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Isohedra with dart-shaped faces

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Dedicated to Helge Tverberg on the occasion of his 65th birthday

Abstract

A polyhedron in $\mathbb{E}^3$ is said to be isohedral (or an isohedron) if its faces are equivalent under the action of its group of symmetries. We use Möbius nets of the three reflection groups of the five Platonic solids to construct isohedra whose faces are dart-shaped, and whose edges lie in planes of reflective symmetry of the polyhedron. This technique for constructing isohedra has only recently been used; it yields many new results in addition to those described in this paper. In the final section we also describe some other isohedra with dart-shaped faces. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

An *isohedron* is a polyhedron in Euclidean 3-space $\mathbb{E}^3$ in which all faces form a single orbit under the group of isometric symmetries of the polyhedron. In the case of convex polyhedra these have been studied for close to two centuries. More recently isohedra that have nonconvex faces (but no selfintersections) have been investigated by Grünbaum and Shephard [7]. However, none of these have faces which are “darts”, that is, simple nonconvex quadrangles. This leads to a number of questions, some of which are still open (see Section 7). Here we solve one of these problems by showing that there exist many isohedra with dart-shaped faces if selfintersections of the polyhedron are allowed. We restrict attention to isohedra whose symmetry groups are the groups of all symmetries of the Platonic solids, that is, to the tetrahedral, octahedral and icosahedral reflection groups. Among these polyhedra we initially consider only those whose edges lie in planes of reflective symmetry.

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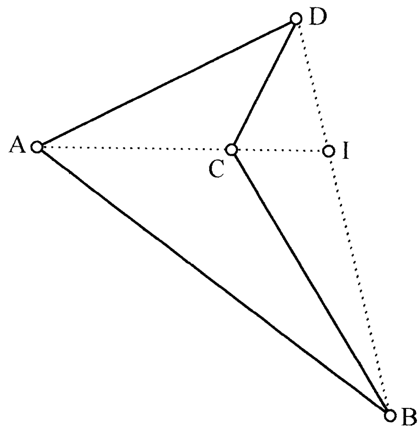


Fig. 1. A dart V with apex A and nadir C; I is the intersection point of the external diagonal with the (extended) internal diagonal. Here $\alpha(V)=2$ and $\gamma(V)=1/3$.

We shall enumerate all possible types. In the case of isohedra with tetrahedral symmetry, these types will be described in some detail. Our description of the octahedral case will be more cursory, and in the case of the icosahedral group only summary results will be given here.

Section 2 of the paper is concerned with darts and their properties. Section 3 gives a detailed description of the types of polyhedra considered, and also of the Möbius nets of the relevant symmetry groups. The main result and its proof will be presented in Section 4, together with more details in the tetrahedral case. Section 5 is devoted to isohedra with the octahedral symmetry group, and Section 6 to those with the icosahedral symmetry group. Section 7 contains comments and open problems.

2. Dart-shaped quadrangles

A *dart* V is any simple (that is, non-selfintersecting) planar non-convex quadrangle. Each dart has one reflex angle (that is, angle greater than 180°); the vertex at that angle is called the *nadir* of the dart, and the opposite vertex is called the *apex*. The *internal* diagonal connects the apex and the nadir; it lies in the interior of the dart. The *external* diagonal connects the *side vertices* of V; it lies in the exterior of the dart.

Each class of affinely equivalent darts can be characterized by two strictly positive real parameters which we shall call the *asymmetry* α and the *concavity* γ. These are defined as follows, see Fig. 1. Let I be the intersection point of the external diagonal of the dart V and the line determined by its internal diagonal. With the labeling of Fig. 1, in which A is the apex, C the nadir of V, and B,D the side vertices, we use $\alpha(V)$ and $\gamma(V)$ for the ratios of the lengths $|BI|/|ID|$ and $|CI|/|AI|$, respectively. It is immediate that $\alpha(V)$ and $\gamma(V)$ are affine invariants and determine the affine class of V unambiguously. Note that $\gamma < 1$, but the value of α is unrestricted and both α and $1/\alpha$

determine the same affine class. One could, of course, select the smaller of α and $1/\alpha$, but we find it advantageous not to do so. Darts V with $\alpha(V)=1$ are called *symmetric* (since the internal diagonal lies in a line of mirror symmetry), and these play a special role in the investigation. The following lemma is basic.

Lemma. *Let a,b,c,d be rays from the origin O contained in an open halfspace, such that no three are coplanar and c is contained in the interior of the convex hull of $\{a,b,d\}$. Then for each point $C\in c$ and for each pair of positive reals α and γ with $\gamma<1$ there exist uniquely determined points $A\in a$, $B\in b$, $D\in d$ such that the quadrangle $ABCD$ is a dart V with apex A and nadir C, satisfying $\alpha(V)=\alpha$, and $\gamma(V)=\gamma$.*

Proof. For any $A\in a$, let I be the intersection of the line through A and C with the plane determined by b and d. Then the ratio $|CI|/|AI|$ is a strictly monotone continuous function of A, which attains arbitrarily small positive values for A sufficiently far from O on the ray a, and values arbitrarily close to 1 for A near O. Hence there is a unique A such that $|CI|/|AI|=\gamma$. Now consider, in the plane of b and d, the ratio $|BI|/|ID|$. This is a strictly monotone continuous function of B, attaining both arbitrarily small positive values and arbitrarily large values. Hence it attains each of the values α and $1/\alpha$ for precisely one point $B\in b$. Thus the dart is uniquely determined by C, α and γ. □

In particular, we observe that whatever rays are chosen, there exists a dart with vertices A, B, C and D on the rays a,b,c,d which is symmetric, that is, $\alpha=1$. Moreover, by similarity it follows that the darts corresponding to different choices of $C\in c$, but the same values of α and γ, are all similar and lie in parallel planes.

3. Polyhedra and Möbius nets of symmetry groups

Although the study of polyhedra is as old as geometry, there is no general agreement as to which collections of polygons should be called polyhedra. Various definitions are appropriate according to the context. Here we shall adopt the following definition.

A finite collection of (plane) polygons (*faces*) will be called a *usual polyhedron* if it satisfies the following properties:

(i) Each edge of a face is also the edge of precisely one other face. These two faces are said to be incident with the edge, as well as with the endpoints of the edge, which are vertices of the polyhedron.
(ii) All faces incident with a vertex form a single combinatorial cycle, in which adjacent faces are both incident with the same edge.
(iii) Not all the faces lie in a plane.
(iv) The collection of faces is strongly connected, that is, any two faces are connected by a chain of faces in which adjacent faces share an edge.

It is sometimes useful to consider collections of polygons that fail to satisfy one or more of the above conditions. We refer to these as *unusual* polyhedra. Among the types of unusual polyhedra we shall encounter we mention here the following:

(i) Polyhedra with the *vertex-pair* property contain pairs of faces that share two distinct vertices but no edge. Note that the Kepler–Poinsot regular polyhedra $\{5/2, 5\}$ and $\{5, 5/2\}$ have the vertex-pair property.
(ii) Polyhedra with the *multicycle* property have one or more vertices such that the faces incident with each form two or more disjoint combinatorial cycles.
(iii) *Compounds*, that is, polyhedra which fail to satisfy the requirement of strong connectivity.

In listing the dart-faced isohedra we shall generally indicate whether the polyhedron is unusual, and if so, in which way. Additional comments on this topic will be found in the last section.

