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Global well-posedness for the micropolar fluid system in critical Besov spaces

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ABSTRACT

We prove the global well-posedness for the 3-D micropolar fluid system in the critical Besov spaces by making a suitable transformation of the solutions and using the Fourier localization method, especially combined with a new L^p estimate for the Green matrix to the linear system of the transformed equation. This result allows to construct global solutions for a class of highly oscillating initial data of Cannone's type. Meanwhile, we analyze the long-time behavior of the solutions and get some decay estimates.

1. Introduction

We consider the incompressible micropolar fluid system in $\mathbb{R}^+ \times \mathbb{R}^3$:

$$\begin{cases} \partial_t u - (\chi + \nu)\Delta u + u \cdot \nabla u + \nabla \pi - 2\chi \nabla \times \omega = 0, \\ \partial_t \omega - \mu \Delta \omega + u \cdot \nabla \omega + 4\chi \omega - \kappa \nabla \operatorname{div} \omega - 2\chi \nabla \times u = 0, \\ \operatorname{div} u = 0, \\ (u, \omega)|_{t=0} = (u_0, \omega_0). \end{cases} \tag{1.1}$$

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Here $u(t,x)$ and $\omega(t,x)$ denote the linear velocity and the velocity field of rotation of the fluid respectively. The scalar $\pi(t,x)$ denotes the pressure of the fluid. The constants κ, χ, ν, μ are the viscosity coefficients. For simplicity, we take $\chi = \nu = \frac{1}{2}$ and $\kappa = \mu = 1$.

Micropolar fluid system was firstly developed by Eringen [11]. It is a type of fluids which exhibits micro-rotational effects and micro-rotational inertia, and can be viewed as a non-Newtonian fluid. Physically, micropolar fluid may represent fluids that consist of rigid, randomly oriented (or spherical particles) suspended in a viscous medium, where the deformation of fluid particles is ignored. It can describe many phenomena that appear in a large number of complex fluids such as the suspensions, animal blood, liquid crystals which cannot be characterized appropriately by the Navier–Stokes system, and that it is important to the scientists working with the hydrodynamic-fluid problems and phenomena. For more background, we refer to [16] and references therein.

If the microstructure of the fluid is not taken into account, that is to say the effect of the angular velocity field of the particle's rotation is omitted, i.e., $\omega = 0$, then Eq. (1.1) reduces to the classical Navier–Stokes equations.

Due to its importance in mathematics and physics, there is a lot of literature devoted to the mathematical theory of the micropolar fluid system. Galdi and Rionero [14] and Lukaszewicz [16] proved the existence of the weak solution. The existence and uniqueness of strong solutions to the micropolar flows and the magneto-micropolar flows either local for large data or global for small data are considered in [2,16,17] and references therein. Recently, inspired by the work of Cannone and Karch [6] on the compressible Navier–Stokes equations, V.-Roa and Ferreira [12] proved the well-posedness of the generalized micropolar fluids system in the pseudo-measure space which is denoted by PM^a-space. The elements of this space have Fourier transform that verifies

$$\sup_{\mathbb{R}^3} |\xi|^a \big|\hat{f}(\xi)\big| < \infty. \tag{1.2}$$

For the well-posedness for the 2D case with full viscosity and partial viscosity one may refer to [16,10] respectively. For the blow-up criterion of smooth solutions and the regularity criterion for weak solutions one refers to [19,18] and references therein.

For the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0, \end{cases} \tag{1.3}$$

Fujita and Kato [13,15] proved the local well-posedness for large initial data and the global well-posedness for small initial data in the homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}$ and the Lebesgue space L^3 respectively. These spaces are all the critical ones, which are related to the scaling of the Navier–Stokes equations: if (u,p) solves (1.3), then

$$\big(u_\lambda(t,x), p_\lambda(t,x)\big) \stackrel{def}{=} \big(\lambda u\big(\lambda^2 t, \lambda x\big), \lambda^2 p\big(\lambda^2 t, \lambda x\big)\big) \tag{1.4}$$

is also a solution of (1.3). The so-called **critical space** is the one such that the associated norm is invariant under the scaling of (1.4). Recently, Cannone [5] (see also [4]) generalized well-posedness to Besov spaces with negative index of regularity. More precisely, he showed that if the initial data satisfies

$$\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}} \leqslant c, \quad p > 3$$

for some small constant c, then the Navier–Stokes equations (1.3) are globally well-posed. Let us emphasize that this result allows to construct global solutions for highly oscillating initial data which may have a large norm in $\dot{H}^{\frac{1}{2}}$ or L^3. A typical example is

$$u_0(x)=\sin\left(\frac{x_3}{\varepsilon}\right)\big(-\partial_2\phi(x),\,\partial_1\phi(x),0\big)$$

where $\phi\in\mathcal{S}(\mathbb{R}^3)$ and $\varepsilon>0$ is small enough. Concerning the compressible Navier–Stokes equations, we have established the global well-posedness in the framework of the hybrid-Besov space, please refer to [8].

In this paper we prove the global well-posedness for the 3-D micropolar fluid system in the critical Besov spaces as the incompressible Navier–Stokes equations (1.3). This result allows to construct global solutions for a class of highly oscillating initial data of Cannone's type. Meanwhile, we analyze the long-time behavior of the solutions and get some decay estimates.

Now let us sketch the main difficulty and the strategy to overcome it.

Applying the Leray projection to the first equation in (1.1), we obtain

$$\begin{cases}\partial_t u-\Delta u+\mathbf{P}(u\cdot\nabla u)-\nabla\times\omega=0,\\ \partial_t\omega-\Delta\omega+u\cdot\nabla\omega+2\omega-\nabla\operatorname{div}\omega-\nabla\times u=0,\\ \operatorname{div}u=0,\\ (u,\omega)|_{t=0}=(u_0,\omega_0).\end{cases}\tag{1.5}$$

Obviously, the system has no scaling invariance compared with the incompressible Navier–Stokes equation. In general there are two ways to achieve the global existence for small data in the critical Besov space as $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ for general p. The first one is Kato's semigroup method which was used in [5], it turns out that the both linear terms $\nabla\times\omega$ and $\nabla\times u$ will play bad roles if they are regarded as the perturbations. The second way is to use the energy method together with the Fourier localization technique, but the linear coupling effect of the system (1.5) is too strong to control unless the coefficients of these two linear terms are sufficiently small, while it is impossible.

To overcome the difficulty from the terms $\nabla\times\omega$ and $\nabla\times u$, we will view them as certain perturbation of the Laplacian operator in some sense. More precisely, we will study the following mixed linear system of Eq. (1.5):

$$\begin{cases}\partial_t u-\Delta u-\nabla\times\omega=0,\\ \partial_t\omega-\Delta\omega+2\omega-\nabla\operatorname{div}\omega-\nabla\times u=0,\end{cases}\tag{1.6}$$

and the action of its Green matrix which is denoted by $G(x,t)$. From [12], we have

$$\widehat{Gf}(\xi,t)=e^{-A(\xi)t}\hat{f}(\xi),\tag{1.7}$$

where

$$A(\xi)=\begin{bmatrix}|\xi|^2I & B(\xi)\\ B(\xi) & (|\xi|^2+2)I+C(\xi)\end{bmatrix}$$

with

$$B(\xi)=i\begin{bmatrix}0 & -\xi_3 & \xi_2\\ \xi_3 & 0 & -\xi_1\\ -\xi_2 & \xi_1 & 0\end{bmatrix}\quad\text{and}\quad C(\xi)=\begin{bmatrix}\xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3\\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3\\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2\end{bmatrix}.$$

It has been shown in [12] that $G(x,t)$ has similar properties in common with the heat kernel, i.e.,

$$\big|\widehat{G}(\xi,t)\big|\leqslant e^{-c|\xi|^2t},\tag{1.8}$$

which means that $\|G(x,t)f\|_{L^2}$ is bounded. However, it is not enough to obtain the estimates of the solution in the Besov space as we wanted. For this purpose, we have to analyze the behavior of the derivative of $\widehat{G}(\xi,t)$ to set up the boundedness of $G(x,t)f$ in L^p. In fact, we have the better property that $\|G(x,t)f\|_{L^p}$ has exponential decay estimate for $\hat{f}$ supported in a ring. But if we directly calculate its derivatives as well as utilizing the estimate (1.8), we only have the rough estimate for example when $|\alpha|=1$,

$$\left|D_\xi^\alpha \widehat{G}(\xi,t)\right| \leqslant e^{-c|\xi|^2 t} t\left(1+|\xi|\right). \tag{1.9}$$

Obviously, the above inequality is not enough for us to deduce that for any couple (t,λ) of positive real numbers and $\operatorname{supp}\hat{f} \subset \lambda\mathcal{C}$ ($\mathcal{C}$ is the annulus in Section 2) such that

$$\left\|G(x,t)f\right\|_{L^p} \leqslant Ce^{-ct\lambda^2}\|f\|_{L^p}, \quad 1 \leqslant p \leqslant \infty, \tag{1.10}$$

except the high frequency case $\lambda \geqslant 1$ and the case $p=2$ is allowed. As for the well-posedness in the pseudo-measure space (see [12]), only the estimate (1.8) is required owing to the speciality of the working space, and its method seems not to work for the derivatives estimate of $\widehat{G}(\xi,t)$.

Although the second equation of (1.5) presents the smoothing effect, there is negative impact from $\nabla\times u$ and $\nabla\times\omega$ which is the main obstacle to establish the well-posedness of (1.1) for highly oscillating data in the more natural Besov space $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$. To this goal, we shall employ sufficiently the structure properties of the systems. In fact, we find that if we make a suitable transformation of the solution, then Eq. (1.5) reduces to a new version. More precisely, the vector field velocity $u=(u_1,u_2,u_3)$ is transformed to an anti-symmetric matrix u_A with

$$u=(u_1,u_2,u_3)\mapsto u_A \overset{def}{=} \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix},$$

and decompose ω into $\omega_d = \Lambda^{-1}\operatorname{div}\omega$ and $\omega_\Omega = \Lambda^{-1}\operatorname{curl}\omega$, here we denote

$$\Lambda^s z \overset{def}{=} \mathcal{F}^{-1}\left(|\xi|^s\hat{z}\right)$$

and the matrix

$$(\operatorname{curl} z)^i_j \overset{def}{=} \left(\partial_j z^i - \partial_i z^j\right)_{1\leqslant i,j\leqslant 3}. \tag{1.11}$$

