The Poisson Kernel on Positively Curved Manifolds

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0. Introduction

Let $M$ a complete Riemannian manifold. We shall denote by $\Delta$ the Laplace-Beltrami operator on $M$ and by $T' = e^\Delta$ the heat diffusion semigroup it generates (cf. Section 2 for details), we shall also denote by

$$P' = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} \frac{u}{\sqrt{u}} T^{1/4_u} du. \quad (0.1)$$

The Poisson semigroup on $M$ (cf. [1]). For every $f \in L^1(M; d \Vol)$ (where we denote by $d \Vol$ or $dV$ throughout the canonical Riemannian volume element) we shall consider the “Hardy-Littlewood” and the Poisson maximal functions

$$Mf(m) = \sup_{t > 0} \frac{1}{\Vol(B_t(m))} \int_{B_t(m)} |f| \, dV,$$

$$Pf(m) = \sup_{t > 0} |Pf(m)|.$$ 

$B_t(m)$ will denote throughout the ball of radius $t$ centered at $m$ in $M$ ($B_t(m) = \{ n \in M; d(m, n) \leq t \}$). We can state then some of the main results that will be proved in this paper.

Theorem 1. Let $M$ be a complete Riemannian manifold of nonnegative Ricci curvature. There exists then $C = C(n)$ a constant that only depends on $n = \dim M$ such that

(i) $Mf \leq CP |f|$, 

(ii) $\Vol |Pf > \lambda| \leq C \frac{\| f \|_1}{\lambda}$, \quad $\lambda > 0$. 


(iii) \( \text{Vol}[Mf > \lambda] \leq C \frac{\|f\|_1}{\lambda}, \quad \lambda > 0. \)

for all \( f \in L^1(M; dV) \).

*Added in proof:* Part (ii) of the above theorem has been proved, and under more general conditions, by Herz cf. [21].

Nonnegative Ricci curvature means of course that the Ricci tensor \( \text{Ric}_{ij} \) (cf. Section 1) is positive semidefinite. Observe that (i) follows from (ii) and (iii), and the proof of (ii) and (iii) is based on the following more precise theorem on the Poisson kernel; i.e., the symmetric kernel \( K_t(x, y) \) \((t > 0; x, y \in M)\) on \( M \) that gives the Poisson semigroup by:

\[
P^t f(x) = \int_M K_t(x, y) f(y) \, dV(y) \quad \forall x \in M.
\]

**Theorem 2.** Let \( M \) be as in Theorem 1 there exists then \( C = C(n) > 0 \) a constant that depends only on the dimension \( n = \dim M \) such that

\[
(i) \quad \frac{1}{C} \leq \frac{K_t(m, m_1)}{K_t(m', m_1)} \leq C, \quad \forall t, t' > 0; \frac{1}{2} \leq t/t' \leq 2;
\]

\[
\forall m, m', m_1 \in M, d(m, m') \leq t.
\]

\[
(ii) \quad \frac{1}{C \text{Vol } B_t(m)} \leq K_t(m, m') \leq \frac{C}{\text{Vol } B_t(m)}, \quad \forall t > 0;
\]

\[
\forall m, m' \in M; d(m, m') \leq t.
\]

\[
(iii) \quad K_t(m, m') \leq \frac{C}{\text{Vol } B_t(m)} \text{Min} \left[ 1, \frac{t}{d(m, m')} \right], \quad \forall t > 0; \forall m, m' \in M.
\]

An immediate corollary of (i) and (ii) above is that

\[
\text{Vol } B_{2t}(m) \leq C \text{Vol } B_t(m). \quad (0.2)
\]

It is also clear that (0.2) directly implies Theorem 1 (iii) by a standard Calderon–Zygmund argument. Inequality (0.2) can be proved also by standard differential geometry methods (this is essentially contained in [2 Sect. 11.10] and is due to Bishop).

The main analytic tool for the proof of all these results will be a Harnack type of inequality which will be a consequence of Yau's work (cf. [3]) and which I will now explain. Toward that end let me denote by \( M_0 = \mathbb{R} \times M \) the product manifold with the product metric \( ds_0^2 = dt^2 + ds^2 \), where \( dt^2 \) is the standard metric on \( \mathbb{R} \) and \( ds^2 \) is the original metric on \( M \). I shall also denote by \( \Delta_0 = \partial^2/\partial t^2 + \Delta \) the corresponding Laplace–Beltrami operator in \( M_0 \) and
for any subregion $\Omega \subset M_0$, I shall say that $u \in C^\infty(\Omega)$ is harmonic in $\Omega$ if $\Delta u = 0$ in $\Omega$. I shall also denote by $M^+ = \{ (t, m) \in M; t > 0 \}$ the “upper half-space” in $M_0$. We can state then:

**Theorem 3.** Let us assume that $M$ is a complete Riemannian manifold with Ricci curvature bounded from below by some constant $-K$ ($K \geq 0$). We can then find a constant $C = C(n)$ that only depends on $n = \dim M$ and such that for all $u = u(t, m)$ nonnegative harmonic function in $M^+$ we have the following estimates:

(i) if $K = 0$ then $|du(t, m)| \leq C \frac{u(t, m)}{t}, \quad t > 0$;

(ii) if $K > 0$ then $|du(t, m)| \leq 3 \sqrt{nK} u(t, m)$

for $t \geq t_0$ sufficiently large depending on $K$ and $n$ ($t_0 = t_0(n, K)$). Here $|du| = |\nabla u| = \text{grad}_0 u$ denotes the length of the covector $du$ in $M_0$, i.e., the length of the gradient of the function $u$.

In view of [4] we also see from the above theorem that the Poisson kernel supplies extremal functions connected with $\lambda_1(M)$, the top of the spectrum of the Laplacian. We shall end up this paper by showing how we can, with the above methods, recover some classical results of Cheeger and Chang on $\lambda_1(M)$.

We also have the following immediately.

**Corollary of Theorem 2.** Let $M$ be as in Theorem 1; then the Green’s function $G(x_0, x)$ exists and is bounded in $x$ in the compacta of $M \setminus x_0$ if and only if

$$\int_1^\infty \frac{t}{\text{Vol } B_t(x_0)} \, dt < +\infty.$$

Let me finally indicate without proof (the proof will appear elsewhere) a rather unexpected result that arises from the present methods.