Let $\mathscr{S}$ be a finite group of isometries of $\mathbb{E}^3$ which leave the origin O fixed. If $\mathscr{S}$ is generated by reflections in a suitable set of mirrors (such a group is sometimes called a Coxeter group) the *Möbius net* $M(\mathscr{S})$ of $\mathscr{S}$ is the triangulation of the unit sphere S centered at O determined by all the mirrors of $\mathscr{S}$. We shall be concerned primarily with the *tetrahedral* group $\mathscr{J}$ (with 6 mirrors and 24 elements), the *octahedral* group $\mathscr{O}$ (with 9 mirrors and 48 elements), and the *icosahedral* group $\mathscr{I}$ (with 15 mirrors and 120 elements). An example of such a net for the octahedral group appears in Fig. 2 of Grünbaum [5], and a net for the icosahedral group in Fig. 7 of Coxeter and Grünbaum [2]; all three are given in Fig. 1 of Shephard [9]. However, in the present context it turns out to be more convenient, instead of the Möbius nets themselves, to use their stereographic projections. Some of these are shown in Fig. 2, the different nets for each group arising from the choice of the center of projection.

The general idea for producing isohedra from Möbius nets is very simple, and it is surprising that it appears not to have been used before the recent paper Grünbaum [5] (see also Shephard [9], Coxeter and Grünbaum [2]). The application of the same idea to isohedra with dart-shaped faces, and to triangle-faced isohedra, was considered by the authors several years ago. Besides the present paper, these discussions led to two others. The case of *deltahedra* (isohedra with equilateral triangles as faces) is treated in Shephard [9], while isohedra with arbitrary triangles as faces are considered in Grünbaum [6].

The steps of the construction of isohedra from Möbius nets are as follows. First select a suitable family $\mathbb{F}$ of triangles from the Möbius net $M(\mathscr{S})$ of the chosen group $\mathscr{S}$. Then choose a plane L that does not pass through O, and project $\mathbb{F}$ from O into L to get a family $\mathbb{F}^*$ of planar triangles; the union of these triangles is a polygonal subset F of L. Finally, create copies of F by applying to F each of the elements of $\mathscr{S}$. Clearly, the complex P generated by all these copies is invariant under $\mathscr{S}$ and, with appropriate choices of $\mathbb{F}$, the complex P will be a polyhedron, hence an isohedron.

However, a number of considerations have to be taken into account when this method is applied.

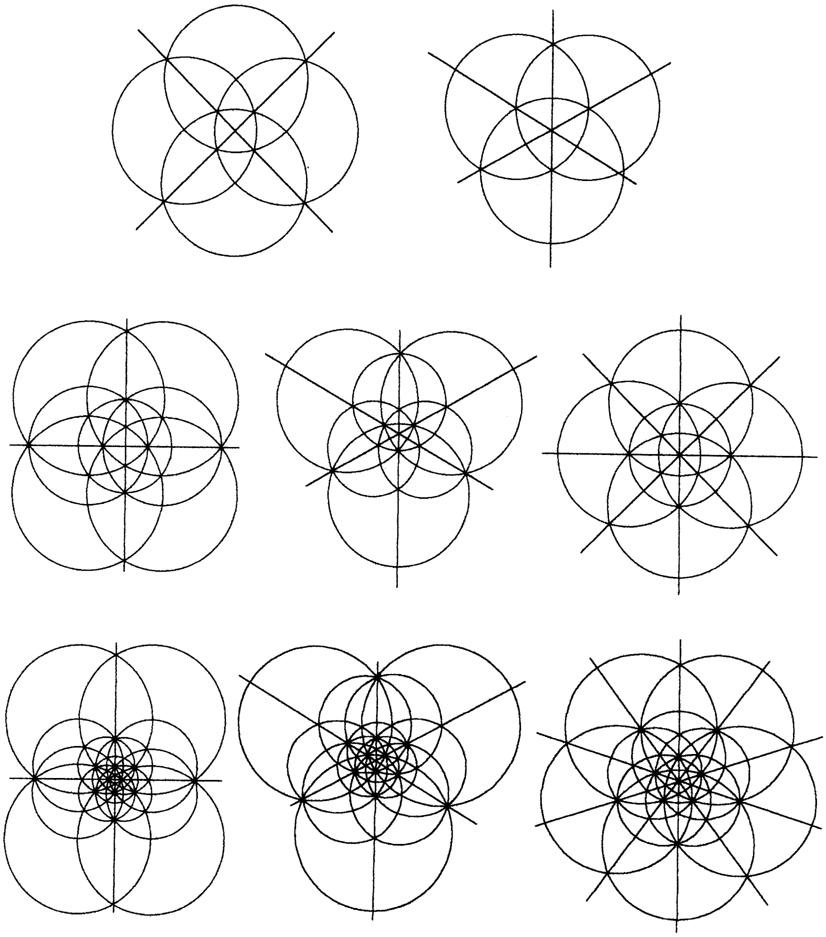


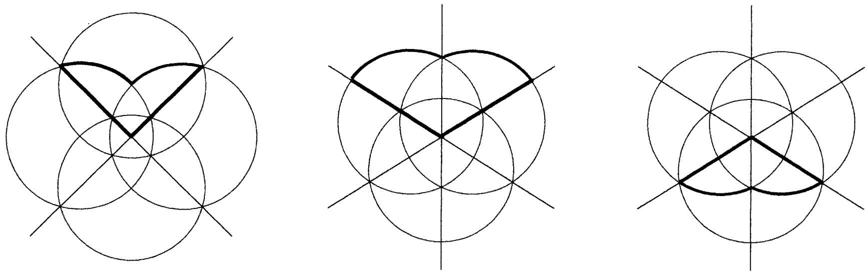
Fig. 2. Stereographic projections of the Möbius nets. Top row: $M(\mathscr{T})$; middle row: $M(\mathscr{O})$; bottom row: $M(\mathscr{I})$.

(i) The family $\mathbb{F}$ has to be contained in an open hemisphere, and the plane L has to be parallel to the bounding plane of such a hemisphere.

(ii) In general, P consists of 24 copies of F for the tetrahedral group $\mathscr{T}$, of 48 copies of F in the case of the octahedral group $\mathscr{O}$, and of 120 copies of F in the case of the icosahedral group $\mathscr{I}$. These copies necessarily share edges as in a polyhedron, but there may be other incidences or properties that make the polyhedron unusual.

(iii) In certain cases some elements of the group may map the set F onto itself (that is, the stabilizer of F in the group may be non-trivial). Then the number of faces and/or vertices will be smaller than in the general case. In particular, this happens if $\mathbb{F}$ is *balanced* (that is, has mirror symmetry) and $\alpha = 1$.

These statements will now be illustrated by examples. Suppose $\mathbb{F}$ consists of a single triangle in the octahedral net $M(\mathscr{O})$. If the plane L is nearly perpendicular, but not per-



DT1 type $\mathscr{T}[2;2;3,3]$ $\Delta = 4$ DT2 type $\mathscr{T}[3;3;2,2]$ $\Delta = 6$ DT3 type $\mathscr{T}[3;3;3,3]$ $\Delta = 4$

Fig. 3. The three types of darts in the Möbius net $M(\mathscr{T})$ of the tetrahedral group $\mathscr{T}$.

pendicular, to a 4-fold axis of the group $\mathcal{O}$ (that is, a ray incident with eight Möbius triangles) then the 48 triangles of P constitute an isohedron isomorphic to the Catalan hexakis octahedron (see, for example, Cundy and Rollett [3, Table II]). We note that here, and throughout, all polyhedra in are considered to be 2-dimensional complexes; if names of traditional polyhedra $\mathbb{E}^3$ are used, it will always be assumed that we are dealing with their boundary complexes. If L is perpendicular to a 4-fold axis, then P will consist of six octuplets of coplanar triangles, namely, the barycentric subdivision of the cube. On the other hand, if $\mathbb{F}$ consists of the eight triangles incident with a 4-fold axis and L perpendicular to this axis, then F is a square, and P is the boundary of a cube.