In light of $\operatorname{div} u = 0$, the system (1.5) can be rewritten as

$$\begin{cases} \partial_t u_A - \Delta u_A - \Lambda\omega_\Omega = -\left(\mathbf{P}(u\cdot\nabla u)\right)_A, \\ \partial_t\omega_\Omega - \Delta\omega_\Omega + 2\omega_\Omega - \Lambda u_A = -\Lambda^{-1}\operatorname{curl}(u\cdot\nabla\omega), \\ \partial_t\omega_d - 2\Delta\omega_d + 2\omega_d = -\Lambda^{-1}\operatorname{div}(u\cdot\nabla\omega), \\ \omega = \Lambda^{-1}\nabla\omega_d - \Lambda^{-1}\operatorname{div}\omega_\Omega, \qquad \operatorname{div} u = 0, \\ (u_A,\omega_\Omega,\omega_d)|_{t=0} = (u_{0,A},\omega_{0,\Omega},\omega_{0,d}), \end{cases} \tag{1.12}$$

where $(\mathbf{P}(u\cdot\nabla u))_A$ is as follows

$$\begin{pmatrix} 0 & u_i\partial_i u_3 + \Lambda^{-2}\partial_3\partial_j(u_i\partial_i u_j) & -u_i\partial_i u_2 - \Lambda^{-2}\partial_2\partial_j(u_i\partial_i u_j) \\ -u_i\partial_i u_3 - \Lambda^{-2}\partial_3\partial_j(u_i\partial_i u_j) & 0 & u_i\partial_i u_1 + \Lambda^{-2}\partial_1\partial_j(u_i\partial_i u_j) \\ u_i\partial_i u_2 + \Lambda^{-2}\partial_2\partial_j(u_i\partial_i u_j) & -u_i\partial_i u_1 - \Lambda^{-2}\partial_1\partial_j(u_i\partial_i u_j) & 0 \end{pmatrix}.$$

Let us observe the associate linear system of Eq. (1.12). Since the third equation of (1.12) is mainly a heat equation, we focus our attention on the first two equations of (1.12). Let us study the following coupling linear system:

$$\begin{cases} \partial_t u_A - \Delta u_A - \Lambda \omega_\Omega = 0, \\ \partial_t \omega_\Omega - \Delta \omega_\Omega + 2\omega_\Omega - \Lambda u_A = 0, \\ (u_A, \omega_\Omega)|_{t=0} = (u_{0,A}, \omega_{0,\Omega}). \end{cases} \tag{1.13}$$

If $\mathcal{G}(x,t)$ denotes by the Green matrix of (1.13), then $\mathcal{G}(x,t)(u_{0,A}, \omega_{0,\Omega})$ is the solution of (1.13). We have

$$\widehat{\mathcal{G}f}(\xi,t) = e^{-\widetilde{A}(\xi)t} \hat{f}(\xi), \tag{1.14}$$

with

$$\widetilde{A}(\xi) = \begin{bmatrix} |\xi|^2 & |\xi| \\ |\xi| & |\xi|^2 + 2 \end{bmatrix}.$$

Then using the Laplace transform, the derivatives of $\widehat{\mathcal{G}}(\xi,t)$ can be exactly and explicitly represented, see Section 3, which helps us to deduce the following crucial estimate

$$\left| D_\xi^\alpha \widehat{\mathcal{G}}(\xi,t) \right| \leqslant C e^{-c|\xi|^2 t} |\xi|^{-|\alpha|}.$$

This allows us to obtain that for any couple (t,λ) of positive real numbers and $\operatorname{supp} \hat{f} \subset \lambda\mathcal{C}$, there holds

$$\left\| \mathcal{G}(x,t) f \right\|_{L^p} \leqslant C e^{-ct\lambda^2} \|f\|_{L^p}, \quad 1 \leqslant p \leqslant \infty,$$

here $\mathcal{C}$ is a ring away from zero, see Proposition 3.5. Let us emphasize that the above inequality is essential to the well-posedness in the Besov spaces.

Definition 1.1. Let $1 \leqslant p \leqslant \infty$, $T > 0$. We denote E_T^p by the space of functions such that

$$\left\| (u,\omega) \right\|_{E_T^p} \overset{def}{=} \left\| (u,\omega) \right\|_{\widetilde{L}^\infty(0,T;\dot{B}^{\frac{3}{p}-1}_{p,\infty})} + \left\| (u,\omega) \right\|_{\widetilde{L}^1(0,T;\dot{B}^{\frac{3}{p}+1}_{p,\infty})} < \infty.$$

If $T = \infty$, we denote E_∞^p by E^p. We refer to Section 2 for the definition of $\widetilde{L}^r(X)$.

Our main results are stated as follows.

Theorem 1.2. *There exist two positive constants η and M such that for all $(u_0, \omega_0) \in \dot{B}^{\frac{3}{p}-1}_{p,\infty}$ with*

$$\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|\omega_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant \eta. \tag{1.15}$$

Then for $1 \leqslant p < 6$, the system (1.1) has a global solution $(u,\omega) \in C((0,\infty); \dot{B}^{\frac{3}{p}-1}_{p,\infty})$ with

$$\left\| (u,\omega) \right\|_{L^\infty(0,\infty;\dot{B}^{\frac{3}{p}-1}_{p,\infty})} \leqslant M\left(\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|\omega_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \right).$$

Moreover, the uniqueness holds in E^p.

Remark 1.3. If we work in the space $\widetilde{L}^\infty(\dot{B}^{\frac{3}{p}-1}_{p,1}) \cap \widetilde{L}^1(\dot{B}^{\frac{3}{p}+1}_{p,1})$, the borderline case $p = 6$ can be achieved. Moreover, the range of p for the existence and the uniqueness can be extended to $[1,\infty)$ and $[1,6]$, respectively. In fact, using the paradifferential calculus, it is easy to see that the nonlinear term $u \cdot \nabla u$ and $u \cdot \nabla \omega$ are bounded in $\widetilde{L}^1(\dot{B}^{\frac{3}{p}-1}_{p,1})$, i.e., in light of $\operatorname{div} u = 0$,

$$\|u \cdot \nabla \omega\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leqslant C\|u\omega\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \leqslant C\|u\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|\omega\|_{\dot{B}^{\frac{3}{p}}_{p,1}}, \quad \text{for } p \in [1,\infty),$$

while $u\omega$ is not continuous from $\dot{B}^{\frac{3}{p}}_{p,\infty} \times \dot{B}^{\frac{3}{p}}_{p,\infty}$ to $\dot{B}^{\frac{3}{p}}_{p,\infty}$.

Theorem 1.4. *If* $(u_0, \omega_0) \in \dot{H}^{\frac{1}{2}}$ *and satisfies* (1.15)*, then the system* (1.1) *has a unique global solution in* $C(\mathbb{R}^+; \dot{H}^{\frac{1}{2}})$.

Remark 1.5. Here we don't impose the $\dot{H}^{\frac{1}{2}}$ smallness condition on the initial data. Especially, this allows us to obtain the global well-posedness of (1.1) for the highly oscillating initial velocity (u_0, ω_0). For example,

$$u_0(x) = \sin\left(\frac{x_3}{\varepsilon}\right)\big(-\partial_2\phi(x), \partial_1\phi(x), 0\big), \qquad \omega_0(x) = \sin\left(\frac{x_1}{\varepsilon}\right)\phi(x), \quad \phi(x) \in \mathcal{S}\big(\mathbb{R}^3\big),$$

which satisfies

$$\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}}, \|\omega_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \ll 1, \quad \text{for } p > 3$$

if $\varepsilon > 0$ is small enough, see Proposition 2.8.

Finally, we prove that the solution has the following decay estimates.

Theorem 1.6. *Let* (u, ω) *be a solution provided by Theorem* 1.2*. Then for all multi-indices* α*, we have*

$$\big\|\big(D_x^\alpha u, D_x^\alpha \omega\big)\big\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C_0 t^{-\frac{|\alpha|}{2}}, \quad t > 0, \tag{1.16}$$

where C_0 *is a constant depending on the initial data.*

Remark 1.7. As a direct consequence of the estimate (1.16) and Sobolev embedding $\dot{B}^{\frac{3}{p_1}-1}_{p_1,\infty} \hookrightarrow \dot{B}^{\frac{3}{p_2}-1}_{p_2,\infty}$ with $p_1 \leqslant p_2$, one knows that for $t > 0$, the solution $(u, \omega) \in C^\infty(\mathbb{R}^3)$. In fact, for $3 < p < 6$, one easily sees that

$$\begin{aligned}
\|D^\beta u\|_{\dot{B}^0_{\infty,1}} &= \sum_{j \in \mathbb{Z}} \big\|\Delta_j\big(D^\beta u\big)\big\|_\infty = \sum_{j \geqslant 0} \big\|\Delta_j\big(D^\beta u\big)\big\|_\infty + \sum_{j<0} \big\|\Delta_j\big(D^\beta u\big)\big\|_\infty \\
&= \sum_{j \geqslant 0} 2^{j\frac{3}{p}} \big\|\Delta_j\big(D^\beta u\big)\big\|_\infty 2^{-j\frac{3}{p}} + \sum_{j<0} 2^{j(\frac{3}{p}-1)} \big\|\Delta_j\big(D^\beta u\big)\big\|_\infty 2^{-j(\frac{3}{p}-1)} \\
&\lesssim \big\|D^\beta u\big\|_{\dot{B}^{\frac{3}{p}}_{p,\infty}} + \big\|D^\beta u\big\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}}, \quad \forall \beta \in \big(\mathbb{Z} \cup \{0\}\big)^3.
\end{aligned}$$

Notation. Throughout this paper, we denote some notations on the matrix $M = (M_{ij})_{1\leqslant i,j\leqslant m}$

$$|M| \stackrel{def}{=} \sum_{i,j} |M_{ij}|,$$

and for a functional space X, we denote $\|M\|_X$ by

$$\|M\|_X \stackrel{def}{=} \sum_{i,j} \|M_{ij}\|_X.$$

The structure of this paper is organized as follows.

In Section 2, we recall some basic facts about the Littlewood–Paley theory and the functional spaces. In Section 3, we analyze Green's matrix of the linear system (1.13) and show some new results concerning its regularizing effect. Section 4 is devoted to the proof of Theorem 1.2. Section 5 is devoted to the proof of Theorem 1.4. In Section 6, we give certain decay rates of the solution.

2. Littlewood–Paley theory and the function spaces

Firstly, we introduce the Littlewood–Paley decomposition. Choose two radial functions $\varphi, \chi \in \mathcal{S}(\mathbb{R}^3)$ supported in $\mathcal{C} = \{\xi \in \mathbb{R}^3,\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$, $\mathcal{B} = \{\xi \in \mathbb{R}^3,\ |\xi| \leqslant \frac{4}{3}\}$ respectively such that

$$\sum_{j\in\mathbb{Z}} \varphi\big(2^{-j}\xi\big) = 1, \quad \text{for all } \xi \neq 0,$$

and

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi\big(2^{-j}\xi\big) = 1, \quad \text{for all } \xi \in \mathbb{R}^3.$$

For $f \in \mathcal{S}'(\mathbb{R}^3)$, the frequency localization operators Δ_j and $S_j (j \in \mathbb{Z})$ are defined by

$$\Delta_j f = \varphi\big(2^{-j}D\big) f, \qquad S_j f = \chi\big(2^{-j}D\big) f.$$

Moreover, we have

$$S_j f = \sum_{k=-\infty}^{j-1} \Delta_k f \quad \text{in } \mathcal{Z}'\big(\mathbb{R}^3\big).$$

Here we denote the space $\mathcal{Z}'(\mathbb{R}^3)$ by the space of tempered distributions modulo polynomials.

With our choice of φ, it is easy to verify that

$$\Delta_j \Delta_k f = 0 \quad \text{if } |j-k| \geqslant 2$$

and

$$\Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j-k| \geqslant 5. \tag{2.1}$$

For more details, please refer to [4,3].