**Theorem.** Let $M$ be a complete Riemannian manifold of non-negative Gaussian curvature and let us assume that $0 \in M$ is a pole of $M$ and that it admits a Green’s function $G(0, y)$ $(y \in M)$. Then this Green’s function satisfies:

$$C^{-1} \leq G(0, y) \left( \int_0^\infty \frac{t}{\text{Vol } B_t(0)} \, dt \right)^{-1} \leq C,$$

where $C$ is a constant that only depends on $\dim M = n$.
1. THE HARNACK INEQUALITY AND THE CURVATURE

$M$ will denote throughout a complete Riemannian manifold assigned with the metric $ds^2$. We shall consider $M^+ = (0, +\infty) \times M$ and assign it with the metric

$$\alpha^2 = \lambda^2(t) \, ds_0^2 + \lambda^2(t) dt^2 + ds^2,$$  \hspace{1cm} (1.1)

where $\lambda(t) > 0$ ($t > 0$) will be some smooth (say, $C^2$) function that will be specified later. Our first task is to compute the curvature of $M^+$ in terms of the curvature of $M$. It is, of course, a matter of direct computation, but I shall outline its salient features. I shall start with some notations which shall be chosen to be the same as in [3].

Let $(e_i)$, $i = 1, 2, \ldots, n$, be a frame field on $M$, let $(\omega_i)$ be the dual frame field of forms which satisfy the structure equations:

$$d\omega_i = \sum_{j=1}^{n} \omega_j \wedge \omega_{ij}, \quad \omega_{ij} = -\omega_{ji};$$

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{n} (\omega_{ik} \wedge \omega_{kj}) = \frac{1}{2} \sum_{k,l=1}^{n} R_{ijkl} \omega_k \wedge \omega_l;$$

where $\omega_{ij}$ are the connection forms and $\Omega_{ij}$ are the curvature forms of $M$. Following [3], for every smooth function $h$ on $M$ I shall denote

$$dh = \sum_{i=1}^{n} h_i \omega_i, \quad \sum_{k=1}^{n} h_k \omega_k = dh - \sum_{k=1}^{n} h_k \omega_k, \hspace{1cm} (1.2)$$

so that $\sum_{i=1}^{n} h_i^2 = |\nabla h|^2 = |\text{grad } h|^2$.

The tangent spaces of $M^+$ can be identified with the products of the tangent spaces of $M$ and the tangent spaces of $(0, +\infty)$. This allows us to identify forms on $M$ with forms on $M^+$ by $\omega(a(\partial/\partial t) + X) = \omega(X)$, with obvious notations and an analogous definition for higher order forms. Observe that the above identification commutes with exterior differentiation $d$. We obtain thus a local frame field and its dual on $M^+$ by setting:

$$\omega_0^+ = \frac{1}{\lambda(t)} \partial, \quad \frac{1}{\lambda(t)} e_1, \ldots, \frac{1}{\lambda(t)} e_n,$$

$$\omega_0^+ = \lambda(t) dt, \quad \omega^+_i = \lambda(t) \omega_i, \ldots, \omega^+_n = \lambda(t) \omega_n.$$

If we set then

$$\omega^+_i = \omega_{ij}, \quad i, j = 1, 2, \ldots, n; \, \omega^+_0 = 0;$$

$$\omega^+_{i,0} = -\omega^+_{0,i} = (\lambda'/\lambda) \omega_i \quad t = 1, 2, \ldots, n, \hspace{1cm} (1.3)$$
we verify at once that the structure equations hold; i.e.,
\[ d\omega^+_i = \sum_{j=0}^{n} \omega^+_j \wedge \omega^+_i; \quad \omega^+_i = -\omega^+_{ji}; \quad i, j = 0, 1, ..., n. \]

It follows that the connection forms of (1.1) are given by (1.3). By differentiation we then obtain the curvature forms of (1.1):
\[
\begin{align*}
\Omega^+_i &= \Omega_i - \left(\lambda'/\lambda\right)^2 \omega_i \wedge \omega_j, \quad i, j = 1, 2, ..., n, \\
\Omega^+_i &= -\Omega^+_{0i} = \left(\lambda'/\lambda\right)' dt \wedge \omega_i, \quad i = 1, 2, ..., n, \\
\Omega^+_{00} &= 0.
\end{align*}
\]

To obtain the curvature tensor of \( M^+ \) we just have to identify coefficients and we obtain:

For \( i, j = 1, 2, ..., n, i \neq j \), we have
\[
R^+_{ijkl} = \frac{1}{\lambda^2} R_{ijkl}; \quad k, l = 1, 2, ..., n; (k, l) \neq (i, j) \text{ or } (j, i);
\]
\[
R^+_{ijij} = \frac{1}{\lambda^2} R_{ijij} - \left(\frac{\lambda'}{\lambda}\right)^2;
\]
\[
R^+_{ijkl} = 0; \quad \text{if either } k \text{ or } l = 0.
\]

For \( i = 1, 2, ..., n \) we have
\[
R^+_{i00i} = -R^+_{i0i0} = \frac{1}{\lambda^2} \left(\frac{\lambda'}{\lambda}\right)'
\]
\[
R^+_{i0kl} = 0 \quad \text{for all other choice of } k, l = 0, 1, ..., n.
\]

This together with the fact that \( R^+_{ikl} = 0, \ i, k, l = 0, 1, ..., n \) completely determines the tensor. If we then denote by \( \text{Ric}_{ij} = \sum_{m=1}^{n} R_{imjm} \) and by \( \text{Ric}'_{ij} = \sum_{m=0}^{n} R_{imjm} \), we finally obtain in matrix notations:

\[
\text{Ric}^+ = \frac{1}{\lambda^2} \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & \alpha_0 & 0 \\ \vdots & \text{Ric} & \vdots \\ 0 & 0 & \alpha_n \end{pmatrix}; \quad (1.4)
\]

where
\[
\alpha_0 = -\frac{n}{\lambda^2} \left(\frac{\lambda'}{\lambda}\right)'; \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = -\frac{n}{\lambda^2} \left(\frac{\lambda'}{\lambda}\right)^2 - \frac{1}{\lambda^2} \left(\frac{\lambda'}{\lambda}\right)' \quad (1.5)
\]
If we denote by $\Delta$ the Laplacian of $M$ we then obtain by direct computation the Laplacian of the metric (1.1) on $M^+$:

$$\Delta^+ = \frac{1}{\lambda^2} \left( \Delta + \frac{\partial^2}{\partial t^2} \right) + (n-1) \frac{\lambda'}{\lambda^3} \frac{\partial}{\partial t}.$$ 

We also have for every $h$ smooth function on $M^+$

$$dh = \frac{\partial h}{\partial t} dt + \sum_{i=1}^{n} h_i \omega_i = \sum_{i=0}^{n} h_i^+ \omega_i^+,$$

where the $h_i (i = 1, ..., n)$ are defined as in (1.2) by "differentiating in the $m$-directions," i.e., by considering $t > 0$ as a fixed parameter

$$d_m h = \sum_{i=1}^{n} h_i \omega_i.$$ 

This gives us

$$h_0^+ = \frac{1}{\lambda} \frac{\partial h}{\partial t}; \quad h_i^+ = \frac{1}{\lambda} h_i; \quad i = 1, 2, ..., n.$$ 