4. The tetrahedral group

Starting with the Möbius net $M(\mathscr{T})$ of the tetrahedral symmetry group $\mathscr{T}$, it is easy to verify that we can select from $M(\mathscr{T})$, in exactly three essentially different ways, a family $\mathbb{F}$ of triangles that is contained in an open hemisphere and is bounded by a dart-shaped spherical quadrangle Q, the edges of which lie along great circles of the net. These three darts, all of which are balanced, are shown in Fig. 3. The rays through the four vertices of such a dart form a family of the type specified in the Lemma of Section 2. Hence, for every choice of positive α and γ with $\gamma < 1$, there exists a planar dart V with these parameters, whose central projection is Q. The images of V under the reflections of the tetrahedral group yield the isohedron. The values of α affect not only the shape, but in many cases the combinatorial type as well. (We say that two polyhedra have the same combinatorial type if they have the same incidences between their elements, that is, vertices, edges and faces; notice that this refers to the combinatorial structure, and not to the position of a point representing a vertex relative to the sets representing edges or faces.) In particular, the value $\alpha = 1$ (the symmetric case) often results in exceptional properties, and in unusual isohedra.

Below each part of Fig. 3 we show a *type symbol* for the dart. This is written as $\mathscr{T}[r;s;t,u]$, where $\mathscr{T}$ stands for "tetrahedral group". The apex of the dart lies on

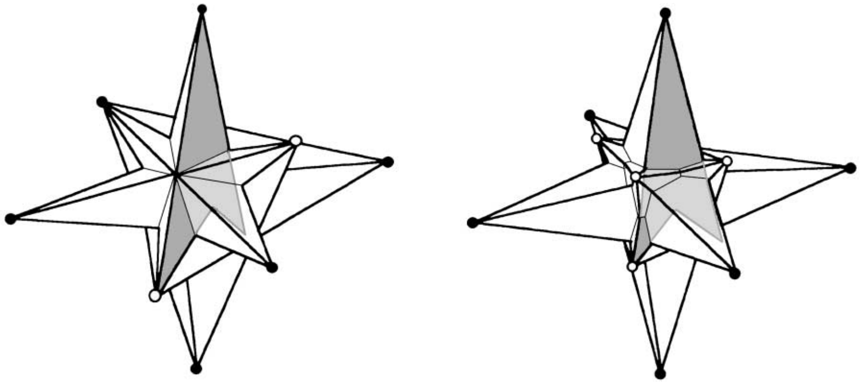


Fig. 4. Isohedra with darts DT1 of type $\mathscr{J}[2;2;3,3]$. (a) $\alpha = 1/2$, $\gamma = 1/5$. Six vertices of the transitivity class formed by the apexes are shown by solid dots; the six vertices at which the nadirs meet are inside the polyhedron, and are not shown. Two of the four vertices formed by the one class of side vertices of the darts are shown by hollow dots, while the other two, and all four of the other transitivity class, are hidden. The thin lines indicate intersections of the faces of the polyhedron which are not edges. One face is shown as the shaded area; the lighter part is inside the polyhedron. The same convention is used in the other illustrations. The coordinates of the vertices of this face, starting with its apex, are: $(0, 0, 1)$, $(3/8, 3/8, -3/8)$, $(0, 1/5, 0)$, $(-3/16, 3/16, -3/16)$. (b) $\alpha = 1$, $\gamma = 1/5$. The faces are balanced darts, and the polyhedron has octahedral symmetry. The coordinates of the vertices of the emphasized face are $(0, 0, 1)$, $(1/4, 1/4, -1/4)$, $(0, 1/5, 0)$, $(-1/4, 1/4, -1/4)$.

an r-fold symmetry axis, the nadir on an s-fold symmetry axis, and the side vertices on t-fold and u-fold symmetry axes. It is convenient to put the integers t and u in lexicographic order. We also indicate the number Δ of spherical triangles that form $\mathbb{F}$. In many cases this is the number of intersections of a ray from the center of the isohedron with the faces. It is thus analogous to the "density" that is often defined for star-polygons.

Theorem 1. *Each of the three types of spherical darts in the tetrahedral Möbius net leads to isohedra with tetrahedral symmetry and dart-shaped faces, in which every edge lies in a plane of mirror symmetry of the polyhedron. In each case, all values of α and $\gamma < 1$ are possible. The seven combinatorial types of isohedra which result from darts are described below.*

Darts DT1, type $\mathscr{J}$[2;2;3,3]. For spherical darts of this type, the resulting isohedra are unusual with the vertex-pair property for every value of α and $\gamma < 1$. All have the same combinatorial type. If $\alpha \neq 1$ the isohedron has 20 vertices and 24 faces. There are two distinct transitivity classes of six vertices each at the 2-fold symmetry axes of the polyhedron, and two transitivity classes of four vertices each at 3-fold symmetry axes. In the symmetric case ($\alpha = 1$) the isohedron has symmetric faces, octahedral symmetry group, and the eight vertices at 3-fold symmetry axes form one transitivity class. Both possibilities are illustrated in Fig. 4.

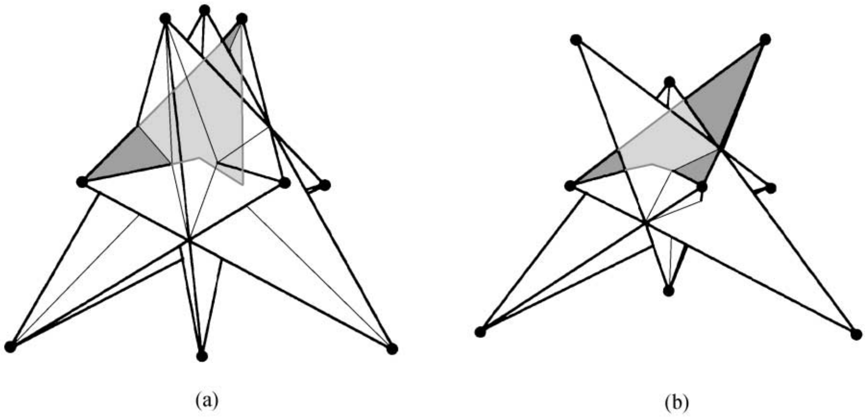


Fig. 5. Examples of isohedra with darts DT2 of type $\mathscr{I}[3;3;2,2]$. (a) $\alpha=1/2$, $\gamma=3/20$. The polyhedron has 24 faces and 20 vertices; the vertices are of one face at points $(1,1,1)$, $(-18/17,0,0)$, $(-3/20,-3/20,3/20)$, $(0,-9/17,0)$. (b) $\alpha=1$, $\gamma=3/20$. The polyhedron has 12 faces and 14 vertices; the vertices of one face are at points $(1,1,1)$, $(-12/17,0,0)$, $(-3/20,-3/20,3/20)$, $(0,-12/17,0)$.

Darts DT2, *type* $\mathscr{I}$[3;3;2,2]. In the case $\alpha\neq 1$ the isohedra are all of the same combinatorial type, with 20 vertices and 24 faces. There are two distinct transitivity classes of six vertices each on the 2-fold symmetry axes of the polyhedron, and two transitivity classes of four vertices each on the 3-fold symmetry axes. In the symmetric case ($\alpha=1$) the isohedra are of another combinatorial type, with 14 vertices and 12 faces. Examples of polyhedra of both types are shown in Fig. 5.

Darts DT3, *type* $\mathscr{I}$[3;3;3,3]. For spherical darts of this type, the resulting isohedra have all vertices on axes of 3-fold symmetry; they are unusual and have the vertex-pair property. If $\alpha\neq 1$ the isohedra are of two combinatorial types; one is represented in Fig. 6 by (a) and (e), and the other in (c). In the symmetric case ($\alpha=1$) there are also two combinatorial types, one shown in parts (b) and (f) of Fig. 6, the other in (d). Types (c) and (d) are unusual in other ways as well, as described below.