In the sequel, we will constantly use the Bony's decomposition from [1]:

$$fg = T_f g + T_g f + R(f,g), \tag{2.2}$$

with

$$T_f g = \sum_{j\in\mathbb{Z}} S_{j-1} f \Delta_j g, \qquad R(f,g) = \sum_{j\in\mathbb{Z}} \Delta_j f \widetilde{\Delta}_j g, \qquad \widetilde{\Delta}_j g = \sum_{|j'-j|\leqslant 1} \Delta_{j'} g.$$

Let us first recall the general definition of a Besov space.

Definition 2.1. Let $s\in\mathbb{R}$, $1\leqslant p,q\leqslant+\infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} \stackrel{def}{=} \left\{ f\in \mathcal{Z}'\left(\mathbb{R}^3\right):\ \|f\|_{\dot{B}^s_{p,q}} < +\infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} \stackrel{def}{=} \left\| 2^{ks} \left\| \Delta_k f(t) \right\|_{L^p} \right\|_{\ell^q}.$$

If $p=q=2$, $\dot{B}^s_{2,2}$ is equivalent to the homogeneous Sobolev space $\dot{H}^s$.
Now let us recall Chemin–Lerner's space-time space [3].

Definition 2.2. Let $s\in\mathbb{R}$, $1\leqslant p,q,r\leqslant\infty$, $I\subset\mathbb{R}$ is an interval. The homogeneous mixed time-space Besov space $\widetilde{L}^r(I;\dot{B}^s_{p,q})$ is space of distributions such that

$$\widetilde{L}^r\left(I;\dot{B}^s_{p,q}\right) \stackrel{def}{=} \left\{ f\in\mathcal{D}\left(I;\mathcal{Z}'\left(\mathbb{R}^d\right)\right);\ \|f\|_{\widetilde{L}^r(I;\dot{B}^s_{p,r})} < +\infty \right\},$$

where

$$\|f(t)\|_{\widetilde{L}^r(I;\dot{B}^s_{p,q})} \stackrel{def}{=} \left\| 2^{sj} \left(\int_I \left\| \Delta_j f(\tau) \right\|_p^r \mathrm{d}\tau \right)^{\frac{1}{r}} \right\|_{\ell^q(\mathbb{Z})}.$$

For convenience, we sometimes use $\widetilde{L}^r_t(\dot{B}^s_{p,q})$ and $\widetilde{L}^r(\dot{B}^s_{p,q})$ to denote $\widetilde{L}^r(0,t;\dot{B}^s_{p,q})$ and $\widetilde{L}^r(0,\infty;\dot{B}^s_{p,q})$, respectively. The direct consequence of Minkowski's inequality is that

$$L^r_t\left(\dot{B}^s_{p,q}\right) \subseteq \widetilde{L}^r_t\left(\dot{B}^s_{p,q}\right) \quad \text{if } r\leqslant q \quad \text{and} \quad \widetilde{L}^r_t\left(\dot{B}^s_{p,q}\right) \subseteq L^r_t\left(\dot{B}^s_{p,q}\right) \quad \text{if } r\geqslant q.$$

Let us state some basic properties about the Besov spaces.

Lemma 2.3. *(See [3].)*

(i) *If $s<\frac{3}{p}$ or $s=\frac{3}{p}$ and $r=1$, then $(\dot{B}^s_{p,q}, \|\cdot\|_{\dot{B}^s_{p,q}})$ is a Banach space.*
(ii) *We have the equivalence of norms*

$$\left\| D^k f \right\|_{\dot{B}^s_{p,q}} \sim \|f\|_{\dot{B}^{s+k}_{p,q}}, \quad \textit{for } k\in\mathbb{Z}^+.$$

(iii) *Interpolation*: *for* $s_1, s_2 \in \mathbb{R}$ *and* $\theta \in [0,1]$, *one has*

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,q}} \leqslant \|f\|^{\theta}_{\dot{B}^{s_1}_{p,q}} \|f\|^{(1-\theta)}_{\dot{B}^{s_2}_{p,q}}.$$

The following Bernstein's lemma will be repeatedly used throughout this paper.

Lemma 2.4. *(See [3].) Let* $1 \leqslant p \leqslant q \leqslant +\infty$. *Then for any* $\beta, \gamma \in (\mathbb{N} \cup \{0\})^3$, *there exists a constant* C *independent of* f, j *such that*

$$\operatorname{supp} \hat{f} \subseteq \{|\xi| \leqslant A_0 2^j\} \quad \Rightarrow \quad \|\partial^\gamma f\|_{L^q} \leqslant C 2^{j|\gamma|+3j(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p},$$
$$\operatorname{supp} \hat{f} \subseteq \{A_1 2^j \leqslant |\xi| \leqslant A_2 2^j\} \quad \Rightarrow \quad \|f\|_{L^p} \leqslant C 2^{-j|\gamma|} \sup_{|\beta|=|\gamma|} \|\partial^\beta f\|_{L^p}.$$

Lemma 2.5. *(See [7].) Let* $2 \leqslant p < +\infty$. *Then for any* f *with* $\operatorname{supp} \hat{f} \subseteq \{A_1 2^j \leqslant |\xi| \leqslant A_2 2^j\}$, *there exists a constant* c *independent of* f, j *such that*

$$c 2^{2j} \int_{\mathbb{R}^3} |f|^p \, dx \leqslant \int_{\mathbb{R}^3} (-\Delta f)|f|^{p-2} f \, dx.$$

Lemma 2.6. *(See [3].)*

(i) *Let* (s, p, r_1) *be such that* $\dot{B}^s_{p,r_1}$ *is a Banach space. Then the paraproduct* T *maps continuously* $L^\infty \times \dot{B}^s_{p,r_1}$ *into* $\dot{B}^s_{p,r}$. *Moreover, if* t *is negative and* r_2 *such that*

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \leqslant 1,$$

and if $\dot{B}^{s+t}_{p,r}$ *is a Banach space, then* T *maps continuously* $\dot{B}^t_{\infty,r_1} \times \dot{B}^s_{p,r_2}$ *into* $\dot{B}^{s+t}_{p,r}$.

(ii) *Let* (p_k, r_k) *(for* $k \in \{1,2\}$*) be such that*

$$s_1 + s_2 > 0, \qquad \frac{1}{p} \leqslant \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1 \quad \text{and} \quad \frac{1}{r} \leqslant \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1.$$

The operator R *maps* $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ *into* $\dot{B}^{\sigma_{12}}_{p,r}$ *with*

$$\sigma_{12} := s_1 + s_2 - 3\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right),$$

provided that $\sigma_{12} < 3/p$, *or* $\sigma_{12} = 3/p$ *and* $r = 1$.

With the help of the above lemma, we can obtain

Lemma 2.7. *Let* $1 \leqslant p \leqslant \infty$. *Then there hold*:

(a) *If* $s_1, s_2 \leqslant \frac{3}{p}$ *and* $s_1 + s_2 > 3\max(0, \frac{2}{p} - 1)$, *then*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,1}} \leqslant C \|f\|_{\dot{B}^{s_1}_{p,1}} \|g\|_{\dot{B}^{s_2}_{p,1}}.$$

(b) *If* $s_1 < \frac{3}{p}$, $s_2 < \frac{3}{p}$, *and* $s_1 + s_2 > 3\max(0, \frac{2}{p} - 1)$, *then*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,\infty}} \leqslant C\|f\|_{\dot{B}^{s_1}_{p,\infty}}\|g\|_{\dot{B}^{s_2}_{p,\infty}}.$$

(c) *If* $s_1 \leqslant \frac{3}{p}$, $s_2 < \frac{3}{p}$, *and* $s_1 + s_2 \geqslant 3\max(0, \frac{2}{p} - 1)$, *then*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,\infty}} \leqslant C\|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,\infty}}.$$

Proposition 2.8. *(See [8].) Let* $\phi \in \mathcal{S}(\mathbb{R}^3)$ *and* $p > 3$. *If* $\phi_\varepsilon(x) \overset{def}{=} e^{i\frac{x_1}{\varepsilon}}\phi(x)$, *then for any* $\varepsilon > 0$,

$$\|\phi_\varepsilon\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C\varepsilon^{1-\frac{3}{p}},$$

here C *is a constant independent of* ε.

The following proposition describes the smoothing effect of the solution for linear heat equation, the proof can be found in [3].

Proposition 2.9. *Let* $s \in \mathbb{R}$, *and* $p, r \in [1, \infty]$, $\nu_1 > 0$, $\nu_2 \geqslant 0$. *Assume that* $u_0 \in \dot{B}^s_{p,q}$, $f \in L^1_t\dot{B}^s_{p,q}$. *Then the equation*

$$\begin{cases} \partial_t u - \nu_1\Delta u + \nu_2 u = f, \\ u|_{t=0} = u_0, \end{cases}$$

has a unique solution u *satisfying*

$$\|u\|_{\widetilde{L}^r_t\dot{B}^{s+\frac{2}{r}}_{p,q}} \leqslant C\big(\|u_0\|_{\dot{B}^s_{p,q}} + \|f\|_{\widetilde{L}^1_t\dot{B}^s_{p,q}}\big).$$

3. The linearized equations of the microfluid system

In this section, we are devoted to analyzing the Green matrix of Eq. (1.6) by Laplace transform with respect to the time variable. We define Laplace transform of $f(t)$

$$(\mathcal{L}f)(p) = \int_0^\infty e^{-pt}f(t)\,\mathrm{d}t.$$

For the sake of convenience, we denote $(f)^L(p)$ by the Laplace transform $(\mathcal{L}f)(p)$ of $f(t)$.

First, let us introduce a notation: if $(M_{ij})_{\{1\leqslant i,j\leqslant 2\}}$ is a matrix, $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$ are vectors, then we denote

$$(M_{ij})_{\{1\leqslant i,j\leqslant 2\}}\begin{pmatrix} f \\ g \end{pmatrix} \overset{def}{=} \begin{pmatrix} M_{11}f + M_{12}g \\ M_{21}f + M_{22}g \end{pmatrix}.$$

Taking Fourier transform of (1.13) yields that

$$\begin{cases} \partial_t \widehat{u_A} + |\xi|^2 \widehat{u_A} - |\xi| \widehat{\omega_\Omega} = 0, \\ \partial_t \widehat{\omega_\Omega} + \big(|\xi|^2 + 2\big)\widehat{\omega_\Omega} - |\xi| \widehat{u_A} = 0, \\ (\widehat{u_A}, \widehat{\omega_\Omega})|_{t=0} = (\widehat{u_{0,A}}, \widehat{\omega_{0,\Omega}}). \end{cases} \tag{3.1}$$

In what follows, we will use the Laplace transform to get the explicit expression of $\widehat{\mathcal{G}}(\xi, t)$, where $\mathcal{G}(x, t)$ denotes the Green matrix of (1.13).