We then have on $M^+$:

$$\sum_{k=0}^{n} h_{ik}^+ \omega_k^+ = dh_i^+ - \sum_{k=0}^{n} h_{ik}^+ \omega_{ki}^+$$

and by direct computation we obtain

$$h_{00}^+ = \frac{1}{\lambda^2} \frac{\partial^2 h}{\partial t^2} - \frac{\lambda'}{\lambda^3} \frac{\partial h}{\partial t},$$

$$h_{0i}^+ = \frac{1}{\lambda^2} \left( \frac{\partial h}{\partial t} \right)_i - \frac{\lambda'}{\lambda^3} h_i, \quad i = 1, 2, ..., n;$$

where as before we set

$$\sum_{i=1}^{n} \left( \frac{\partial h}{\partial t} \right)_i \omega_i = d_m \left( \frac{\partial h}{\partial t} \right) = \text{differential in the "M-directions."}$$

If we denote by $\nabla^+$ the gradient of $M^+$ we have

$$|\nabla^+ h| = \frac{1}{\lambda} \left( \left| \frac{\partial h}{\partial t} \right|^2 + \sum_{i=1}^{n} |h_i|^2 \right)^{1/2},$$

(1.7)
which together with (1.6) gives the estimates

\[
\left| \frac{\partial^2 h}{\partial t^2} \right| \leq |\lambda' \nabla^2 h| + \lambda^2 |h_{oo}|, \tag{1.8}
\]

\[
\left| \frac{\partial h}{\partial t} \right| \leq \lambda^2 |h_{0i}| + \left| \frac{\lambda'}{\lambda} h_i \right| ;
\]

\[
\sum_{i=1}^{n} \left| \frac{\partial h}{\partial t} \right| ^2 \leq 2\lambda^4 \sum_{i=1}^{n} |h_{0i}|^2 + 2|\lambda' \nabla h|^2. \tag{1.9}
\]

Let us now apply these estimates to some \( h = u(t, m) \) that satisfies \( \Delta_0 u = (\partial^2/\partial t^2 + \Delta) u = 0 \) and for which we therefore have (for \( n \geq 2 \)):

\[
\frac{1}{n-1} \Delta^+ u = \frac{\lambda'}{\lambda^3} \frac{\partial u}{\partial t},
\]

\[
\frac{1}{(n-1)^2} |\nabla \Delta^+ u|^2 = \frac{1}{\lambda^2} \left| \left( \frac{\lambda'}{\lambda^3} \right) \frac{\partial u}{\partial t} + \left( \frac{\lambda'}{\lambda^3} \right)^2 \frac{\partial^2 u}{\partial t^2} \right|^2 + \left( \frac{\lambda'}{\lambda^3} \right) \sum_{i=1}^{n} \left| \frac{\partial u}{\partial t} \right| ^2.
\]

We then obtain by a trivial computation that uses (1.7), (1.8), and (1.9), the following:

**Lemma 1.1.** Let us assume that \( u \) satisfies \( \Delta_0 u = 0 \) and let \( \lambda(t) \) satisfy

\[
\sup_{t > 0} \left\{ \left| \frac{\lambda'(t)}{\lambda^3(t)} \right| + \left| \frac{\lambda''(t)}{\lambda^3(t)} \right| \right\} = c < + \infty \tag{1.10}
\]

we then have (for \( n \geq 2 \)):

\[
[1/(n-1)] \| \Delta^+ u \| \leq C_1 |\nabla^+ u|,
\]

\[
[1/(n-1)] |\nabla \Delta^+ u| \leq C_4 \sqrt{\sum_{i=0}^{n} |u_{0i}|^2} + C_5 |\nabla^+ u|,
\]

where \( C_1, C_4, C_5 \) only depend on \( c \).

If we use (1.4) and (1.5) we also obtain

**Lemma 1.2.** Let us assume that \( \lambda(t) \) satisfies (1.10) and also

\[
\sup_{t} \left( 1/\lambda \right) = T < + \infty.
\]

Let us also assume that the Ricci transformation of \( M \) is bounded from below
bu -K \[i.e., that \ \text{Ric}(x,x) > -K|x|^2\] for some \(K \geq 0\). Then the Ricci transformation of \(M^+ \text{Ric}^+\) is bounded from below by

\[-K^+ = -T^2K - nC.\]

where \(C\) only depends on \(c\).

Condition (1.10) is certainly verified for the following two choices of \(A\)

\[\lambda_0(t) = 1; \quad \lambda_\infty(t) = 1/t; \quad t > 0.\]

The metric arising from \(\lambda_0\) is, however, not complete. We shall proceed to construct a one-parameter family of metrics that satisfy (1.10):

\[\lambda_T(t) = \frac{\phi_T(t)}{t} = \frac{\phi(t)}{t}, \quad 1 \leq T < \infty. \quad (1.11)\]

By direct computation we see that (1.10) is verified if in (1.11), \(\phi\) satisfies:

\[\phi(t) \geq c_0 > 0, \quad t \geq 0; \quad (1.12)\]
\[|\phi'(t)| \leq c_1|\phi^2(t)/t|, \quad t \geq 0; \quad (1.13)\]
\[|\phi''(t)| \leq c_2|\phi^3(t)/t^2|, \quad t \geq 0; \quad (1.14)\]

for three positive constants \(c_0, c_1, c_2\).

Let us now fix some positive number, \(T \geq 1\); we can then construct a function \(\alpha(t) = \alpha_T(t) \in C^\infty\) that satisfies

\[\alpha(t) = 0, \quad t \leq T; \quad \alpha(t) > 0, \quad t > T; \quad \alpha(t) = 1/t, \quad t > \sqrt{2}T\]

\[\alpha(t) \leq C, \quad t \geq 0; \quad \alpha'(t) \leq C/T, \quad t \geq 0; \quad \int_T^\infty \frac{\alpha(t)}{t} dt = 1,\]

where \(C = 10^{10}\) is numerical and independent of \(T\). Such an \(\alpha\) clearly exists. We shall define \(\varphi(t) = \varphi_T(t)\), then, by

\[\frac{1}{\varphi(t)} = \int_T^\infty \frac{\alpha(t)}{t} dt. \quad (1.15)\]

It is then clear that \(\varphi(t)\) is a positive \(C^\infty\) increasing function s.t.

\[\varphi(t) = 1, \quad t \leq T; \quad \varphi(t) = t, \quad t \geq \sqrt{2}T; \quad \varphi' = \frac{\varphi^2\alpha}{t}, \quad t > 0.\]
Inequalities (1.12) and (1.13) are then automatically verified by the choice of $a$. We also have:

$$
\varphi'' = \frac{2\varphi\varphi' a}{t} + \frac{\varphi^2 \varphi'}{t} - \frac{\varphi^2 a}{t^2},
$$

which implies that (1.14) is certainly verified in the range $T \leq t \leq \sqrt{2}T$, but outside this range (1.14) is automatically verified. So the choice (1.15) of $\varphi(t)$ will verified (1.12), (1.13), (1.14) uniformly for the parameter $T \geq 1$. If we set then $\lambda_f(t) = \varphi_f(t)/t$ we obtain the required family of metrics which are, in addition, complete and which also satisfy:

$$
\sup_t \left[ \frac{1}{\lambda_f(t)} \right] \leq \sqrt{2}T.
$$

We are finally in a position to give the proof of Theorem 3.