Although all these isohedra are of one of the seven combinatorial types listed above, many additional distinctions are possible if the shapes of the polyhedra are taken into account. We have not attempted to distinguish these as there are no generally accepted methods of classification applicable to nonconvex and selfintersecting polyhedra. We illustrate this remark with reference to darts DT3. For each value of α there is a critical value $\gamma^*=\gamma^*(\alpha)$ of γ at which the appearance of the isohedron changes (see Fig. 6). For smaller values of γ the apexes of the faces are visible on the outside while the nadirs are in the interior. For values of γ larger than the critical value, the nadirs are visible and the apexes are inside. The isohedra are unusual with the vertex-pair property for all values of α and γ, but polyhedra with $\gamma=\gamma^*(\alpha)$ have coplanar faces as well, and if $\alpha\neq 1$ then there are also multicycle vertices.

Another possible distinction of the shapes of the isohedra is illustrated by those with darts DT2. Here the convex hull may have either 10 or four vertices; for each α, the

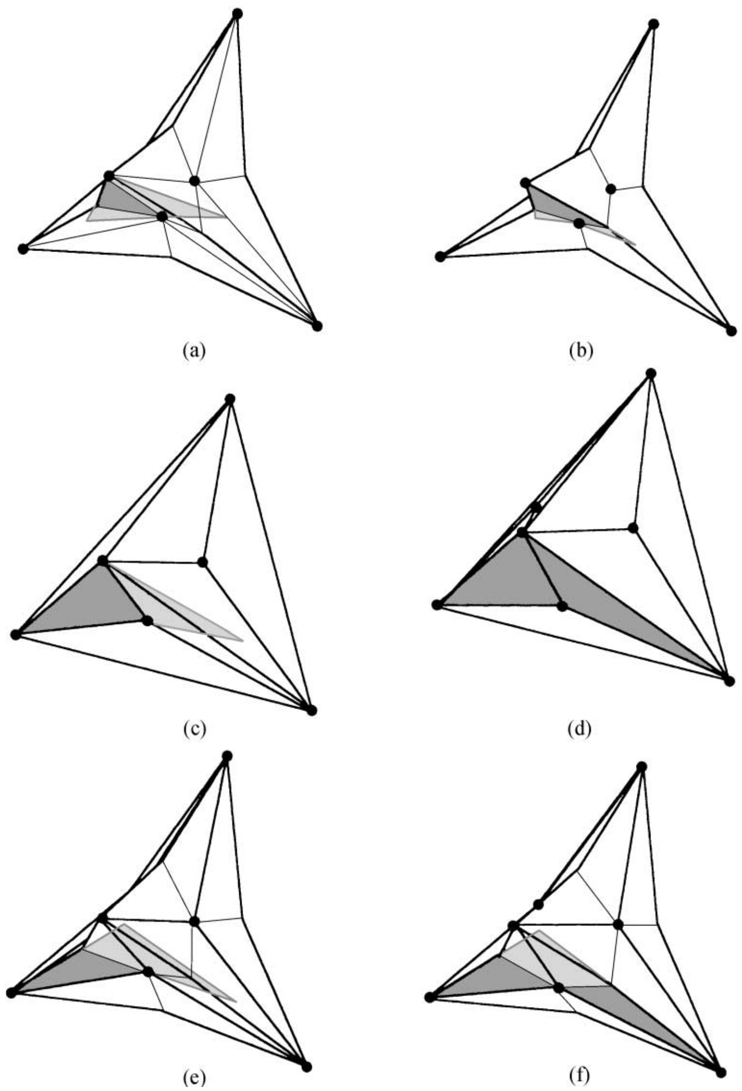


Fig. 6. Isohedra with darts DT3 of type $\mathscr{J}[3;3;3,3]$. All vertices are on 3-fold axes of symmetry. The data on the polyhedra are as follows: (a) $\alpha = 1/2$, $\gamma = 3/20$. The polyhedron has 24 faces and 16 vertices; the vertices of one face are $(1,1,1)$, $(9/17,-9/17,-9/71)$, $(3/20,3/20,-3/20)$, $(-9/34,9/34,-9/34)$. (b) $\alpha = 1$, $\gamma = 3/20$. The polyhedron has 12 faces and 12 vertices; the vertices of one face are $(1,1,1)$, $(6/17,-6/17,-6/17)$, $(3/20,3/20,-3/20)$, $(-6/17,6/17,-6/17)$. (c) $\alpha = 1/2$, $\gamma = 1/4$. The polyhedron has 24 faces and 12 vertices; the vertices of one face are $(1,1,1)$, $(1,-1,-1)$, $(1/4,1/4,-1/4)$, $(-1/2,1/2,-1/2)$. (d) $\alpha = 1$, $\gamma = 1/3$. The polyhedron has 12 faces and 8 vertices; the vertices of one face are $(1,1,1)$, $(21/13,-21/13,-21/13)$, $(7/20,7/20,-7/20)$, $(-21/26,21/26,-21/26)$. (e) $\alpha = 1/2$, $\gamma = 7/20$. The polyhedron has 24 faces and 16 vertices; the vertices of one face are $(1,1,1)$, $(21/13,-21/13,-21/13)$, $(7/20,7/20,-7/20)$, $(-21/26,21/26,-21/26)$. (f) $\alpha = 1$, $\gamma = 9/20$. The polyhedron has 12 faces and 12 vertices; the vertices of one face are $(1,1,1)$, $(1,-1,-1)$, $(1/3,1/3,-1/3)$, $(-1,1,-1)$.

transition occurs at a well-determined value of γ. The two parts of Fig. 5 illustrate the possibilities. If $\alpha = 1$, the transition occurs for $\gamma = 3/7$.

Clearly, many other distinctions could be made.

5. The octahedral group

There are 23 different kinds of darts possible in the Möbius net of the octahedral group. They are shown in Fig. 7. Because of the large number of darts it is impractical to present as detailed an account of the isohedra as we did for the tetrahedral group in the previous section. However, we shall mention some aspects which do not occur in the tetrahedral case.

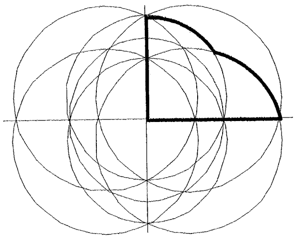
In the octahedral case, the spherical darts need not be balanced. This occurs in 16 cases, because only seven darts are balanced, namely DO9, DO11, DO12, DO14, DO15, DO18 and DO22 in the notation of Fig. 7.

The other difference is that here the collection of polygons obtained by the action of the group can, in certain cases, be an unusual polyhedron in that it is not strongly connected. Hence, according to a widely accepted terminology, we obtain not one isohedron, but a *compound* of two isohedra. This happens in three of the balanced cases, namely for darts DO9, DO11, DO22. Each of the resulting compounds is the union of one of the isohedra with tetrahedral symmetry and its reflection in the origin. Examples of isohedra of the four other balanced types are shown in Fig. 8. In Fig. 9 we show a few examples of isohedra arising from asymmetric darts. We note that for asymmetric darts the values α and $1/\alpha$ yield, in general, polyhedra of different appearance; this is illustrated in Fig. 10.

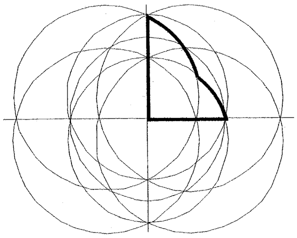
6. The icosahedral group

In the icosahedral Möbius net there are 136 distinct darts, of which 26 are balanced. Clearly there are too many to show in diagrams analogous to Figs. 3 and 7, or to describe in detail. A listing of types would not be very informative, so we shall here describe a straightforward procedure by which all these darts can be determined. Naturally, the same method can be used for the tetrahedral and octahedral nets.