Let

$$\sum_{\phi} = \big\{ z \in \mathbb{C} \backslash \{0\},\ |\arg z| < \phi \big\}.$$

Assume that $p \in \sum_\phi$ for some $\phi \in [0, \pi/2)$, then we have

$$\begin{cases} p(\widehat{u_A})^L + |\xi|^2 (\widehat{u_A})^L - |\xi| (\widehat{\omega_\Omega})^L = \widehat{u_{0,A}}, \\ p(\widehat{\omega_\Omega})^L + \big(|\xi|^2 + 2\big)(\widehat{\omega_\Omega})^L - |\xi| (\widehat{u_A})^L = \widehat{\omega_{0,\Omega}}, \end{cases} \tag{3.2}$$

that is,

$$\begin{pmatrix} (\widehat{u_A})^L(\xi, t) \\ (\widehat{\omega_\Omega})^L(\xi, t) \end{pmatrix} = \begin{pmatrix} p + |\xi|^2 & -|\xi| \\ -|\xi| & p + |\xi|^2 + 2 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{u_{0,A}} \\ \widehat{\omega_{0,\Omega}} \end{pmatrix}.$$

Setting $\lambda^2 = p + |\xi|^2$, we see that

$$\begin{pmatrix} (\widehat{u_A})^L \\ (\widehat{\omega_\Omega})^L \end{pmatrix} = \frac{1}{\det} \begin{pmatrix} \lambda^2 + 2 & |\xi| \\ |\xi| & \lambda^2 + 2 \end{pmatrix} \begin{pmatrix} \widehat{u_{0,A}} \\ \widehat{\omega_{0,\Omega}} \end{pmatrix}$$

with

$$\det \overset{def}{=} \lambda^4 + 2\lambda^2 - |\xi|^2.$$

Then we have the explicit expression of the solution of (3.1):

$$\begin{pmatrix} \widehat{u_A} \\ \widehat{\omega_\Omega} \end{pmatrix} = \left\{ \mathcal{L}^{-1}\left(\frac{\lambda^2}{\det}\right) I + \mathcal{L}^{-1}\left(\frac{1}{\det}\right) \begin{pmatrix} 2 & |\xi| \\ |\xi| & 0 \end{pmatrix} \right\} \begin{pmatrix} \widehat{u_{0,A}} \\ \widehat{\omega_{0,\Omega}} \end{pmatrix}, \tag{3.3}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transformation with respect to p, and I is the identity matrix. Denote

$$\mathcal{A}(\xi, t) \overset{def}{=} \frac{e^{(-1-\sqrt{1+|\xi|^2})t} - e^{(-1+\sqrt{1+|\xi|^2})t}}{2\sqrt{1+|\xi|^2}},$$

$$\mathcal{B}(\xi, t) \overset{def}{=} \frac{e^{(-1-\sqrt{1+|\xi|^2})t} + e^{(-1+\sqrt{1+|\xi|^2})t}}{2}.$$

Note that

$$\int_0^\infty e^{-pt} \big[\mathcal{A}(\xi, t) + \mathcal{B}(\xi, t)\big] e^{-|\xi|^2 t}\, dt = \frac{\lambda^2}{(\lambda^2 + 1 - \sqrt{1+|\xi|^2})(\lambda^2 + 1 + \sqrt{1+|\xi|^2})} = \frac{\lambda^2}{\det},$$

and

$$-\int_0^\infty e^{-pt}\mathcal{A}(\xi,t)e^{-|\xi|^2t}\,\mathrm{d}t = \frac{1}{(\lambda^2+1-\sqrt{1+|\xi|^2}\,)(\lambda^2+1+\sqrt{1+|\xi|^2}\,)} = \frac{1}{\det},$$

we obtain the following proposition.

Proposition 3.1. *There exists a unique solution* $(\widehat{u_A}, \widehat{\omega_\Omega})$ *of Eq.* (3.1) *which is given by*

$$\begin{pmatrix} \widehat{u_A} \\ \widehat{\omega_\Omega} \end{pmatrix} = e^{-|\xi|^2t}\big(\widehat{\mathcal{G}}_1(\xi,t)+\widehat{\mathcal{G}}_2(\xi,t)\big)\begin{pmatrix} \widehat{u_{0,A}} \\ \widehat{\omega_{0,\Omega}} \end{pmatrix} \tag{3.4}$$

with

$$\widehat{\mathcal{G}}_1(\xi,t) \overset{def}{=} \mathcal{A}(\xi,t)\mathcal{R}(\xi), \qquad \widehat{\mathcal{G}}_2(\xi,t) \overset{def}{=} \mathcal{B}(\xi,t)I,$$

where

$$\mathcal{R}(\xi) = \begin{pmatrix} -1 & -|\xi| \\ -|\xi| & 1 \end{pmatrix}.$$

Next we will derive the pointwise estimates for $\widehat{\mathcal{G}}_1(\xi,t)$, $\widehat{\mathcal{G}}_2(\xi,t)$ and their derivatives.

Lemma 3.2. *For multi-indices* α, *there exists a positive constant* C *independent of* ξ, t *such that*

$$\begin{aligned} &|\xi|^{|\alpha|}\big|D_\xi^\alpha\widehat{\mathcal{G}}_1(\xi,t)\big|, |\xi|^{|\alpha|}\big|D_\xi^\alpha\widehat{\mathcal{G}}_2(\xi,t)\big| \\ &\quad\leqslant C\big(1+e^{\frac{|\xi|^2}{2}t}\big)\big(\big(|\xi|^2t\big)^{|\alpha|}+\big(|\xi|^2t\big)^{|\alpha|-1}+\cdots+|\xi|^2t+1\big). \end{aligned} \tag{3.5}$$

Proof. The mean value theorem tells us that there exists a constant $\theta\in[0,1]$ such that

$$\sqrt{1+|\xi|^2}-1 = \frac{1}{2}|\xi|^2\big(1+|\xi|^2\theta\big)^{-\frac{1}{2}},$$

which implies that

$$e^{(-1+\sqrt{1+|\xi|^2}\,)t} \leqslant e^{\frac{|\xi|^2}{2}t}. \tag{3.6}$$

Using the Leibnitz's formula yields that

$$\begin{aligned} D_\xi^\alpha\widehat{\mathcal{G}}_1(\xi,t) = &\sum_{|\alpha|=N,\ |\alpha_1|+|\alpha_2|=|\alpha|} D_\xi^{\alpha_1}\big(e^{(-1-\sqrt{1+|\xi|^2}\,)t}+e^{(-1+\sqrt{1+|\xi|^2}\,)t}\big) \\ &\times D_\xi^{\alpha_2}\left(\frac{1}{2\sqrt{1+|\xi|^2}}\begin{pmatrix} -1 & -|\xi| \\ -|\xi| & 1 \end{pmatrix}\right). \end{aligned} \tag{3.7}$$

For simplicity, we only show the case of $|\alpha|=1$ in details, the other cases ($|\alpha|>1$) can be done by the same argument. Noting that

$$1+|\xi| \leqslant 2\sqrt{1+|\xi|^2},$$

one gets

$$D_\xi\big(e^{(-1-\sqrt{1+|\xi|^2})t}\big)\frac{(1+|\xi|)}{\sqrt{1+|\xi|^2}} \leqslant Ce^{-|\xi|t}t|\xi|.$$

In addition, due to (3.6) and notations in Section 1, we obtain

$$D_\xi\big(e^{(-1+\sqrt{1+|\xi|^2})t}\big)\frac{(1+|\xi|)}{\sqrt{1+|\xi|^2}} \leqslant Ce^{\frac{|\xi|^2t}{2}}t|\xi|,$$

$$\big(e^{(-1-\sqrt{1+|\xi|^2})t}+e^{(-1+\sqrt{1+|\xi|^2})t}\big)D_\xi\left(\frac{1}{\sqrt{1+|\xi|^2}}\right)\big(1+|\xi|\big) \leqslant C\big(e^{-|\xi|t}+e^{\frac{|\xi|^2t}{2}}\big)|\xi|^{-1},$$

and

$$\begin{aligned}&\big(e^{(-1-\sqrt{1+|\xi|^2})t}+e^{(-1+\sqrt{1+|\xi|^2})t}\big)\frac{1}{\sqrt{1+|\xi|^2}}\left|D_\xi\begin{pmatrix}-1 & -|\xi|\\ -|\xi| & 1\end{pmatrix}\right|\\ &\quad\leqslant C\big(e^{-|\xi|t}+e^{\frac{|\xi|^2t}{2}}\big)\big(\sqrt{1+|\xi|^2}\big)^{-1} \leqslant C\big(e^{-|\xi|t}+e^{\frac{|\xi|^2t}{2}}\big)|\xi|^{-1}.\end{aligned}$$

Combining the four above inequalities with (3.7) ($|\alpha|=1$), we have

$$\big|D_\xi\widehat{\mathcal{G}}_1(\xi,t)\big| \leqslant C\big(1+e^{\frac{|\xi|^2t}{2}}\big)\big(t|\xi|+|\xi|^{-1}\big).$$

Similarly, we can deduce that

$$\begin{aligned}\big|D_\xi^\alpha\widehat{\mathcal{G}}_1(\xi,t)\big| \leqslant{}& C\big(1+e^{\frac{|\xi|^2}{2}t}\big)\big(|\xi t|^{|\alpha|}+|\xi|^{|\alpha|-2}t^{|\alpha|-1}+|\xi|^{|\alpha|-4}t^{|\alpha|-2}+\cdots\\ &+|\xi|^{-|\alpha|+2}t+|\xi|^{-|\alpha|}\big),\end{aligned}$$

from which the estimate (3.5) holds. $\square$

Thanks to Proposition 3.1, we have

Proposition 3.3. *Let* $(u_A,\omega_\Omega)(t)=\mathcal{G}(x,t)(u_{0,A},\omega_{0,\Omega})$ *be the solution of* (1.13)*, then the Green matrix* $\widehat{\mathcal{G}}(\xi,t)$ *satisfies*

$$\widehat{\mathcal{G}}(\xi,t)=e^{-|\xi|^2t}\big(\widehat{\mathcal{G}}_1(\xi,t)+\widehat{\mathcal{G}}_2(\xi,t)\big),$$

where $\widehat{\mathcal{G}}_1(\xi,t)$ *and* $\widehat{\mathcal{G}}_2(\xi,t)$ *are defined in Proposition* 3.1.

Lemma 3.4. *For any multi-indices* α*, there exists a positive constant* C *independent of* ξ*,* t *such that*

$$\big|D_\xi^\alpha\widehat{\mathcal{G}}(\xi,t)\big| \leqslant Ce^{-\frac{1}{3}|\xi|^2t}|\xi|^{-|\alpha|}. \tag{3.8}$$

Proof. Noting that for $c > \tilde{c} > 0$, $k > 0$, we have

$$e^{-c|\xi|^2 t}\big(t|\xi|^2\big)^k \leqslant e^{-\tilde{c}|\xi|^2 t}.$$

Then using the Leibniz formula, the estimate

$$\big|\partial_\xi^\gamma \big(e^{-|\xi|^2 t}\big)\big| \leqslant C|\xi|^{-|\gamma|} e^{-\frac{11}{12}|\xi|^2 t},$$

and Lemma 3.2, the estimate (3.8) follows easily from the explicit expression of $\widehat{\mathcal{G}}(\xi, t)$. □

Using this lemma, we can obtain the following smoothing effect on Green's matrix $\mathcal{G}$, which will play an important role in this paper.