Let $u \geq 0$ satisfy $\Delta_0 u = 0$ in $\mathcal{M}^+$ and let us fix the metric (1.1) on $\mathcal{M}^+$ with $\lambda = \lambda_f$ as in (1.11) for some fixed $T \geq 1$.

Let us then, with the notations of [3], set

$$
g(x) = \frac{u(x) + b}{\sqrt{\nabla^2 u^2(x) + a}}, \quad a, b > 0
$$

(Definition (2.8) in [3]), where $f = u$ satisfies (2.7) of [3] with $C_2 = C_3 = 0 = C_6 = C_7$ and $C_1, C_4, C_5$ as in Lemma 1.1 (multiplied by $n-1$).

Furthermore a lower bound of the Ricci curvature of $\mathcal{M}^+$ can then be given by Lemma 1.2 and (1.16), we obtain

$$
-K^+ = -2T^2 K - nC
$$

for such a bound, where $C$ is numerical and where $-K (K > 0)$ is a lower bound of Ricci on $\mathcal{M}$.

We then obtain from inequality (2.26) of Yau in [3] that:

$$
(n - 1) C_1 + \left[ 1 + 2\sqrt{a} + (k(n-1)C_4 - 1)\frac{2n+1}{2n} \right] \frac{1}{\inf g} \inf g \geq 0,
$$

where $C, C_1, C_4, C_5$ are positive numerical constants, and where the above inequality holds for all $a, k > 0$ sufficiently small. For $a$ and $k$ sufficiently small the coefficient of $1/\inf g$ is negative and that of $\inf g$ is positive.
If $K = 0$ the above inequality is independent of $T$ and we obtain

\[ \inf g \geq C = C(n) > 0, \]

where $C(n)$ only depends on the dimension and is, particular, independent of $T$.

To obtain then Theorem 3(i) we first fix some $t < T$ and then let $a, b \to 0$. For $K > 0$ we are again going to let $a, b \to 0$ but set $t = T$ and make $T$ large ($T \to \infty$); Theorem 3(ii) follows.

It is of course clear that inequality (1.17) more generally implies the following estimate that is valid for all $t > 0$ and $K > 0$

\[ |du(t, m)| \leq (C_1 \sqrt{K} + C_2/t) u(t, m), \]

where $C_1$ and $C_2$ only depend on $n$, but we shall have no use for that estimate.

Let us finish this by observing that Theorem 1(i) immediately implies Theorem 2(i) by integrating the logarithm of the relevant function.

2. The Heat Diffusion Semigroup and the Poisson Kernel

First of all, let us recall that for us the Laplace-Beltrami operator is $\Delta f = \text{div} \text{ grad} f$ as in [5, Chap. X, Sect. 2], which is the sign convention opposite the one usually adopted by geometers [so that with our convention the euclidean Laplacian is $\Delta = \sum_{i=1}^n (\partial^2/\partial x_i^2)$].

One can then use the Hille-Yosida theory to construct the heat diffusion semigroup $T^t = e^{t\Delta}$ which, under the assumption that the Ricci curvature of $M$, is bounded from below by some fixed constant $-K$ ($K > 0$) ($M$ is assumed to be complete), is a Markovian Feller semigroup on $M$; i.e., it satisfies

\[ T^t1 = 1; \quad T^tC_0(M) \subseteq C_0(M). \] (2.1)

This can be found in [6]. In fact the construction of that semigroup is classical and certainly much older than [6] (e.g., cf. [7], where it is essentially due to Ito). The problem is the "nonexplosion of Brownian motion on $M" and that is where the global condition on the Ricci curvature plays a role (cf. [6, 7] for details; cf. also the end of this paragraph). The above semigroup is symmetric, i.e.,

\[ \int_M (T^t\varphi) \psi \, dV = \int_M \varphi(T^t\psi) \, dV; \quad t > 0; \varphi, \psi \in C_0(M). \]
From that symmetry condition and the Markovian property we deduce that it preserves \( dV \), i.e.

\[
\int_M T' \varphi \, dV = \int_M \varphi \, dV, \quad \forall \varphi \in C_0(M).
\]

This in turn implies that \( T' \) acts on \( L^1(M; dV) \), i.e., that

\[
\| T' \varphi \|_1 \leq \| \varphi \|_1, \quad \forall \varphi \in C_0(M),
\]

which, by interpolation, finally implies that

\[
\| T' \varphi \|_p \leq \| \varphi \|_p; \quad 1 \leq p \leq +\infty, \varphi \in C_0(M).
\]

In other words, \( T'(t > 0) \) has all the good properties of [1 Chap. III].

The Poisson semigroup \( P'(t > 0) \) can then be defined as in (0.1) and the corresponding kernels can also be defined by

\[
T' \varphi(x) = \int_M p_t(x, y) \varphi(y) \, dV; \quad P' \varphi(x) = \int_M K_t(x, y) \varphi(y) \, dV.
\]

Both \( p_t \) and \( K_t \) are symmetric and for each fixed \( x \in M \) both \( p(x, \cdot) \) and \( K(x, \cdot) \) belong to \( C^\infty([0, \infty) \times M] \). This can be seen from the differential equations these kernels satisfy cf. [7] (or from the more recent methods developed by P. Malliavin cf. [8]). Indeed for every fixed \( x \in M \) the function \( u(t, y) = K_t(x, y) (t, y) \in M \) is a positive harmonic function there; i.e., it satisfies \( \Delta_b u = 0 \).

From this, by a direct application of Theorem 3, we obtain the basic Harnack inequality of Theorem 2(i) when the Ricci curvature of \( M \) is non-negative.

The rest of this section will be devoted to the remaining parts of Theorem 2.