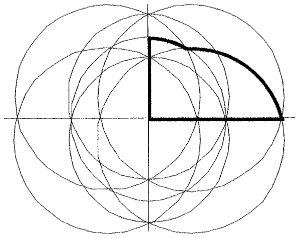
We are looking for darts for which the apex is on an r-fold symmetry axis, and the nadir is on an s-fold symmetry axis. To begin with, suppose that $r = 2$. In the appropriate projected Möbius net (that is, the one in which a point corresponding to an axis of 2-fold symmetry lies at the center A' of the diagram in Fig. 11) choose two non-collinear rays m_1 and m_2 of the net meeting at A', and let R denote the region (with angle less that 180°) bounded by these two rays. In the simple case ($r = 2$) which we are considering there are clearly four choices for m_1 and m_2, but as these are equivalent, only one needs to be considered. Any vertex of the net lying in the interior of R may be chosen to be the nadir of the dart; label it C'. It is easy to see



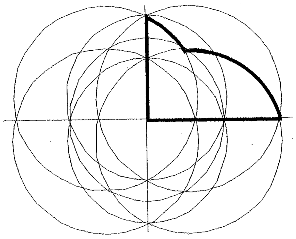
DO1 type ↻[2;2;3,4] $\Delta = 10$



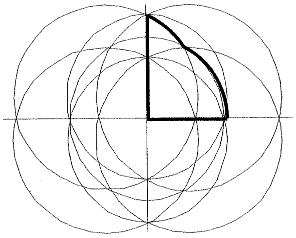
DO2 type ↻[2;2;4,4] $\Delta = 6$



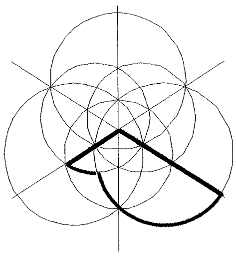
DO3 type ↻[2;3;2,3] $\Delta = 18$



DO4 type ↻[2;3;3,4] $\Delta = 9$

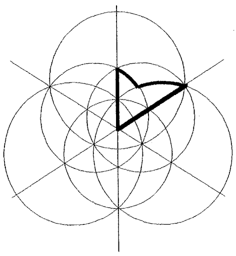


DO5 type ↻[2;3;4,4] $\Delta = 7$

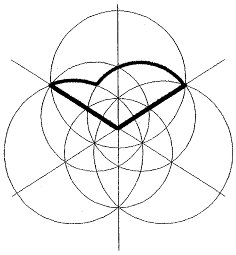


DO6 type ↻[3;2;2,3] $\Delta = 12$

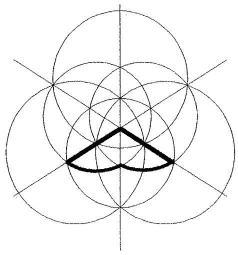
(a)



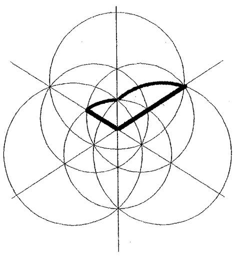
DO7 type ↻[3;2;3,4] $\Delta = 5$



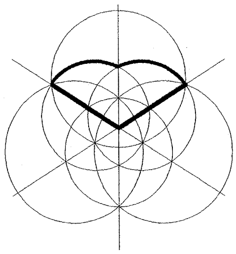
DO8 type ↻[3;2;4,4] $\Delta = 11$



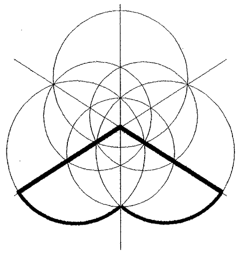
DO9 type ↻[3;3;3,3] $\Delta = 8$



DO10 type ↻[3;3;3,4] $\Delta = 6$



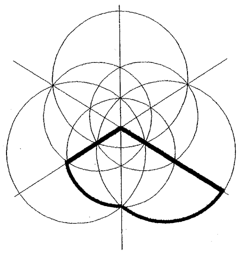
DO11 type ↻[3;3;4,4] $\Delta = 12$

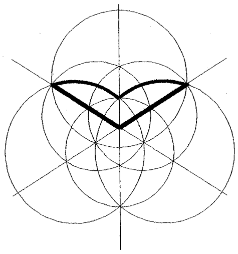


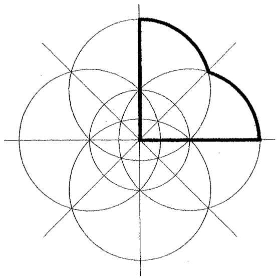
DO12 type ↻[3;4;22] $\Delta = 14$

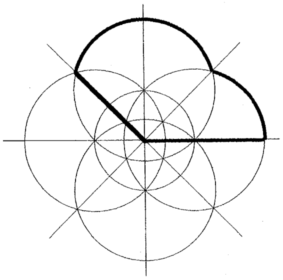
(b)

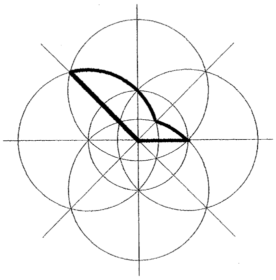
Fig. 7. The 23 types of dart-shaped quadrangles in the octahedral Möbius net, arranged lexicographically by their type symbol.

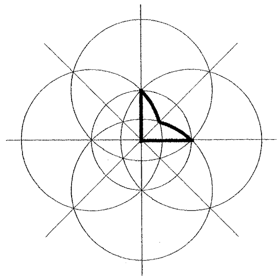

DO13 type $\circlearrowleft[3;4;2,3]$ $\Delta = 13$


DO14 type $\circlearrowleft[3;4;4,4]$ $\Delta = 8$

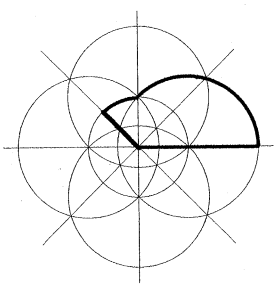

DO15 type $\circlearrowleft[4;3;2,2]$ $\Delta = 10$

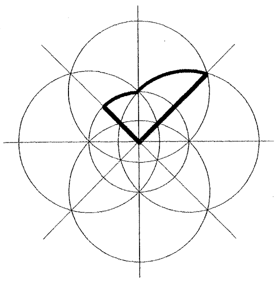

DO16 type $\circlearrowleft[4;3;2,3]$ $\Delta = 15$

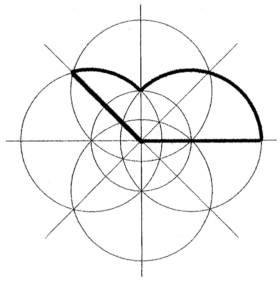

DO17 type $\circlearrowleft[4;3;3,4]$ $\Delta = 8$

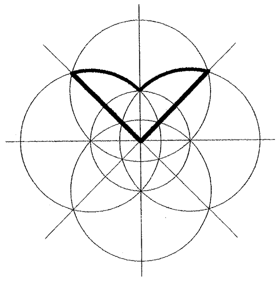

DO18 type $\circlearrowleft[4;3;4,4]$ $\Delta = 4$

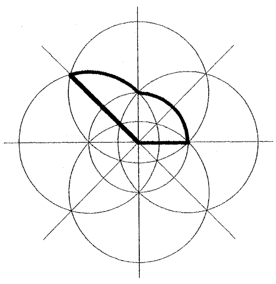
(c)


DO19 type $\circlearrowleft[4;4;2,2]$ $\Delta = 12$


DO20 type $\circlearrowleft[4;2;2,3]$ $\Delta = 7$


DO21 type $\circlearrowleft[4;4;2,3]$ $\Delta = 13$


DO22 type $\circlearrowleft[4;4;3,3]$ $\Delta = 8$


DO23 type $\circlearrowleft[4;4;3,4]$ $\Delta = 10$

(d)

Fig. 7. (*continued*)