Proposition 3.5. *Let $\mathcal{C}$ be a ring centered at* 0 *in $\mathbb{R}^3$. There exist two positive constants c and C, for any real $p \in [1, \infty]$, any couple (t, λ) of positive real numbers such that if* $\operatorname{supp} \hat{u} \subset \lambda\mathcal{C}$, *then we have*

$$\big\|\mathcal{G}(x, t)u\big\|_{L^p} \leqslant C e^{-c\lambda^2 t} \|u\|_{L^p}. \tag{3.9}$$

Proof. We will adopt the spirit of the proof for heat operators as in [3]. For the completeness, here we will present the proof.

Let $\phi \in \mathcal{D}(\mathbb{R}^3 \setminus \{0\})$, which equals 1 near the ring $\mathcal{C}$. Set

$$g_\lambda(t, x) \stackrel{def}{=} (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \phi\big(\lambda^{-1}\xi\big) \widehat{\mathcal{G}}(\xi, t)\, \mathrm{d}\xi.$$

To prove (3.9), it suffices to show

$$\big\|g_\lambda(x, t)\big\|_{L^1} \leqslant C e^{-c\lambda^2 t}. \tag{3.10}$$

Thanks to (3.8) and the support property of ϕ, we infer that

$$\int_{|x|\leqslant\lambda^{-1}} \big|g_\lambda(x, t)\big|\, \mathrm{d}x \leqslant C \int_{|x|\leqslant\lambda^{-1}} \int_{\mathbb{R}^3} \big|\phi\big(\lambda^{-1}\xi\big)\big| \big|\widehat{\mathcal{G}}(\xi, t)\big|\, \mathrm{d}\xi\, \mathrm{d}x \leqslant C e^{-c\lambda^2 t}. \tag{3.11}$$

Set $L_x =^{def} \frac{x\cdot\nabla_\xi}{i|x|^2}$. Noting that $L_x(e^{ix\cdot\xi}) = e^{ix\cdot\xi}$, we get by integration by part that

$$\begin{aligned} g_\lambda(x, t) &= \int_{\mathbb{R}^3} L_x^4\big(e^{ix\cdot\xi}\big)\phi\big(\lambda^{-1}\xi\big)\widehat{\mathcal{G}}(\xi, t)\, \mathrm{d}\xi \\ &= (-1)^4 \int_{\mathbb{R}^3} e^{ix\cdot\xi} \big(L_x^*\big)^4 \big(\phi\big(\lambda^{-1}\xi\big)\widehat{\mathcal{G}}(\xi, t)\big)\, \mathrm{d}\xi. \end{aligned}$$

From the Leibniz formula and (3.8),

$$\big|\big(L_x^*\big)^4\big(\phi\big(\lambda^{-1}\xi\big)\widehat{\mathcal{G}}(\xi, t)\big)\big| \leqslant C|\lambda x|^{-4} \sum_{|\gamma|=4,\ |\beta|\leqslant|\gamma|} \lambda^{|\beta|} \big|\big(\nabla^{\gamma-\beta}\phi\big)\big(\lambda^{-1}\xi\big)\big| e^{-\frac{1}{3}|\xi|^2 t} |\xi|^{-|\beta|}.$$

Then we obtain, for any ξ with $|\xi| \sim \lambda$,

$$\left|\left(L_x^*\right)^4\left(\phi\left(\lambda^{-1}\xi\right)\widehat{\mathcal{G}}(\xi,t)\right)\right| \leqslant C|\lambda x|^{-4} e^{-\frac{1}{3}|\xi|^2 t},$$

which implies that

$$\int_{|x|\geqslant\frac{1}{\lambda}} \left|g_\lambda(x,t)\right| \mathrm{d}x \leqslant Ce^{-c\lambda^2 t}\lambda^3 \int_{|x|\geqslant\frac{1}{\lambda}} |\lambda x|^{-4} \mathrm{d}x \leqslant Ce^{-c\lambda^2 t}.$$

This together with (3.11) gives (3.10). □

Proposition 3.6. *Let $\mathcal{C}$ be a ring centered at* 0 *in* $\mathbb{R}^3$, $G(x,t)$ *is the Green matrix of the system* (1.6)*, defined by* (1.7)*. Then there exist two positive constants c and C such that for any couple* (t,λ) *of positive real numbers, if* $\operatorname{supp}\hat{u} \subset \lambda\mathcal{C}$*, then*

$$\left\|G(x,t)u\right\|_{L^2} \leqslant Ce^{-c\lambda^2 t}\|u\|_{L^2}. \tag{3.12}$$

Proof. Thanks to Plancherel theorem and (1.8), we get

$$\left\|G(x,t)u\right\|_{L^2} = \left\|\widehat{G}(\xi,t)\hat{u}(\xi)\right\|_{L^2} \leqslant C\left\|e^{-c|\xi|^2 t}\hat{u}(\xi)\right\|_2 \leqslant Ce^{-c\lambda^2 t}\|u\|_2,$$

where we have used the support property of $\hat{u}(\xi)$. □

4. Proof of Theorem 1.2

4.1. A priori estimate

In this section, we will derive a priori estimate for the system (1.5).

Proposition 4.1. *Let* $1 \leqslant p < 6$, $T > 0$. *Assume that* (u,ω) *is a smooth solution of the system* (1.5) *on* $[0,T]$, *then we have*

$$\left\|(u,\omega)\right\|_{E_T^p} \leqslant C\left(\left\|(u_0,\omega_0)\right\|_{E_0^p} + \left\|(u,\omega)\right\|_{E_T^p}^2\right), \tag{4.1}$$

where

$$E_T^p = L_T^\infty\left(\dot{B}_{p,\infty}^{\frac{3}{p}-1}\right) \cap \widetilde{L}_T^1\left(\dot{B}_{p,\infty}^{\frac{3}{p}-1}\right), \qquad E_0^p = \dot{B}_{p,\infty}^{\frac{3}{p}-1}.$$

Proof. Let us consider the following frequency localized system:

$$\begin{cases} \partial_t \Delta_j u_A - \Delta\Delta_j u_A - \Lambda\Delta_j\omega_\Omega = \Delta_j F, \\ \partial_t \Delta_j\omega_\Omega - \Delta\Delta_j\omega_\Omega + 2\Delta_j\omega_\Omega - \Lambda\Delta_j u_A = \Delta_j H, \\ (\Delta_j u_A, \Delta_j\omega_\Omega)|_{t=0} = (\Delta_j u_{0,A}, \Delta_j\omega_{0,\Omega}), \end{cases} \tag{4.2}$$

with

$$F = -\left(\mathbf{P}(u\cdot\nabla u)\right)_A, \qquad H = -\Lambda^{-1}\operatorname{curl}(u\cdot\nabla\omega) \quad \text{and} \quad \operatorname{div} u = 0.$$

In terms of the Green matrix $\mathcal{G}$, the solution of (4.2) can be expressed as

$$\begin{pmatrix} \Delta_j u_A(t) \\ \Delta_j \omega_\Omega(t) \end{pmatrix} = \mathcal{G}(x,t) \begin{pmatrix} \Delta_j u_{0,A} \\ \Delta_j \omega_{0,\Omega} \end{pmatrix} + \int_0^t \mathcal{G}(x,t-\tau) \begin{pmatrix} \Delta_j F(\tau) \\ \Delta_j H(\tau) \end{pmatrix} \mathrm{d}\tau. \tag{4.3}$$

Applying Proposition 3.5 to the above equation to get

$$\begin{aligned} \|\Delta_j u_A\|_{L^p} + \|\Delta_j \omega_\Omega\|_{L^p} &\leqslant C e^{-c2^{2j}t} \left(\left\|\Delta_j u_A^0\right\|_{L^p} + \left\|\Delta_j \omega_\Omega^0\right\|_{L^p} \right) \\ &\quad + C \int_0^t e^{-c2^{2j}(t-\tau)} \left(\left\|\Delta_j F(\tau)\right\|_{L^p} + \left\|\Delta_j H(\tau)\right\|_{L^p} \right) \mathrm{d}\tau. \end{aligned} \tag{4.4}$$

Taking L^r norm with respect to t gives

$$\begin{aligned} &\|\Delta_j u_A\|_{L^r_T L^p} + \|\Delta_j \omega_\Omega\|_{L^r_T L^p} \\ &\quad \leqslant C 2^{-\frac{2j}{r}} \left(\|\Delta_j u_{0,A}\|_{L^p} + \|\Delta_j \omega_{0,\Omega}\|_{L^p} + \|\Delta_j F\|_{L^1_T L^p} + \|\Delta_j H\|_{L^1_T L^p} \right). \end{aligned}$$

Multiplying $2^{j(\frac{3}{p}-1+\frac{2}{r})}$ on both sides, then taking supremum over $j \in \mathbb{Z}$, we derive

$$\begin{aligned} &\|u_A\|_{\widetilde{L}^r_T \dot{B}^{\frac{3}{p}-1+\frac{2}{r}}_{p,\infty}} + \|\omega_\Omega\|_{\widetilde{L}^r_T \dot{B}^{\frac{3}{p}-1+\frac{2}{r}}_{p,\infty}} \\ &\quad \leqslant C \left(\|u_{0,A}\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|\omega_{0,\Omega}\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|F\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|H\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \right). \end{aligned}$$

According to the boundness of Riesz transform on the homogeneous Besov space and Lemma 2.7, we have

$$\begin{aligned} \left\| \left(\mathbf{P}(u \cdot \nabla u) \right)_A \right\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} &\leqslant C \|u \cdot \nabla u\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C \|u\|_{\widetilde{L}^4_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \|\nabla u\|_{\widetilde{L}^{\frac{4}{3}}_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}}, \\ \left\| \Lambda^{-1} \operatorname{curl}(u \cdot \nabla \omega) \right\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} &\leqslant C \|u \cdot \nabla \omega\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C \|u\|_{\widetilde{L}^4_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \|\nabla \omega\|_{\widetilde{L}^{\frac{4}{3}}_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}}. \end{aligned} \tag{4.5}$$

From Proposition 2.8, we infer that

$$\begin{aligned} \|\omega_d\|_{\widetilde{L}^r_T \dot{B}^{\frac{3}{p}-1+\frac{2}{r}}_{p,\infty}} &\leqslant C \left(\|\omega_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \left\| \Lambda^{-1} \operatorname{div}(u \cdot \nabla \omega) \right\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \right) \\ &\leqslant C \left(\|\omega_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|u\|_{\widetilde{L}^4_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \|\nabla \omega\|_{\widetilde{L}^{\frac{4}{3}}_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \right). \end{aligned} \tag{4.6}$$

Thanks to the interpolation

$$\begin{aligned} \left(\widetilde{L}^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}, \widetilde{L}^1_T \dot{B}^{\frac{3}{p}+1}_{p,\infty} \right)_{\frac{3}{4}} &= \widetilde{L}^4_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}, \\ \left(\widetilde{L}^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}, \widetilde{L}^1_T \dot{B}^{\frac{3}{p}+1}_{p,\infty} \right)_{\frac{1}{4}} &= \widetilde{L}^{\frac{4}{3}}_T \dot{B}^{\frac{3}{p}+\frac{1}{2}}_{p,\infty}, \end{aligned} \tag{4.7}$$

which together with (4.5), (4.6) and Lemma 2.3(ii) imply

$$\begin{aligned}&\left\|(u_A,\omega_\Omega,\omega_d)\right\|_{\widetilde{L}^r_T\dot{B}^{\frac{3}{p}-1+\frac{2}{r}}_{p,\infty}}\\&\quad\leqslant C\big(\|u\|_{\widetilde{L}^\infty_T\dot{B}^{\frac{3}{p}-1}_{p,\infty}}+\|u\|_{\widetilde{L}^1_T\dot{B}^{\frac{3}{p}+1}_{p,\infty}}\big)\big(\left\|(u,\omega)\right\|_{\widetilde{L}^\infty_T\dot{B}^{\frac{3}{p}-1}_{p,\infty}}+\left\|(u,\omega)\right\|_{\widetilde{L}^1_T\dot{B}^{\frac{3}{p}+1}_{p,\infty}}\big).\end{aligned}\tag{4.8}$$