Let us start by considering the distance function

\[
r(x) = d(x, y), \quad x \in M,
\]

for some fixed \( y \in M \). \( r \) is then a Hölder continuous function on \( M \) (i.e., \( |r(x_1) - r(x_2)| \leq d(x_1, x_2) \)) and satisfies the following key:

**Lemma 2.1.** Let us suppose that \( M \) is a complete manifold with a nonnegative Ricci tensor; then the Laplacian of \( r^2 \) taken in the distribution sense satisfies

\[
\Delta r^2 \leq 2n;
\]
i.e., we have

\[ \int_M r^2(x) \Delta \varphi(x) \, dV(x) \leq 2n \int_M \varphi(x) \, dV(x) \]

for all \( \varphi \in C_0^\infty(M) \), \( \varphi \geq 0 \).

The above lemma was observed in its present form by Yau [cf. very end of [9]) but the basic ideas for its proof are due to Cheeger and Gromoll (cf. [10]). From that lemma, I shall deduce the following.

**Proposition 2.1.** Let \( M \) be a complete manifold of nonnegative Ricci curvature, then for every fixed \( x \in M \) the distance function \( r(y) = d(x, y) \) satisfies:

\[ \int_M p_t(x, y) r^2(y) \, dV(y) \leq 2nt; \quad t \geq 0, \quad n = \dim M \]  

(2.2)

(i.e. \( T^r \Delta r^2 \|_{\infty} \leq 2nt, \ t \geq 0 \)).

Before we embark on the proof of the proposition let me draw some of its consequences. Statement (2.2) implies that for all \( \Gamma, t > 0 \) and \( x \in M \) we have

\[ \int_{d(t, x) \geq \Gamma} p_t(x, \xi) \, dV(\xi) \leq 2nt/\Gamma^2, \]

which by the definition of the Poisson kernel implies that:

\[ \int_{d(t, x) \geq \Gamma} K_t(x, \xi) \, dV(\xi) \leq \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \min \left[ 1, \frac{nt^2}{2u\Gamma^2} \right] du \leq C \sqrt{n} \frac{t}{\Gamma} \]  

(2.3)

with \( C \) numerical. This, in particular, together with Theorem 2(i), which has already been proved in Section 1, implies Theorem 2(ii) at once.

The following lemma is contained in [22]. Unfortunately I was not aware of that reference when I wrote this paper, so here it is:

**Lemma 2.1.** For every fixed \( m \in M \) and \( \varepsilon > 0 \) and for every compact subset \( K \subset M \) we can find some \( \varphi_j \in C_0^\infty(M) \) \( (j = 1, \ldots) \) and \( C > 0 \) depending only on \( K \) s.t. \( \varphi_j(x) \to_{j \to \infty} d^2(x, m) \) uniformly on \( K \); \( \Delta \varphi_j(x) \leq C, j \geq 1, x \in K; \lim_{j \to \infty} \Delta \varphi_j(x) \leq 2n \) almost everywhere \( x \in K \).

This is but a standard variant of the classical Friedrich's Lemma and of the fact that \( r(x) = d(x, m) \) is Hölder continuous.
Proof. For \( \psi \in C(\mathbb{R}^n) \) let us denote

\[
J_\epsilon \psi(x) = \int_{\mathbb{R}^n} \psi(x - \epsilon y) \varphi(y) \, dy,
\]

where \( \psi \in C^0_c(\mathbb{R}^n) \) is some fixed positive function that satisfies \( \int_{\mathbb{R}^n} \psi \, dx = 1 \). For any \( a \in C^1(\mathbb{R}^n) \) and \( L = a(\partial^2/\partial x_i \partial x_j) = a D_i D_j \) \((i, j = 1, 2, \ldots, n)\) it is clear then that

\[
J_\epsilon L \psi(x) - LJ_\epsilon \psi(x) = \int_{\mathbb{R}^n} (D_j \psi)(x - \epsilon y) \left[ \frac{a(x) - a(x - \epsilon y)}{\epsilon} (D_i \varphi)(y) + (D_i a)(x - \epsilon y) \varphi(y) \right] \, dy
\]

for all \( f \in C^1(\mathbb{R}^n) \) which means that we can estimate \( \sup_{|x| < R} |J_\epsilon L f(x) - LJ_\epsilon f(x)| \) in terms of \( \sup_{|x| < 2R} |\nabla f(x)| \) and \( \sup_{|x| < 2R} |\nabla a(x)| \), provided that \( \epsilon \) is small enough. The upshot is that if \( f \) is a Hölder continuous function on some domain \( \Omega \subset \mathbb{R}^n \) then if we compute the differential operator \( Lf \) in the distribution sense we have:

\[
\|J_\epsilon f - I f\|_{L^2(\Omega)} \to_{\epsilon \to 0} 0, \quad \|J_\epsilon f - I f\|_{L^\infty(\Omega)} \leq C = C(\Omega), (2.4)
\]

for every \( K \subset \Omega \) and every \( \epsilon \) sufficiently small; and (2.4) clearly also holds for any second-order differential operator \( L \) with \( C^1 \) coefficients in \( \Omega \) (if \( L \) only involves first derivatives the proof of (2.4) is even easier, since it only uses the boundedness of \( f \), cf. [11, 5.2.2]).

Now let \( \Omega \) be an open relatively compact coordinate patch on \( M \). It follows from the above that there exists \( J_m \), a sequence of regularising mappings from the space of compactly supported measures of \( \Omega \) to \( C^\infty(\Omega) \) that has the property that if \( f \) is a Hölder continuous function of \( M \) supported in \( \Omega \) and such that \( \Delta f \) in the distribution sense is the measure \( \Delta f = \Phi dV + d\mu \), with \( \Phi \in L^1(M; dV) \) and \( \mu \) singular, then we have:

\[
\begin{align*}
(i) & \quad \|J_m f - f\|_{L^\infty(\Omega)} \to_{m \to \infty} 0, \quad \|J_m f - J_m \Delta f\|_{L^\infty(\Omega)} \leq C_\Omega \|f\|_A, \quad m \geq 1, \\
(ii) & \quad \lim_{m \to \infty} \Delta J_m f = \lim_{m \to \infty} J_m \Delta f = \Phi \quad \text{a.e. on } M,
\end{align*}
\]

where \( C_\Omega \) is a constant that only depends on \( \Omega \) and \( \|f\|_A \) is the Hölder norm of \( f \); and where we identify, as usual, a function \( \varphi \) on \( M \) with the measure \( \varphi \, dV \). To construct the operators \( J_m \) from the convolution \( J_\epsilon \) in \( \mathbb{R}^n \) it is worth observing that, at least for \( \Omega \) small enough, we can map \( \Omega \) into a relatively compact subset \( \bar{\Omega} \subset \mathbb{R}^n \) by a smooth mapping that transports the measure \( dV \) to the Lebesgue measure \( dx \). It is enough then to regularise by convolution in \( \mathbb{R}^n \) and to transport back in \( \Omega \).