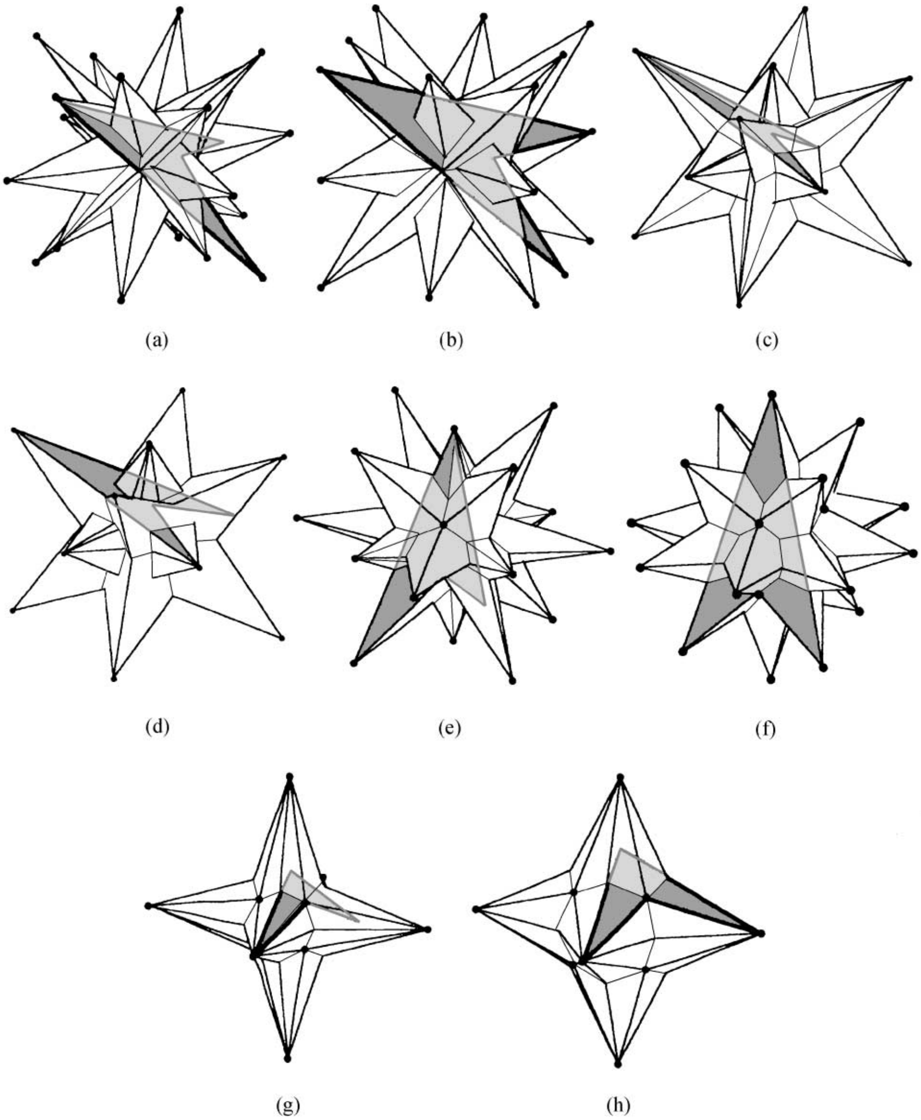


Fig. 8. Examples of isohedra having octahedral symmetry, with dart-shaped faces arising from balanced darts, and with edges in planes of symmetry. (a) DO12. $\alpha=1/2$, $\gamma=1/3$. The polyhedron has 48 faces and 38 vertices; the vertices of one face are at points $(1,-1,1)$, $(0,3/2,-3/2)$, $(0,1/3,0)$, $(-3/4,3/4,0)$. (b) DO12. $\alpha=1$, $\gamma=1/3$. The polyhedron has 24 faces and 26 vertices; the vertices of one face are at points $(1,-1,1)$, $(0,1,-1)$, $(0,1/3,0)$, $(-1,1,0)$. (c) DO14. $\alpha=1/2$, $\gamma=1/4$. The polyhedron has 48 faces and 26 vertices; the vertices of one face are at points $(1,-1,1)$, $(0,1.,0)$, $(0,0,0.25)$, $(-0.5,0,0)$. (d) DO14. $\alpha=1$, $\gamma=1/3$. The polyhedron has 24 faces and 20 vertices; the vertices of one face are at points $(1,-1,1)$, $(0,1.,0)$, $(0,0,0.333333)$, $(-1.,0,0)$. (e) DO15. $\alpha=1/2$, $\gamma=1/4$. The polyhedron has 48 faces and 38 vertices; the vertices of one face are at points $(0,0,1)$, $(1,0,-1)$, $(1/4,1/4,-1/4)$, $(0,1/2,-1/2)$. (f) DO15. $\alpha=1$, $\gamma=1/4$. The polyhedron has 24 faces and 26 vertices; the vertices of one face are at points $(0,0,1)$, $(2/3,0,-2/3)$, $(1/4,1/4,-1/4)$, $(0,2/3,-2/3)$. (g) DO18. $\alpha=1/2$, $\gamma=1/2$. The polyhedron has 48 faces and 26 vertices; the vertices of one face are at points $(0,0,1)$, $(3/2,0,0)$, $(1/3,1/3,1/3)$, $(0,3/4,0)$. (h) DO18. $\alpha=1$, $\gamma=1/2$. The polyhedron has 24 faces and 14 vertices; the vertices of one face are at points $(0,0,1)$, $(1,0,0)$, $(1/3,1/3,1/3)$, $(0,1,0)$.

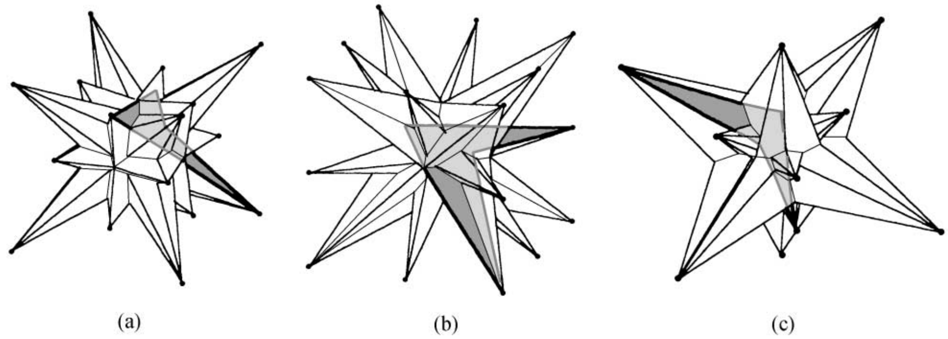


Fig. 9. Several of the isohedra that arise from unbalanced darts in the octahedral Möbius net. (a) DO3. $\alpha = 1/2$, $\gamma = 1/3$. The polyhedron has 48 faces and 40 vertices; the vertices of one face are at points $(1, 0, 1)$, $(-3/2, 3/2, -3/2)$, $(-1/3, 1/3, 1/3)$, $(-3/4, 0, 3/4)$. (b) DO13. $\alpha = 1/2$, $\gamma = 1/2$. The polyhedron has 48 faces and 28 vertices; the vertices of one face are at points $(1, -1, 1)$, $(3, 3, -3.)$, $(0, 1., 0)$, $(-3, 3, 0)$. (c) DO17. The polyhedron has 48 faces and 28 vertices; the vertices of one face are at points $\alpha = 1/2$, $\gamma = 1/2$. $(0, 0, 1)$, $(0, -3, 0)$, $(1/4, -1/4, 1/4)$, $(3/4, 3/4, -3/4)$.