On the other hand, noting that $\omega=\Lambda^{-1}\nabla\omega_d-\Lambda^{-1}\operatorname{div}\omega_\Omega$ and

$$\|u\|_{\widetilde{L}^r_T\dot{B}^{\frac{3}{p}-1+\frac{2}{r}}_{p,\infty}}=\sum_{i=1}^{3}\|u_i\|_{\widetilde{L}^r_T\dot{B}^{\frac{3}{p}-1+\frac{2}{r}}_{p,\infty}}\leqslant\|u_A\|_{\widetilde{L}^r_T\dot{B}^{\frac{3}{p}-1+\frac{2}{r}}_{p,\infty}},$$

taking $r=\infty$ and $r=1$ in (4.8), then adding up the resulting equations, we have

$$\left\|(u,\omega)\right\|_{E^p_T}\leqslant C\big(\left\|(u_0,\omega_0)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}}+\left\|(u,\omega)\right\|^2_{E^p_T}\big).$$

The proof is completed. □

4.2. Approximate solutions and uniform estimates

The construction of approximate solutions is based on the following local existence theorem.

Theorem 4.2. *(See [19].) Let $s>3/2$. Assume that $(u_0,\omega_0)\in H^s(\mathbb{R}^3)$ with $\operatorname{div}u_0=0$, then there is a positive time $T(\|(u_0,\omega_0)\|_{H^s})$ such that a unique solution $(u,\omega)\in C([0,T);H^s)\cap C^1((0,T);H^s)\cap C((0,T);H^{s+2})$ of system (1.1) exists.*

Moreover, if there exists an absolute constant $M>0$ such that if

$$\lim_{\varepsilon\to 0}\sup_{j\in\mathbb{Z}}\int_{T-\varepsilon}^{T}\left\|\Delta_j(\nabla\times u)\right\|_\infty\,\mathrm{d}t=\delta<M$$

then $\delta=0$, and the solution (u,ω) can be extended past time $t=T$.

Let us consider a sequence $(\phi_n)_{n\in\mathbb{N}}\in\mathcal{S}$ such that ϕ_n is uniformly bounded with respect to n and such that $\phi_n\equiv 1$ in a neighborhood of the ball $B(0,n)$. Then for the initial data u_0, ω_0, we define an approximate sequence $u_{0,n}=\phi_n(S_nu_0)$, and $\omega_{0,n}=\phi_n(S_n\omega_0)\in H^s$ such that

$$\lim_{n\to\infty}\left\|\phi_n(S_nu_0)-u_0\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}}=0,\qquad\lim_{n\to\infty}\left\|\phi_n(S_n\omega_0)-\omega_0\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}}=0.\tag{4.9}$$

Then Theorem 4.2 ensures that there exists a maximal existence time $T_n>0$ such that the system (1.5) with the initial data $(u_{0,n},\omega_{0,n})$ has a unique solution (u^n,ω^n) satisfying

$$\big(u^n,\omega^n\big)\in C\big([0,T_n);H^s\big)\cap C^1\big((0,T_n);H^s\big)\cap C\big((0,T_n);H^{s+2}\big).$$

On the other hand, using the definition of the Besov space and Lemma 2.4, it is easy to check that

$$\big(u^n,\omega^n\big)\in C\big([0,T_n);\dot{B}^{\frac{3}{p}-1}_{p,\infty}\big)\cap L^1\big(0,T_n;\dot{B}^{\frac{3}{p}+1}_{p,\infty}\big).$$

From (4.9) and (1.15) we find that

$$\left\|(u_{0,n},\omega_{0,n})\right\|_{\dot B^{\frac{3}{p}-1}_{p,\infty}}\leqslant C_0\eta,$$

for some constant C_0. Given a constant M to be chosen later on, let us define

$$T_n^*\overset{def}{=}\sup\left\{t\in[0,T_n);\ \left\|\left(u^n,\omega^n\right)\right\|_{L^\infty_t\dot B^{\frac{3}{p}-1}_{p,\infty}\cap\widetilde{L}^1_t\dot B^{\frac{3}{p}+1}_{p,\infty}}\leqslant M\eta\right\}.$$

Firstly, we claim that

$$T_n^*=T_n,\quad \forall n\in\mathbb{N}.$$

Using the continuity argument, it suffices to show that for all $n\in\mathbb{N}$,

$$\left\|\left(u^n,\omega^n\right)\right\|_{\widetilde{L}^\infty_{T_n^*}\dot B^{\frac{3}{p}-1}_{p,\infty}\cap\widetilde{L}^1_{T_n^*}\dot B^{\frac{3}{p}+1}_{p,\infty}}\leqslant\frac{3}{4}M\eta. \tag{4.10}$$

In fact, we apply Proposition 4.1 to obtain

$$\left\|\left(u^n,\omega^n\right)\right\|_{E^p_{T_n^*}}\leqslant C\left(C_0\eta+(M\eta)^2\right). \tag{4.11}$$

If we set $M=4CC_0$, and choose η small enough such that

$$8C^2C_0\eta\leqslant 1,$$

then the inequality (4.10) follows from (4.11). In conclusion, we construct a sequence of approximate solution (u^n,ω^n) of (1.5) on $[0,T_n)$ satisfying

$$\left\|\left(u^n,\omega^n\right)\right\|_{E^p_{T_n}}\leqslant M\eta, \tag{4.12}$$

for any $n\in\mathbb{N}$. Next, we claim that

$$T_n=+\infty,\quad \forall n\in\mathbb{N}.$$

According to Theorem 4.2, it remains to prove $\nabla\times u^n\in\widetilde{L}^1_{T_n}\dot B^0_{\infty,\infty}$. From (4.12) we know that

$$\left\|\nabla\times u^n\right\|_{\widetilde{L}^1_{T_n}\dot B^{\frac{3}{p}}_{p,\infty}}\leqslant\left\|\nabla u^n\right\|_{\widetilde{L}^1_{T_n}\dot B^{\frac{3}{p}}_{p,\infty}}\leqslant M\eta,$$

this combined with the embedding $\widetilde{L}^1_{T_n}\dot B^{\frac{3}{p}}_{p,\infty}\hookrightarrow\widetilde{L}^1_{T_n}\dot B^0_{\infty,\infty}$ implies that $\nabla\times u^n\in\widetilde{L}^1_{T_n}\dot B^0_{\infty,\infty}$, thus the continuation criterion in Theorem 4.2 has been verified.

4.3. Existence

We will use the compactness argument to prove the existence of the solution. Due to (4.12), it is easy to see that

- u^n, ω^n is uniformly bounded in $\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{3}{p}-1}_{p,\infty}) \cap \widetilde{L}^1(0,\infty;\dot{B}^{\frac{3}{p}+1}_{p,\infty})$.

Let u^n_L, ω^n_L be a solution of

$$\begin{cases} \partial_t u^n_L - \Delta u^n_L = 0, & u^n_L(0) = v_{0,n}, \\ \partial_t \omega^n_L - \Delta \omega^n_L + 2\omega^n_L = 0, & \omega^n_L(0) = \omega_{0,n}. \end{cases}$$

It is easy to verify that u^n_L, ω^n_L tends to the solution of

$$\begin{cases} \partial_t u_L - \Delta u_L = 0, & u_L(0) = u_0, \\ \partial_t \omega_L - \Delta \omega_L + 2\omega_L = 0, & \omega_L(0) = \omega_0 \end{cases} \tag{4.13}$$

in $\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{3}{p}-1}_{p,\infty}) \cap \widetilde{L}^1(0,\infty;\dot{B}^{\frac{3}{p}+1}_{p,\infty})$.

We set $\widetilde{u}^n \stackrel{def}{=} u^n - u^n_L$ and $\widetilde{\omega}^n \stackrel{def}{=} \omega^n - \omega^n_L$. Firstly, we claim that $(\widetilde{u}^n, \widetilde{\omega}^n)$ is uniformly bounded in $C^{\frac{1}{2}}_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-2}_{p,\infty}) \times C^{\frac{1}{2}}_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-1}_{p,\infty} + \dot{B}^{\frac{3}{p}-2}_{p,\infty})$. In fact, let us recall that

$$\partial_t \widetilde{u}^n = \Delta \widetilde{u}^n - \mathbf{P}\big(u^n \cdot \nabla u^n\big) - \nabla \times \omega^n.$$

Thanks to Lemma 2.7, we have

$$\big\|\mathbf{P}\big(u^n u^n\big)\big\|_{L^2 \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C \big\|u^n\big\|_{L^4 \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \big\|u^n\big\|_{L^4 \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}},$$

combined with $\Delta \widetilde{u}^n \in \widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-2}_{p,\infty})$ and $\nabla \times \omega^n \in L^\infty(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-2}_{p,\infty})$ implies $\partial_t \widetilde{u}^n \in \widetilde{L}^2_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-2}_{p,\infty})$, thus $\widetilde{u}^n$ is uniformly bounded in $C^{\frac{1}{2}}_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-2}_{p,\infty})$. On the other hand, since

$$\partial_t \widetilde{\omega}^n = \Delta \widetilde{\omega}^n - 2\widetilde{\omega}^n - u^n \cdot \nabla \omega^n - \nabla \times u^n,$$

by the same argument as used in the proof of $\partial_t \widetilde{u}^n$, we get $\partial_t \widetilde{\omega}^n \in \widetilde{L}^2_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-1}_{p,\infty} + \dot{B}^{\frac{3}{p}-2}_{p,\infty})$, which implies $\widetilde{u}^n$ is uniformly bounded in $C^{\frac{1}{2}}_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-1}_{p,\infty} + \dot{B}^{\frac{3}{p}-2}_{p,\infty})$.

