3, x_5, x_7, x_{11}, x_{12}\};$$

$$\text{B : high} = \{x_1, x_2, x_4, x_{15}\}, \quad \text{middle} = \{x_6, x_8, x_9, x_{10}, x_{13}, x_{14}\}, \quad \text{low} = \{x_3, x_5, x_7, x_{11}, x_{12}\};$$

$$\text{C : high} = \{x_1, x_2, x_8, x_{10}, x_{15}\}, \quad \text{middle} = \{x_4, x_6, x_9, x_{13}, x_{14}\}, \quad \text{low} = \{x_3, x_5, x_7, x_{11}, x_{12}\};$$

$$\mathrm{D}: \mathrm{high} = \{x_1, x_2, x_8, x_{15}\}, \quad \mathrm{middle} = \{x_4, x_6, x_9, x_{10}, x_{13}, x_{14}\}, \quad \mathrm{low} = \{x_3, x_5, x_7, x_{11}, x_{12}\}.$$

For attribute "surrounding":

$$\mathrm{A}: \mathrm{quiet} = \{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{noisy} = \{x_1, x_2, x_4, x_7, x_8, x_{10}, x_{11}, x_{12}\}, \quad \mathrm{quitenoisy} = \{x_{15}\};$$

$$\mathrm{B}: \mathrm{quiet} = \{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{noisy} = \{x_2, x_4, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{15}\}, \quad \mathrm{quitenoisy} = \{x_1\};$$

$$\mathrm{C}: \mathrm{quiet} = \{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{noisy} = \{x_2, x_4, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{15}\}, \quad \mathrm{quitenoisy} = \{x_1\};$$

$$\mathrm{D}: \mathrm{quiet} = \{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{noisy} = \{x_1, x_4, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{15}\}, \quad \mathrm{quitenoisy} = \{x_2\}.$$

For attribute "structure":

$$\mathrm{A}: \mathrm{reasonable} = \{x_1, x_2, x_3, x_7, x_{11}, x_{12}\}, \quad \mathrm{ordinary} = \{x_5, x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{unreasonable} = \{x_4, x_8, x_{10}, x_{15}\}$$

$$\mathrm{B}: \mathrm{reasonable} = \{x_1, x_7, x_{11}, x_{12}\}, \quad \mathrm{ordinary} = \{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{unreasonable} = \{x_2, x_4, x_8, x_{10}, x_{15}\};$$

$$\mathrm{C}: \mathrm{reasonable} = \{x_2, x_3, x_7, x_{11}, x_{12}, x_{15}\}, \quad \mathrm{ordinary} = \{x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{unreasonable} = \{x_1, x_4, x_5, x_8, x_{10}\}$$

$$\mathrm{D}: \mathrm{reasonable} = \{x_5, x_7, x_{11}, x_{12}, x_{15}\}, \quad \mathrm{ordinary} = \{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \quad \mathrm{unreasonable} = \{x_1, x_2, x_4, x_8, x_{10}\}.$$

Assume the evaluation of every specialist is of the same importance. If we want to combine these evaluations together without losing information, we should union the evaluations given by every specialist for every attribute. Then for every attribute we get a cover instead of a partition, which embodies a kind of uncertainty.

For attribute "price" we get

$$\mathbf{C}_1 = \{\{x_1, x_2, x_4, x_8, x_{10}, x_{15}\}, \{x_4, x_6, x_8, x_9, x_{10}, x_{13}, x_{14}\}, \{x_3, x_5, x_7, x_{11}, x_{12}\}\}.$$

For attribute "surrounding" we get

$$\mathbf{C}_2 = \{\{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \{x_1, x_2, x_4, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{15}\}, \{x_1, x_2, x_{15}\}\}.$$

For attribute "structure" we get

$$\mathbf{C}_3 = \{\{x_1, x_2, x_3, x_5, x_7, x_{11}, x_{12}, x_{15}\}, \{x_3, x_5, x_6, x_9, x_{13}, x_{14}\}, \{x_1, x_2, x_3, x_4, x_5, x_8, x_{10}, x_{15}\}\}.$$

Let $(U, \mathbf{\Delta})$ be a covering information system, where $U = \{x_1, x_2, \ldots, x_{15}\}$ and $\mathbf{\Delta} = \{\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3\}$.