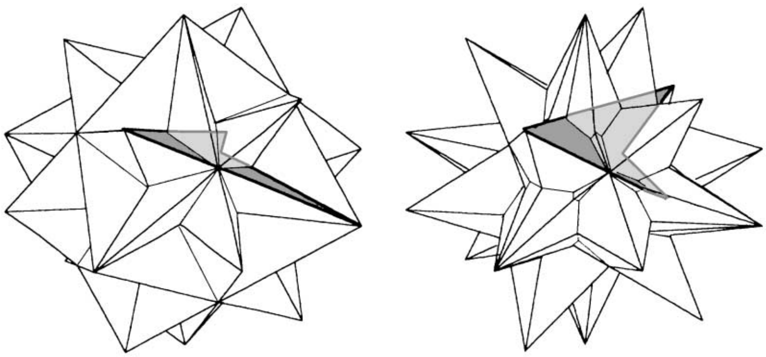


Fig. 10. Both isohedra shown have arisen from asymmetric darts of type DO1, with $\gamma = 1/3$. Their faces are affinely equivalent, and correspond to $\alpha = 1/2$ and $\alpha = 2$, respectively.

that if $s = 2$, so that C' lies on two circles, there is exactly one dart with apex A', nadir C' and two edges lying in m_1 and m_2. If $s = 3$, there exist exactly three such darts, and if $s = 5$, there are 10 such darts. One of the latter is indicated in Fig. 11. In our example with $r = 2$, the region R contains in its interior six points on 2-fold axes, three points on 3-fold axes and one point on a 5-fold axes. Hence there are a total of

$$6 \times 1 + 3 \times 3 + 1 \times 10 = 25$$

darts with $r = 2$.

An analogous procedure can be applied for darts with $r = 3$ or 5, using the center A' of an appropriate projection of the Möbius net as the apex of the dart we are constructing. When $r = 3$ there are three essentially different choices for the region R, and if $r = 5$ there are six essentially different choices. We examine each of these in turn. After choosing the nadir C' the dart can be completed as above, but the number

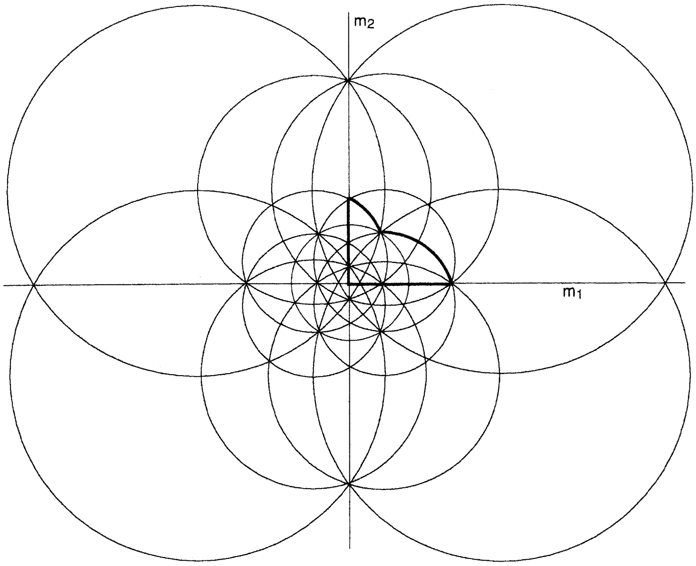


Fig. 11. An illustration of the counting procedure outlined in the text. To avoid clutter, the vertices have not been labeled. The point A' is at the center of the diagram.

of ways in which this can be done depends on C' and has to be determined in each case. In particular, the number of possibilities is reduced if C' lies on one of the lines of the net that pass through A', as well as in some other circumstances. Examining all possible choices for C' for a given A' we obtain the following results:

For the three choices of the regions R in case $r = 3$, there are altogether

9 different choices of C' on 2-fold axes, yielding 9 darts;

8 different choices of C' on 3-fold axes, yielding 20 darts;

4 different choices of C' on 5-fold axes, yielding 16 darts;

hence there are 45 different darts with $r = 3$.

Similarly, for the six choices of the regions R in case $r = 5$, there are altogether

10 different choices of C' on 2-fold axes, yielding 10 darts;

15 different choices of C' on 3-fold axes, yielding 16 darts;

8 different choices of C' on 5-fold axes, yielding 40 darts;

hence there are 66 different darts with $r = 5$.

It follows that there are altogether $25+45+66 = 136$ different darts in the icosahedral Möbius net.

A slight variant of the procedure just described leads to an enumeration of balanced darts. We find that there are none if $r = 2$; for $r = 3$ there are ten, and for $r = 5$ there are sixteen, for a total of 26 balanced darts in the icosahedral Möbius net.

Diagrams of most of these polyhedra are very complicated, and tend to be unintelligible. Hence we show in Fig. 12 just one simple example, with a balanced dart, and with 60 faces that have reflective symmetry.

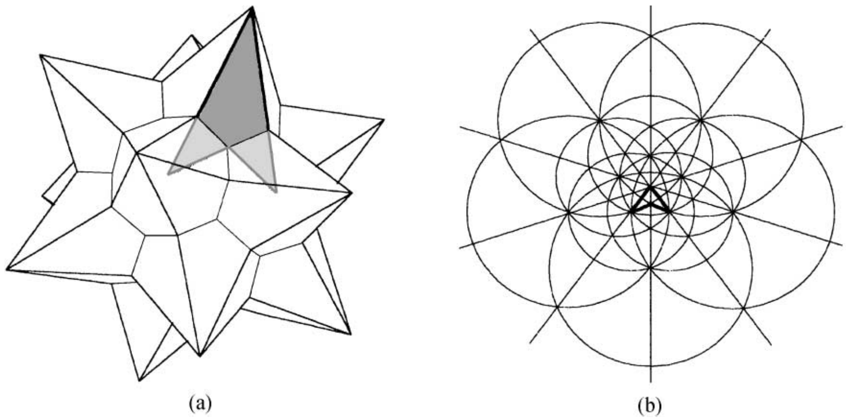


Fig. 12. An example of a dart-faced isohedron with icosahedral symmetry is shown in (a); it is obtained from the dart indicated in the Möbius net in (b).

7. Comments

7.1. Suppose a dart F in the Möbius net $\mathscr{M}$ is of type $\mathscr{M}[r;s;t,u]$ and has apex A and nadir C. Then the following facts are easily verified. There is a unique dart F' of type $\mathscr{M}[s;r;t,u]$ with apex C and nadir A, whose edges lie in the same great circles as the edges of F, see Fig. 13. Moreover, let the *convex kernel* K of F denote the convex quadrangle contained in F and determined by the great circles that contain the edges of F. If the vertices of K lie on axes of multiplicities r, p, s, q (in a cyclic order), then the other pair p, q of axes also determines a pair of darts, of types $\mathscr{M}[p;q;t,u]$ and $\mathscr{M}[q;p;t,u]$. Naturally, these four darts need not be all different. We have found this connection between darts and convex quadrangles very useful in checking our results, ensuring that no possibilities have been overlooked, especially in the icosahedral case. We note in passing that there are 42 different convex quadrangles in the icosahedral case; in the tetrahedral case there are two convex quadrangles, and in the octahedral case nine.