Let $\{\chi_j\}_{j\in\mathbb{N}}$ be a sequence of smooth functions supported in the ball $B(0, j+1)$ and equal to 1 on $B(0, j)$. The claim ensures that for any $j \in \mathbb{N}$, $\{\chi_j \widetilde{u}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-2}_{p,\infty})$, and $\{\chi_j \widetilde{\omega}^n\}_{k\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}_{loc}(\mathbb{R}^+;\dot{B}^{\frac{3}{p}-1}_{p,\infty} + \dot{B}^{\frac{3}{p}-2}_{p,\infty})$. Observe that for any $\chi \in C^\infty_0(\mathbb{R}^+ \times \mathbb{R}^3)$, for $\varepsilon \in (0,1)$, the map: $(\widetilde{u}^n, \widetilde{\omega}^n) \mapsto (\chi \widetilde{u}^n, \chi \widetilde{\omega}^n)$ is compact from

$$\big(\dot{B}^{\frac{3}{p}-2}_{p,\infty} \cap \dot{B}^{\frac{3}{p}-1-\varepsilon}_{p,\infty}\big) \times \big(\big(\dot{B}^{\frac{3}{p}-1}_{p,\infty} + \dot{B}^{\frac{3}{p}-1-\varepsilon}_{p,\infty}\big)\big) \quad \text{into } \dot{B}^{\frac{3}{p}-2}_{p,\infty} \times \big(\dot{B}^{\frac{3}{p}-1}_{p,\infty} + \dot{B}^{\frac{3}{p}-2}_{p,\infty}\big),$$

see [3]. By applying Ascoli's theorem and Cantor's diagonal process, there exists some distribution $(\widetilde{u}, \widetilde{\omega}) \in L^\infty \dot{B}^{\frac{3}{p}-1}_{p,\infty} \cap L^1 \dot{B}^{\frac{3}{p}+1}_{p,\infty}$ such that for any $j \in \mathbb{N}$,

$$\begin{aligned} &\chi_j \widetilde{u}^n \to \chi_j \widetilde{u} \quad \text{in } C_{loc}\big(\mathbb{R}^+; \dot{B}^{\frac{3}{p}-2}_{p,\infty}\big), \\ &\chi_j \widetilde{\omega}^n \to \chi_j \widetilde{\omega} \quad \text{in } C_{loc}\big(\mathbb{R}^+; \dot{B}^{\frac{3}{p}-1}_{p,\infty} + \dot{B}^{\frac{3}{p}-2}_{p,\infty}\big). \end{aligned} \tag{4.14}$$

With (4.14), it is a routine process to verify that $(\widetilde{u} + u_L, \widetilde{\omega} + \omega_L)$ satisfies the system (1.5) in the sense of distribution.

Here we show as an example the case of the term $u^n \cdot \nabla u^n$. Let $\psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ and $j \in \mathbb{N}$ such that $\operatorname{supp} \psi \subset [0, j] \times B(0, j)$. We write

$$u^n \cdot \nabla u^n - u \cdot \nabla u = \big(u^n - u\big) \cdot \nabla u^n + u \cdot \nabla\big(u^n - u\big).$$

We will only give the estimate of the first term with help of Bony's decomposition, and the similar argument can be applied to the term $u \cdot \nabla(u^n - u)$. Thanks to $\operatorname{div} u = 0$ and Lemma 2.6,

$$\begin{aligned} &\big\|T_{u^n - u} u^n\big\|_{L^\infty \dot{B}^{\frac{3}{p}-3}_{p,\infty}} + \big\|T_{u^n}\big(u^n - u\big)\big\|_{L^\infty \dot{B}^{\frac{3}{p}-3}_{p,\infty}} \\ &\quad \leqslant C\big\|u^n - u\big\|_{L^\infty \dot{B}^{-2}_{\infty,\infty}} \big\|u^n\big\|_{L^\infty \dot{B}^{\frac{3}{p}-1}_{p,\infty}} + C\big\|u^n\big\|_{L^\infty \dot{B}^{-1}_{\infty,\infty}} \big\|u^n - u\big\|_{L^\infty \dot{B}^{\frac{3}{p}-2}_{p,\infty}} \\ &\quad \leqslant C\big\|u^n - u\big\|_{L^\infty \dot{B}^{\frac{3}{p}-2}_{p,\infty}} \big\|u^n\big\|_{L^\infty \dot{B}^{\frac{3}{p}-1}_{p,\infty}}, \end{aligned}$$

where in the last inequality we have used the embedding $\dot{B}^{s_1}_{p,\infty} \subseteq \dot{B}^{s_2}_{\infty,\infty}$ for $s_1 - \frac{3}{p} = s_2$. And

$$\big\|R\big(u^n - u, u^n\big)\big\|_{L^1 \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C\big\|u^n - u\big\|_{L^\infty \dot{B}^{\frac{3}{p}-2}_{p,\infty}} \big\|u^n\big\|_{L^1 \dot{B}^{\frac{3}{p}+1}_{p,\infty}}.$$

The other nonlinear terms can be treated in the same way.

4.4. Uniqueness

In this subsection, we prove the uniqueness of the solution. Assume that $(u^1, \omega^1) \in E_T^p$ and $(u^2, \omega^2) \in E_T^p$ are two solutions of the system (1.1) with the same initial data. Then we have $(\delta u, \delta\omega) = (u^1 - u^2, \omega^1 - \omega^2)$ satisfies

$$\begin{cases} \partial_t \delta u - \Delta \delta u = \delta F, \\ \partial_t \delta\omega - \Delta \delta\omega - \nabla \operatorname{div} \delta\omega + 2\delta\omega = \delta H, \\ (\delta u, \delta\omega)|_{t=0} = (0, 0), \end{cases} \tag{4.15}$$

where

$$\begin{aligned} \delta F &= \nabla \times \delta\omega - \mathbf{P}\big(\delta u \cdot \nabla u^1\big) - \mathbf{P}\big(u^2 \cdot \nabla \delta u\big), \\ \delta H &= \nabla \times \delta u - \delta u \cdot \nabla \omega^1 - u^2 \cdot \nabla \delta\omega. \end{aligned}$$

Let us apply the operator $\Lambda^{-1} \operatorname{div}$ and $\Lambda^{-1} \operatorname{curl}$ (for definition see (1.11)) to the second equation of (4.15) respectively, then we have

$$\begin{cases} \partial_t \delta\omega_d - 2\Delta \delta\omega_d + 2\delta\omega_d = \Lambda^{-1} \operatorname{div} \delta H, \\ \partial_t \delta\omega_\Omega - \Delta \delta\omega_\Omega + 2\delta\omega_\Omega = \Lambda^{-1} \operatorname{curl} \delta H, \\ \delta\omega = \Lambda^{-1} \nabla \delta\omega_d - \Lambda^{-1} \operatorname{div} \delta\omega_\Omega, \end{cases}$$

here $\delta\omega_d = \Lambda^{-1}\operatorname{div}\delta\omega$ and $\delta\omega_\Omega = \Lambda^{-1}\operatorname{curl}\delta\omega$. Applying Proposition 2.9 to the two equations of the above system respectively, one obtains

$$\|\delta\omega_d\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}}_{p,\infty}\cap\widetilde{L}^2_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|\delta\omega_\Omega\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}}_{p,\infty}\cap\widetilde{L}^2_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C\big\|\Lambda^{-1}\operatorname{div}\delta H\big\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}-2}_{p,\infty}} + C\big\|\Lambda^{-1}\operatorname{curl}\delta H\big\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}-2}_{p,\infty}}.$$

Again using Proposition 2.9 to the first equations of (4.15), and combining the above inequality we get

$$\big\|\big(\delta u(t),\delta\omega(t)\big)\big\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}}_{p,\infty}} + \big\|\big(\delta u(t),\delta\omega(t)\big)\big\|_{\widetilde{L}^2_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C\big\|\big(\delta F(\tau),\delta H(\tau)\big)\big\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}-2}_{p,\infty}}. \tag{4.16}$$

From Lemma 2.7 and $\operatorname{div} u = 0$, we infer that

$$\|\delta F\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}-2}_{p,\infty}} + \|\delta H\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}-2}_{p,\infty}} \leqslant C\|\delta u\|_{\widetilde{L}^{\frac{4}{3}}_t\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}}\big\|\big(\omega^1,u^1,u^2\big)\big\|_{\widetilde{L}^4_t\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} + C\big\|(\delta\omega,\delta u)\big\|_{\widetilde{L}^2_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}}t^{\frac{1}{2}}.$$

Then we have

$$\begin{aligned}
&\big\|\big(\delta u(t),\delta\omega(t)\big)\big\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}}_{p,\infty}} + \big\|\big(\delta u(t),\delta\omega(t)\big)\big\|_{\widetilde{L}^2_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}}\\
&\quad\leqslant C\big(\|\delta u\|_{\widetilde{L}^1_t\dot{B}^{\frac{3}{p}}_{p,\infty}} + \|\delta u\|_{\widetilde{L}^2_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}}\big)\big\|\big(\omega^1,u^1,u^2\big)\big\|^{\frac{1}{2}}_{\widetilde{L}^2_t\dot{B}^{\frac{3}{p}}_{p,\infty}}\big\|\big(\omega^1,u^1,u^2\big)\big\|^{\frac{1}{2}}_{\widetilde{L}^\infty_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}}\\
&\qquad + Ct^{\frac{1}{2}}\big\|(\delta\omega,\delta u)\big\|_{\widetilde{L}^2_t\dot{B}^{\frac{3}{p}-1}_{p,\infty}}.
\end{aligned} \tag{4.17}$$

If t is taken small enough such that $\|(\omega^1,u^1,u^2)\|_{\widetilde{L}^2_t\dot{B}^{\frac{3}{p}}_{p,\infty}}$ and $t^{\frac{1}{2}}$ sufficiently small, then we conclude that $(\delta u,\delta\omega) = 0$ on $[0,T]$, and a continuity argument ensures that $(u^1,\omega^1) = (u^2,\omega^2)$ on $[0,\infty)$.

5. The proof of Theorem 1.4

To prove Theorem 1.4, we will use the Green matrix of the linear system (1.6). Let us return to (1.5). Due to $\operatorname{div} u = 0$, we have

$$\begin{aligned}
\begin{pmatrix} u\\ \omega\end{pmatrix} &= G(x,t)\begin{pmatrix} u_0\\ \omega_0\end{pmatrix} - \int_0^t G(x,t-\tau)\nabla\cdot\begin{pmatrix}\mathbf{P}(uu)\\ u\omega\end{pmatrix}\mathrm{d}\tau\\
&= \begin{pmatrix} G_{ij}(t)u_0^j\\ G_{(i+3)j}(t)\omega_0^j\end{pmatrix} - \int_0^t\begin{pmatrix} G_{ij}\partial_k\mathbf{P}(u_ku_j) + G_{i(j+3)}\partial_k(u_k\omega_j)\\ G_{(i+3)j}\partial_k\mathbf{P}(u_ku_j) + G_{(i+3)(j+3)}\partial_k(u_k\omega_j)\end{pmatrix}\mathrm{d}\tau\\
&\overset{def}{=} G(t)(u_0,\omega_0) + \begin{pmatrix} B(u,\omega)\\ \widetilde{B}(u,\omega)\end{pmatrix},\quad i = 1,2,3,
\end{aligned} \tag{5.1}$$

here $G_{ij}(x,t)$ is the element of the Green matrix $G(x,t)$, and the summation convention over repeated indices $1\leqslant j,k\leqslant 3$ is used.