Let now \( K \subset M \) be a compact subset of \( M \), let \( K \subset \bigcup \Omega \) be a finite covering of \( K \) by relatively compact open coordinate patches and let \( \{\omega_j\} \) be
an associated smooth partition of unity (i.e., $\text{supp } \theta_j \subset \Omega_j$; $\sum \theta_j = 1$ on $K$) and let us denote by $J_m^{(j)}$ the regularising operators associated to $\Omega_j$. The fact that $\Delta r^2 \leq 2n$ (as in the lemma), in the distribution sense implies that we have $\Delta(\theta_j r^2) \leq C \; (\forall j)$ in the distribution sense (i.e., the measure $\Delta(\theta_j r^2)$ is bounded above by the measure $Cd \text{Vol}$), where $C$ only depends on $K$ and the particular $\theta_j$ chosen. It also follows by the fact that $J_m^{(j)}$ are chosen to be convolution operators by a positive function that they also satisfy:

$$(iii) \quad J_m^{(j)} \Delta(\theta_j r^2) \leq C, \quad \forall m, \; \forall j;$$

where the $C$ again only depends on $K$ and the $\{\theta_j\}$. The above is a pointwise one-sided inequality.

It is clear from (i), (ii), and (iii) that the sequence

$$\varphi_m = \sum_j J_m^{(j)}(\theta_j r^2)$$

satisifies all the conditions of the lemma.

To cope with the singularity of $r$ at $\infty$, I shall resort to probability theory.

Let $z(t), \; t \geq 0,$ denote the continuous-path diffusion on $M$ generated by the Markov semigroup $T^t$ (the brownian motion on $M$). Let us fix $x \in M$ and let us denote for every $a > 0$,

$$T_a = \inf \{ t; r(z(t)) = d(x, z(t)) > a \}$$

the first exit time from the ball $B_a(x)$.

Let us also denote by $E_x$ the expectation for the probability $P_x$ on the path space s.t. $P_x(z(0) = x) = 1.$ We have then:

**LEMMA 2.2.** $E_x[r^2(z(t \wedge T_a))] \leq 2nt \quad \forall x \in M, \; \forall a \geq 0.$

**Proof:** Let $\varphi_j \in C_0^\infty(K)$ be the sequence of Lemma 2.1 that corresponds to $K = B_{2a}(x).$ Dynkin's formula (cf. [12]) applies now because of the strong Markov property (which follows by the Fellerian nature of $T^t$) and if we suppose in addition, as we may, that $\varphi_j(x) = 0 \; (j \geq 1)$ we have:

$$E_x[\varphi_j(z(t \wedge T_a))] = E_x \int_0^{t \wedge T_a} \Delta \varphi_j(z(s)) \, ds \quad (j \geq 1).$$

A passage to limit gives then

$$E_x[r^2(z(t \wedge T_a))] \leq 2nt,$$

which completes the proof. In that passage to the limit we have to decompose $B_a(x)$ into a good set, $G_j = \{ \Delta \varphi_j \leq 2n + \varepsilon \}$, and a bad set, $B_j$, where we can only assert that $\Delta \varphi_j \leq C_a = \text{constant depending on } a$, but
where $\lim_j B_j$ is a set of measure zero. For every $j \geq 0$ the above integral is bounded by

$$(2n + \varepsilon) t + C_a \left( \int_{B_j} \left( \int_0^t p_t(x, y) \, dt \right) \, dV(y) \right).$$

This, by interchanging the order of the two integrations, gives the required result.

To complete the proof of our proposition, it is enough to observe that

$$\lim_{a \to \infty} \mathbb{E}_x [r^2(z(t))] \leq 2nt.$$  

To prove Theorem 2(iii) we first observe that Theorem 2(i) (under the assumption that $K = 0$) gives us:

$$K_{t}(x, y) \leq \frac{C}{\text{Vol} B_{t}(y)} \int_{B_{t}(y)} K_{t}(x, \xi) \, dV(\xi); \quad t > 0; x, y \in M, \quad (2.5)$$

where $C$ only depends on the dimension. This gives us immediately the uniform estimate. If we set $d(x, y) > 2t$ in (2.5) and use (2.3) we obtain the behavior at infinity. This completes the proof of Theorem 2.

Remarks. (i) If we make the weaker hypothesis that

$$\text{Ric}(X, X) \geq -l|X|^2 (r + 1), \quad \forall X \in T_x(M), \ r = d(x, y), \quad (2.6)$$

for some $l > 0$, we can show by the same method as in Lemma 2.1 that we have

$$\Delta r^2 \leq Cl[1 + r^2]$$

in the distribution sense (where $C$ only depends on $n = \dim M$).

(ii) Lemma 2.2 provides an alternative approach as well as a generalisation of (2.1). Indeed we can first construct the brownian motion on $M$ using Ito's approach (cf. [7]) and then use Lemma 2.2 or its obvious generalisation that we can obtain if we impose condition (2.6) to deduce that under condition (2.6) the above brownian motion satisfies (2.1) (and, in particular, there is no explosion). This generalises some of the results of [6].

The proof of the corollary to Theorem 2 is an immediate consequence of
that theorem and of the formula
\[ G(x_0, x) = \int_0^\infty p_t(x_0, x) \, dt = \int_0^\infty tK_t(x_0, x) \, dt. \]
Indeed this formula shows that:
\[
\int_{B_r(x_0)} G(x_0, x) \, dV(x) = \int_{B_r(x_0)} \left[ \int_0^r + \int_r^\infty tK_t(x_0, x) \, dt \right] \, dV(x)
\leq \frac{r^2}{2} + \frac{Ct}{\text{Vol} B_t(x_0)} \int_r^\infty \frac{dV(x)}{t} \quad \forall r > 0.
\]

3. The Weak-Type Estimate

We shall now construct on the cartesian product \( M_0 = \mathbb{R} \times M \) the diffusion that is the cartesian product of standard brownian motion on \( \mathbb{R} \) with the diffusion generated by \( T' t > 0 \) on \( M \). The infinitesimal generator of that diffusion on \( M_0 \) is the Laplace-Beltrami operator \( \Delta_0 = \partial^2/\partial t^2 + \Delta \) of \( M_0 \) (cf. [13] for details on that process). We shall denote by \( z(t) = (y(t), x(t)) \in \mathbb{R} \times M, t > 0 \), the paths of that diffusion, and for every \( a \in \mathbb{R}, \)
\[
T_a = \inf \{ t/y(t) \leq a \}.
\]
For every \( z \in M_0 \) I shall also denote by \( \mathbb{P}_z \) the probability on the path space for which \( \mathbb{P}_z[z(0) = z] = 1 \). Let us as before denote by \( M^+ = (0, +\infty) \times M \) the upper half-space; then for every \( x_0 \in M \) the function
\[
h(z) = h_{x_0}(z) - K_{x_0}(x, x_0), \quad z = (y, x) \in M^+,
\]is harmonic there in the sense of Markov process theory. (For any \( \Omega \subset M_0 \) we say that \( \varphi(z) \) is Markov harmonic in \( \Omega \) if the process \( \{\varphi(z(t)); t < T\} \) is a local martingale under every probability \( \mathbb{P}_z \) \( z \in M_0 \), where \( T \) is the exit time from \( \Omega \).) This is equivalent to saying that \( \Delta_0 h = 0 \) in \( M^+ \).