7.2. The isohedra with dart-shaped faces described above are not the only kinds possible. Another family of isohedra with dihedral symmetry also exists. The idea should be clear from the illustrations in Fig. 14. Let P be a regular n-gon centered at O. Let Q be a right prism whose base is P and whose top is the regular n-gon P' centered at O'. Choose any three vertices of P, say A, B and D, which define a triangle with O in its interior. Let B' and D' be the vertices of P' corresponding to (that is, vertically above) the vertices B and D of P, and suppose the triangle $AB'D'$ cuts OO' in C. Then one face F of the isohedron is $AB'CD'$ (with apex A and nadir C) and the other faces are the images of this under the symmetries of Q (that is, the dihedral group of order $2n$). In contrast to the isohedra described above, members of this family depend both on n and on another integer parameter d, which is analogous to the "density"

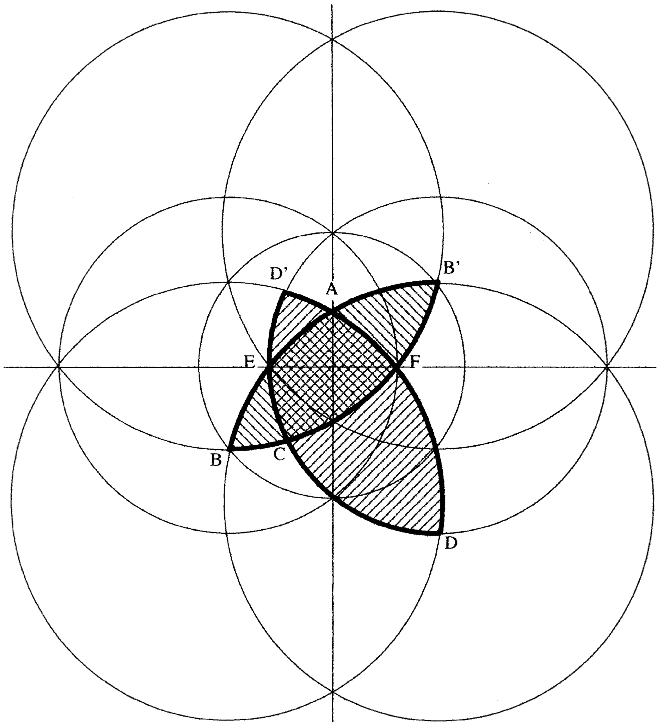


Fig. 13. An illustration of the quadruplets of darts that are associated with each other. In the instance shown the four darts are all different. The darts are: $ABCD$, $EDFB'$, $CB'AD'$ and $FD'EB$.

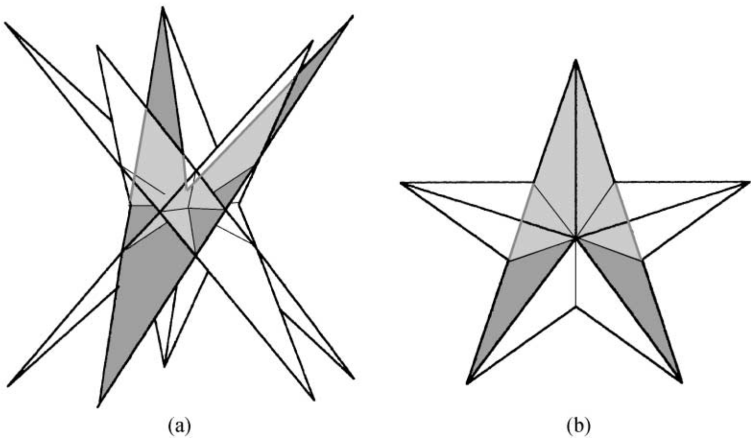


Fig. 14. Isohedra with dart-shaped faces and dihedral symmetry group D_n. (a) and (b) are views of the same polyhedron from side and from above; the polyhedron corresponds to $n = 5$, $d = 2$.

used for star polygons. Given these two integers, the isohedron is determined uniquely up to affinity. In some cases this isohedron is unusual, a compound of two parts each with the symmetry of an antiprism and itself an isohedron with dart-shaped faces.

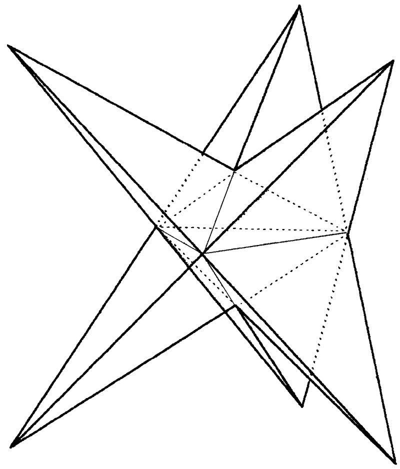


Fig. 15. The dart-faced isohedron shown by Brückner (1900). It was dihedral symmetry, and corresponds to $n=3,\ d=1$.

Conjecture 1. *There are no usual isohedra with dart-shaped faces beyond the following*:

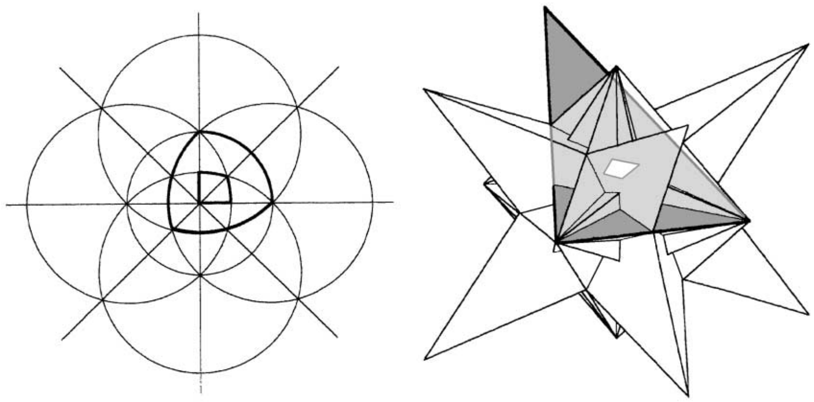
(i) *Isohedra with tetrahedral*, *octahedral or icosahedral symmetry group and edges in planes of mirror symmetry*;
(ii) *The isohedra with dihedral symmetry group mentioned above*, *and illustrated in Fig.* 14, *and their parts in the unusual case*.

Note that the edges of the isohedra in (ii) do not lie in planes of mirror symmetry; each face is mapped into its neighbor by a half-turn about an axis passing through the midpoint of the longer common edge.

7.3. It is well known that every quadrangle (including darts) can be used to tile the plane isohedrally. However, it is also known that no bounded convex region in the plane can be tiled by darts—not even if they are allowed to have different shapes (see Schwenk [8], Gale [4]). In Grünbaum and Shephard [7] it was proved that there is no acoptic isohedron (that is, one free of selfintersections) with dart-shaped faces. As we have seen, if selfintersections are allowed, dart-faced isohedra exist. However, there are still some open questions, and we suggest the following conjecture which would greatly strengthen the result just quoted by admitting non-isohedral polyhedra as well as polyhedra of any genus:

Conjecture 2. *There is no acoptic polyhedron with dart-shaped faces.*

7.4. It is remarkable that we found only one reference to a dart-faced polyhedron in the literature. On p. 105 of Brückner [1], he mentions that the isohedron shown in Fig. 21 of his Plate X (see Fig. 15) has as faces six quadrangles, each of which has a reflex angle. However, he neither asks any questions about the existence of other dart-faced polyhedra, nor does he offer any comments on the subject.



(a) (b)

Fig. 16. (a) A triangle with a quadrilateral hole, in the octahedral Möbius net; it can be used to construct self-intersecting isohedral holyhedra with 24 faces. (b) A holyhedron of this type.

7.5. The only paper of which we are aware that deals with isohedra having all edges in planes of symmetry is Unkelbach [10]. In this paper he discovered a remarkable acoptic hexecontahedron with rhombic faces. This can easily be obtained using the method of Möbius nets described here. Other applications are in the papers Grünbaum [5], Coxeter and Grünbaum [2], Shephard [9] and Grünbaum [6].

7.6. Some years ago J.H. Conway asked whether there exist any *holyhedra*, that is, polyhedra which have "a hole in every face". More specifically, he asked that each face has one or more holes such that each hole (together with its boundary) is contained in the interior of the face. A very ingenious and tremendously complicated affirmative answer was recently given by Vinson [11]. The polyhedron is acoptic, as was, apparently, implied in the original question. However, if one admits selfintersecting polyhedra then it is easy to construct isohedra with faces that have holes, by using Möbius nets. In Fig. 16 we show in the Möbius net a triangular face with a quadrangular hole in its interior, with all edges along the lines of the net. Since the face has a line of symmetry, the isohedron generated from it will have only 24 faces. By a similar construction in the icosahedral Möbius net we can construct an isohedral polyhedron in which each face has three disjoint holes. We conjecture that there is no such polyhedron with faces that have more than three holes.

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