Thus $\mathbf{\Delta}_{x_1} = \mathbf{\Delta}_{x_2} = \mathbf{\Delta}_{x_5} = \{x_1, x_2, x_5\}$, $\mathbf{\Delta}_{x_3} = \mathbf{\Delta}_{x_5} = \{x_3, x_5\}$, $\mathbf{\Delta}_{x_4} = \mathbf{\Delta}_{x_8} = \mathbf{\Delta}_{x_{10}} = \{x_4, x_8, x_{10}\}$, $\mathbf{\Delta}_{x_6} = \mathbf{\Delta}_{x_9} = \mathbf{\Delta}_{x_{13}} = \mathbf{\Delta}_{x_{14}} = \{x_6, x_9, x_{13}, x_{14}\}$, $\mathbf{\Delta}_{x_7} = \mathbf{\Delta}_{x_{11}} = \mathbf{\Delta}_{x_{12}} = \{x_7, x_{11}, x_{12}\}$.

Let $V = \{y_1, y_2, y_3, y_4, y_5\}$. Define a mapping f: $U \to V$ as follows:

x_1, x_2, x_{15}	x_3, x_5	x_4, x_8, x_{10}	x_6, x_9, x_{13}, x_{14}	x_7, x_{11}, x_{12}
y_1	y_2	y_3	y_4	y_5

Then $f(\mathbf{\Delta}) = \{f(\mathbf{C}_1), f(\mathbf{C}_2), f(\mathbf{C}_3)\}$, where

$$f(\mathbf{C}_1) = \{\{y_1, y_3\}, \{y_3, y_4\}, \{y_2, y_5\}\},$$
$$f(\mathbf{C}_2) = \{\{y_2, y_4\}, \{y_1, y_3, y_5\}, \{y_1\}\},$$
$$f(\mathbf{C}_3) = \{\{y_1, y_2, y_5\}, \{y_2, y_4\}, \{y_1, y_2, y_3\}\}.$$

Hence $(V, f(\mathbf{\Delta}))$ is the f-induced covering information system of $(U, \mathbf{\Delta})$. We can verify that f is a homomorphism from $(U, \mathbf{\Delta})$ to $(V, f(\mathbf{\Delta}))$ and verify that $f(\mathbf{C}_3)$ is superfluous in $f(\mathbf{\Delta})$ if and only if $\mathbf{C}_3$ is superfluous in $\mathbf{\Delta}$ and that $\{f(\mathbf{C}_1), f(\mathbf{C}_2)\}$ is a reduct of $f(\mathbf{\Delta})$ if and only if $\{\mathbf{C}_1, \mathbf{C}_2\}$ is a reduct of $\mathbf{\Delta}$. In this example the covering information system $(U, \mathbf{\Delta})$ has great size. After data compression by homomorphism, its image system $(V, f(\mathbf{\Delta}))$ becomes relatively smaller and has the same reducts as the original database. Therefore, we can reduce the original system by reducing the smaller image system.

6. Conclusions

This paper point out that a covering mapping between covering approximation spaces can be explained as a mapping between the given covering information systems. A homomorphism is a special covering mapping between two covering information systems. The data compression in a covering information system includes two aspects of the operations on data; one is to reduce stored and transferred data volume with a many-to-one homomorphism between two covering information systems. The other is to reduce data dimension via attribute reduction. Under the condition of homomorphism, we investigate some invariant properties of covering information systems and prove that a massive covering information system can be compressed into a relatively small-scale covering information system by searching for a homomorphism. By compression with homomorphism we can get the smaller image system that has the same reducts as a given original database. Thus we can perform equivalent attribute reductions in the smaller compressed image database. We believe that these results will have useful applications in extraction of decision rules, attribute reduction and reasoning about data.

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