It follows that we can define the associated Doob h-processes in \( M^+ \) for every \( x_0 \in M \), which amounts to conditioning the motion to exit \( M^+ \) at \( x_0 \) (i.e., \( z(T_0) = x_0 \)). We shall denote by \( \mathbb{P}^h_z (z \in M^+) \) the corresponding probabilities of these processes (i.e. \( \mathbb{P}^h_z [z(0) = z, z \in M^+] \) (cf. [14] for the definition and details on these processes, cf. also [15], where much of what we are about to do has been carried out in a classical setting). Here is the basic estimate that we shall need.

**Proposition 3.1.** Let us assume that \( M \) is a complete manifold with nonnegative Ricci curvature; then there exists \( C = C(n) > 0 \) that only
depends on \( n = \dim M \) such that for all \( h = h_{x_0} \) \((x_0 \in M)\) as in (3.1) and all \( x \in M \) and \( a > 2b > 0 \) we have
\[
\mathbb{P}_{h}^{0}[d(z(T_b), x_0) \leq b] \geq C, \quad z = (a, x) \in M^+.
\] (3.2)

If we denote by:
\[
u^* = \nu^*(\omega) = \sup_{0 < t < T_0} |\nu(t)|, \quad \omega \in \Omega,
\]
the 'brownian maximal function' of \( \nu \in C^1(M^+) \) we then have the following.

**COROLLARY 3.1.** Let \( M \) be as in the proposition and let \( \nu \geq 0 \) satisfy
\[
\Delta_h \nu = 0 \text{ in } M^+. \quad \text{Let } x, x_0 \in M \text{ and } a \geq 2b > 0 \text{ be fixed and let } C > 0 \text{ be a positive number that satisfies}
\]
\[
\frac{1}{C} \leq \frac{\nu(t, x_0)}{\nu(t, \xi)} \leq C; \quad \forall t > 0, \forall \xi \in M, d(x_0, \xi) \leq t.
\] (3.3)

Let us finally suppose that
\[
\sup_{0 < t < b} |\nu(t, x_0)| \geq CA
\] (3.4)
for some \( A > 0 \). We then have:
\[
P_{h_{x_0}}^{(a,x)}[\nu^* > A] > \delta,
\]
where \( \delta \) is a positive number that only depends on the dimension of \( M \).

**Proof of the proposition.** Let us fix \( h = h_{x_0}, b > 0 \), and for all \( z = (y, x) \in M^+ \) \((y \geq b)\) let us define
\[
\phi(z) = \mathbb{P}_{h}^{0}[d(z(T_b), x_0) < b].
\]
It is clear that for all \( x \in M \), such that \( d(x, x_0) < b \), we have
\[
\phi(y, x) \to 1 \quad \text{as} \quad y \to b, y > b.
\] (3.5)
It is also clear from the general theory (cf. [12]) that \( \phi(z) \) is Markov harmonic (for the \( h \)-process) in the domain \( M_b = \{z = (y, x) \in M^+; y > b\} \).

Now the general form of "\( h \)-harmonic" functions is known and can easily be deduced from "ordinary" harmonic functions (cf. [14]). We have
\[
\phi(z) = \frac{f(z)}{h(z)},
\]
where \( \Delta_h f = 0 \) in \( M_b^+ \). We deduce then from (3.5) that for all \( x \in M \) s.t. \( d(x, x_0) < b \) we have:
\[
\lim \{f(y, x); y \to b, y > b\} \geq \min_{d(x, x_0) < b} K_b(x, x_0) = L.
\]
This, by harmonic majorisation, gives that:
\[
f(z) \geq L \int_{d(t, x_0) < b} K_{y-b}(x, \xi) \, dV(\xi), \quad z = (y, x) \in M_b^-.
\]

By Theorem 2 (i) and (ii) there exist then two constants \(C_1, C_2 > 0\) only depending on the dimension such that for all \(z = (y, x)\) with \(y > 2b\) we have
\[
f(z) \geq C_1 L K_{y-b}(x, x_0) \, \text{Vol} B_b(x_0) \geq C_2 K_y(x, x_0) = C_2 h_{x_0}(z).
\]
This clearly completes the proof of the proposition.

**Proof of the corollary.** If the corollary fails it means that for all \(\delta > 0\) we can find some \(x, x_0 \in M\) \(a, b, A > 0\) as in the corollary such that
\[
P^{h_{x_0}}_{(a, x)} [u^* < A] > \delta.
\]
But as soon as \(\delta\) is small enough this contradicts (3.2), (3.3), and (3.4) and proves the corollary.

On the path space \(\Omega\) of the above diffusion let us now define the probability \(P\) coming from the uniform starting measure \(dV\) at level \(y = a > 0\); i.e., we suppose that \(z(0) = (y(0), x(0))\) satisfies \(y(0) = a\) and \(P|x(0) \in E| = \text{Vol}(E)\) for every Borel \(E \subset M\). The above probability has, in general, infinite mass, more precisely we have \(P(\Omega) = \text{Vol} M\), but this fact causes no problems. (The reader at this point is strongly advised to look at [13, Sections 2, 3], where we elaborate on the above probability]. The Doob maximal theorem holds for that "probability":
\[
P[u^* > \alpha] \leq C \|f\|/\alpha, \quad \alpha > 0,
\]
for all \(f \in L^1(M; dV)\) and \(u(y, x) = P^*f(x)\), where \(C\) is an absolute constant and where the inequality above is now significant for small values of \(\alpha\) as well as for large ones.

We can then decompose the path space \(\Omega\) to the subsets
\[
\Omega^x_{x_0} = \{\omega \in \Omega | z(0) = (a, x); z(T_0) = x_0\}
\]
and it is easy to see that the genuine probabilities
\[
P^x_{x_0} = P^{h_{x_0}}_{(a, x)}
\]
on \(\Omega^x_{x_0}\) satisfy
\[
P = \int_M \int_M K_0(x, x_0) \, P^x_{x_0} \, dV(x) \, dV(x_0)
\]
in the sense of weak vector integrals (cf. [14, 15]).
We conclude therefore that for all $\alpha > 0$
\[
\mathbb{P}[u^* > \alpha] = \int_M \int_M K_\alpha(x, x_0) \mathbb{P}_{x_0}^{\ast}[u^* > \alpha] \, dV(x) \, dV(x_0)
\]
\[
\geq \delta \int_M \int_{x_0 \in E} K_\alpha(x, x_0) \, dV(x) \, dV(x_0),
\]
where
\[
E = \{x_0 \in M; \sup_{0 < y < a^2} u(y, x_0) \geq C\alpha\}
\]
with $C$ and $\delta$ as in Corollary 3.1 and depending only on the dimension. We conclude therefore that
\[
\mathbb{P}[u^* > \alpha] \geq \delta \text{Vol} E.
\]
which if we let $a \to \infty$ gives us the required weak-type estimate
\[
\text{Vol}[x_0 \in M; \sup_{t > 0} |u(t, x_0)| > \alpha] \leq C \|f\|/\alpha, \quad \alpha > 0,
\]
with $C$ only depending on the dimension. This gives us Theorem 1(ii). Theorem 1(i) follows from Theorem 2(ii), and from this Theorem 2(iii) follows.

As I already pointed out an alternative and more efficient approach to Theorem 2(ii) can be found in [21].

4. THE TOP OF THE SPECTRUM OF THE LAPLACIAN

The expression that we propose to examine is the top of the spectrum of the Laplacian that is given by
\[
\lambda_1(M) = \inf_{\|d\varphi\|_2 < C_0(M)} \left( \|d\varphi\|_2^2/\|\varphi\|_2^2 \right)
\]
when $M$ is noncompact, and
\[
\lambda_1(M) = \inf \left\{ \|d\varphi\|_2^2/\|\varphi\|_2^2; 0 \neq \varphi \in C^\infty(M), \int_M \varphi \, dV = 0 \right\}
\]
when $M$ is compact (cf. [4, 16, 17]).

Indeed in (4.1) if we set $\varphi(m) = K_\alpha(m, m_0)$ for some fixed $m_0$ and let $t \to \infty$ we obtain at once from Theorem 3(ii) that
\[
\lambda_1(M) \leq 9n(n - 1) K,
\]
provided that the Ricci curvature of $M$ is bounded from below by
(n - 1)(-K) (K > 0). But, although we can easily improve the 9 to 2, the above result is not as good as Cheng's result that gives λ₁(M) ≤ (n - 1)²K/2 (cf. [17]).

Let us now make the assumption that K = 0; i.e., that M is a complete manifold of nonnegative Ricci curvature and let

\[ \varphi(m) = K_{t₁}(m, m₀) - K_{t₂}(m, m₀) \]

for some fixed \( m₀ \in M \) and \( t₁, t₂ > 0 \).

It is clear then from Theorem 3(i) and Theorem 2(iii) that

\[ \| \varphi \|^₂ = K_{2t₁}(m₀, m₀) + K_{2t₂}(m₀, m₀) - 2K_{t₁ + t₂}(m₀, m₀) \]

\[ \| d \varphi \|^₂ \leq C \left[ \frac{K_{2t₁}(m₀, m₀)}{t₁} + \frac{K_{2t₂}(m₀, m₀)}{t₂} + \frac{K_{t₁ + t₂}(m₀, m₀)}{t₁t₂} \right] \]

\[ \lim_{t \to \infty} K_i(m₀, m₀) \leq C₀/\text{Vol} M \]

where \( C \) and \( C₀ \) only depend on \( n = \dim M \).

Now let \( D \) be an arbitrary positive number \( D \leq \text{diameter}(M) \) (the diameter could be finite or not). It is then clear that we can find \( \alpha \) and \( C \), two positive numbers that depend only on \( n = \dim M \) such that

\[ K_{\alpha D}(m₀, m₀) \geq \frac{C}{\text{Vol} B_{\alpha D}(m₀)} \geq \frac{5C₀}{\text{Vol} M} \]

for an appropriate choice of \( m₀ \). If we set \( \alpha D = 2t₁ \) in (4.3) and let \( t₂ \to \infty \) we obtain

\[ \| \varphi \|^₂ \geq \frac{1}{2} K_{2t₁}(m₀, m₀). \]

It follows therefore, by letting \( t₂ \to \infty \) in (4.4), that

\[ \| d \varphi \|^₂ \leq \frac{C}{D \varphi}, \]

where \( C \) only depends on the dimension, and in view of the fact that \( \int_M \varphi \ dV = 0 \) this is just Cheeger's estimate for (4.2) (cf. [18]).

5. A Counterexample

Theorem 1(i) and (iii) breaks down as soon as we allow the curvature to become negative. The failure of Theorem 1(i) is already apparent in the
spaces of constant negative curvature. Indeed in the three-dimensional space of constant curvature $-1$ we have ([20])

$$p_r(x_0, x) = (2\pi r)^{-3/2} e^{-r^2/2} e^{-r^2/2} (r/\sinh r); \quad d(x, x_0) = r.$$  

It follows that for $\delta_0$ the Dirac mass at $x_0$ we have

$$P\delta_0(x) \leq \sup_{t > 0} T^t \delta_0(x) \leq (Ce^{-2r}/\sqrt{r}),$$

$$M\delta_0(x) = \frac{1}{\Vol B_r(x_0)} = Ce^{-2r}$$

for large $r = d(x_0, x)$. In the above example, as in every other two-point homogeneous space, it is the reverse inequality that holds:

$$P f \leq T f = \sup_{t > 0} T^t f \leq M f. \quad (5.1)$$

It would be interesting to prove (5.1) and to study $T f$ in more general spaces.

Observe further that Theorem 1 is invariant by renormalisation of the metric (i.e., multiplication by a constant). It therefore follows that as soon as we obtain a counterexample in some complete manifold, we can easily give an example also in a manifold of bounded curvature. To obtain a counterexample of Theorem 1(iii), in general, it is clearly enough to find an open bounded domain $\Omega \subset M$ in some complete manifold, $M$, and $0 \in \Omega$, some point, such that:

$$\Vol(\Omega) \geq K \Vol(B_r(x)), \quad x \in \Omega, \quad r = d(x, 0), \quad (5.2)$$

with $K$ arbitrarily large. To make (5.2) work we can, for instance, take $M = \mathbb{R}^2$ with the flat metric $A(dx^2 + dy^2)$ and in the disc of radius $B$ replace that metric by the metric $(C^2 - x^2 - y^2)^{-2} (dx^2 + dy^2)$ with $B < C$ and $A = (C^2 - B^2)^{-2}$. If we smooth out that metric and choose $A, B, C$ appropriately we obtain (5.2) with $\Omega$ the disc of radius $B$ and 0, the origin. This can be seen by an easy direct computation.

In seeing the above example I have profited from a conversation with A. Douady. Observe that in [19], Theorem 1(iii) is proved for all symmetric spaces of noncompact type, the proof heavily uses the structure of these spaces and, in view of the above example, this is not surprising.

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