In view of the relationship: $\dot{H}^{\frac{1}{2}} \approx \dot{B}^{\frac{1}{2}}_{2,2}$, we have

$$\begin{aligned}
\|B(u,\omega)\|_{\widetilde{L}^\infty_T \dot{H}^{\frac{1}{2}}} &\leqslant \left\| \int_0^t G(t-\tau)\nabla\cdot\big(\mathbf{P}(uu)+u\omega\big)(\tau)\,\mathrm{d}\tau \right\|_{\widetilde{L}^\infty_T \dot{H}^{\frac{1}{2}}} \\
&\leqslant C\left(\sum_{j\in\mathbb{Z}} 2^j \left(\sup_{t\in[0,T)} \int_0^t \big\|G(t-\tau)\nabla\cdot\Delta_j(uu+u\omega)(\tau)\big\|_{L^2}\,\mathrm{d}\tau \right)^2 \right)^{\frac{1}{2}} \\
&\leqslant C\left\| 2^{\frac{3}{2}j} \sup_{t\in[0,T)} \int_0^t e^{-c2^{2j}t} \big\|\Delta_j(uu+u\omega)(\tau)\big\|_{L^2}\,\mathrm{d}\tau \right\|_{\ell^2} \\
&\leqslant C\big(\|T_u u\|_{\widetilde{L}^4_T \dot{B}^0_{2,2}} + \|T_u\omega + T_\omega u\|_{\widetilde{L}^4_T \dot{B}^0_{2,2}} + \big\|R(u,u)+R(u,\omega)\big\|_{\widetilde{L}^{\frac{4}{3}}_T \dot{B}^1_{2,2}} \big),
\end{aligned}$$

where in the third inequality we have used Lemma 2.4 and Proposition 3.6, in the last inequality we have used Bony's decomposition. From Lemma 2.6, we have

$$\begin{aligned}
\|T_u\omega\|_{\widetilde{L}^4_T \dot{B}^0_{2,2}} &\leqslant C\|u\|_{\widetilde{L}^4_T \dot{B}^{-\frac{1}{2}}_{\infty,\infty}} \|\omega\|_{\widetilde{L}^\infty_T \dot{B}^{\frac{1}{2}}_{2,2}} \leqslant C\|u\|_{\widetilde{L}^4_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \|\omega\|_{\widetilde{L}^\infty_T \dot{B}^{\frac{1}{2}}_{2,2}}, \\
\|T_\omega u\|_{\widetilde{L}^4_T \dot{B}^0_{2,2}} &\leqslant C\|\omega\|_{\widetilde{L}^4_T \dot{B}^{-\frac{1}{2}}_{\infty,\infty}} \|u\|_{\widetilde{L}^\infty_T \dot{B}^{\frac{1}{2}}_{2,2}} \leqslant C\|\omega\|_{\widetilde{L}^4_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \|u\|_{\widetilde{L}^\infty_T \dot{B}^{\frac{1}{2}}_{2,2}},
\end{aligned}$$

and

$$\big\|R(u,\omega)\big\|_{\widetilde{L}^{\frac{4}{3}}_T \dot{B}^1_{2,2}} \leqslant C\|u\|_{\widetilde{L}^{\frac{4}{3}}_T \dot{B}^{\frac{1}{2}}_{\infty,\infty}} \|\omega\|_{\widetilde{L}^\infty_T \dot{B}^{\frac{1}{2}}_{2,2}} \leqslant C\|u\|_{\widetilde{L}^{\frac{4}{3}}_T \dot{B}^{\frac{3}{p}+\frac{1}{2}}_{p,\infty}} \|\omega\|_{\widetilde{L}^\infty_T \dot{B}^{\frac{1}{2}}_{2,2}}.$$

The terms $T_u u$ and $R(u,u)$ can be treated in the same way as $T_u\omega$, $R(u,\omega)$, respectively. Combining the above inequalities, we obtain

$$\big\|B(u,\omega)\big\|_{\widetilde{L}^\infty_T \dot{H}^{\frac{1}{2}}} \leqslant C\big\|(u,\omega)\big\|_{E^p_T} \big\|(u,\omega)\big\|_{\widetilde{L}^\infty_T \dot{H}^{\frac{1}{2}}}. \tag{5.2}$$

Similarly, we have

$$\big\|\widetilde{B}(u,\omega)\big\|_{\widetilde{L}^\infty_T \dot{H}^{\frac{1}{2}}} \leqslant C\big\|(u,\omega)\big\|_{E^p_T} \big\|(u,\omega)\big\|_{\widetilde{L}^\infty_T \dot{H}^{\frac{1}{2}}}. \tag{5.3}$$

From Proposition 3.6, it is easy to verify that

$$\big\|G(t)(u_0,\omega_0)\big\|_{L^\infty_T \dot{H}^{\frac{1}{2}}} \leqslant C\big\|e^{-c2^{2j}t}\big\|_{L^\infty_T} \big\|(u_0,\omega_0)\big\|_{\dot{H}^{\frac{1}{2}}} \leqslant C\big\|(u_0,\omega_0)\big\|_{\dot{H}^{\frac{1}{2}}}. \tag{5.4}$$

It follows from Theorem 1.2 that $\|(u,\omega)\|_{E^p_T} \leqslant \eta$, then if η is sufficiently small such that $\eta C \leqslant \frac{1}{2}$, we have for any $T>0$

$$\big\|(u,\omega)\big\|_{L^\infty_T \dot{H}^{\frac{1}{2}}} \leqslant 2C\big\|(u_0,\omega_0)\big\|_{\dot{H}^{\frac{1}{2}}}.$$

This finishes the existence of the proof of Theorem 1.4.

The uniqueness in $C(\dot H^{\frac12})$. We will adopt the spirit of [4]. Firstly, let us recall the following bilinear estimate from [9]:

Lemma 5.1. *For any $T>0$, the bilinear operators $B(u,v)(t)$, $\widetilde{B}(u,v)(t)$ are bi-continuous from $L^\infty_T(\dot B^{\frac12}_{2,\infty})\times L^\infty_T(\dot H^{\frac12})$ to $L^\infty_T(\dot B^{\frac12}_{2,\infty})$. Furthermore, we have*

$$\|B(u,v)\|_{L^\infty_T\dot B^{\frac12}_{2,\infty}},\ \|\widetilde{B}(u,v)\|_{L^\infty_T\dot B^{\frac12}_{2,\infty}}\leqslant C\|u\|_{\widetilde{L}^\infty_T\dot B^{\frac12}_{2,\infty}}\big\|(e_{k,T})^{\frac14}2^{\frac k2}\|\Delta_j v\|_{L^\infty_T L^2}\big\|_{\ell^2(k\in\mathbb{Z})},$$

here

$$e_{k,T}\overset{def}{=}1-e^{-c2^{2k}T},$$

where $c>0$ is a constant independent of j,T,u,v.

Now let (u,ω) and (v,ϖ) be two solutions in $C(0,T;\dot H^{\frac12})$ with the initial data $(u_0,\omega_0)\in\dot H^{\frac12}$. Using (5.1), we have the difference

$$\begin{aligned}u-v={}&B\big(u-G_{ij}(t)u_0^j,u-v\big)+B\big(G_{ij}(t)u_0^j,u-v\big)+B\big(u-v,v-G_{ij}(t)u_0^j\big)\\&+B\big(u-v,G_{ij}(t)u_0^j\big)+B\big(u-G_{ij}(t)u_0^j,\omega-\varpi\big)+B\big(G_{ij}(t)u_0^j,\omega-\varpi\big)\\&+B\big(u-v,\varpi-G_{(i+3)j}(t)\omega_0^j\big)+B\big(u-v,G_{(i+3)j}(t)\omega_0^j\big),\quad i=1,2,3.\end{aligned}$$

Replacing B by $\widetilde{B}$ in the above equation, we can get the representation of $\omega-\varpi$. One easily verifies by Lemma 5.1 that

$$\begin{aligned}&\sup_{t\in(0,T)}\big(\|(u-v)(t)\|_{\dot B^{\frac12}_{2,\infty}}+\|(\omega-\varpi)(t)\|_{\dot B^{\frac12}_{2,\infty}}\big)\\&\quad\leqslant C\sup_{t\in(0,T)}\big(\|(u-v)(t)\|_{\dot B^{\frac12}_{2,\infty}}+\|(\omega-\varpi)(t)\|_{\dot B^{\frac12}_{2,\infty}}\big)\\&\quad\times\Big(\big\|\big(1-e^{-c2^{2k}T}\big)^{\frac14}2^{\frac k2}\big(\|\Delta_j u_0\|_2+\|\Delta_j\omega_0\|_2\big)\big\|_{\ell^2}\\&\quad+\sup_{t\in(0,T)}\big(\|u-G(t)u_0\|_{\dot H^{\frac12}}+\|v-G(t)u_0\|_{\dot H^{\frac12}}+\|\varpi-G(t)\omega_0\|_{\dot H^{\frac12}}\big)\Big).\end{aligned}$$

With the help of the fact: if T is chosen sufficiently small and $(u_0,\omega_0)\in\dot H^{\frac12}$, then

$$\big\|\big(1-e^{-c2^{2k}T}\big)^{\frac14}2^{\frac k2}\big(\|\Delta_k u_0\|_2+\|\Delta_k\omega_0\|_2\big)\big\|_{\ell^2}\leqslant\frac14$$

and the strong continuity in time of the $\dot H^{\frac12}$ norm of the Duhamel's term of the solution (u,ω) and (v,ϖ), then a small enough time T is to be chosen such that the last factor in the right side is dominated by 1/2, this implies that

$$\|(u-v,\omega-\varpi)(t)\|_{\dot B^{\frac12}_{2,\infty}}\equiv0,\quad\forall t\in[0,T].$$

Then by the standard argument ensures that $u=v$, $\omega=\varpi$ on $[0,\infty)$.

6. The decay estimate

In this section, we will analyze the long-time behavior of the solutions and get some decay estimates.

Set

$$W(T) \stackrel{def}{=} \sup_{0\leqslant t\leqslant T,\ 0<|\alpha|} t^{\frac{|\alpha|}{2}}\big(\big\|D_x^\alpha u\big\|_{\dot B^{\frac{3}{p}-1}_{p,\infty}} + \big\|D_x^\alpha \omega\big\|_{\dot B^{\frac{3}{p}-1}_{p,\infty}}\big).$$

Taking D_x^α on both sides of (4.3), one gets

$$\begin{pmatrix} \Delta_j D_x^\alpha u_A \\ \Delta_j D_x^\alpha \omega_\Omega \end{pmatrix} = D_x^\alpha \mathcal{G}(\cdot,t)\begin{pmatrix} \Delta_j u_{0,A} \\ \Delta_j \omega_{0,\Omega} \end{pmatrix} + \int_0^t D_x^\alpha \mathcal{G}(\cdot,t-\tau)\begin{pmatrix} \Delta_j F(\tau) \\ \Delta_j H(\tau) \end{pmatrix} \mathrm{d}\tau.$$

Applying Lemma 2.4 to the above equation, we have

$$\begin{aligned} &\big\|\Delta_j D_x^\alpha u_A\big\|_{L^p} + \big\|\Delta_j D_x^\alpha \omega_\Omega\big\|_{L^p} \\ &\quad\leqslant Ce^{-c2^{2j}t}2^{j|\alpha|}\big(\|\Delta_j u_{0,A}\|_{L^p} + \|\Delta_j \omega_{0,\Omega}\|_{L^p}\big) + \mathcal{I} + \mathcal{II} \end{aligned} \tag{6.1}$$

where

$$\mathcal{I} = C\int_0^{t/2} 2^{j|\alpha|}\big(\big\|\mathcal{G}(t-\tau)\Delta_j F(\tau)\big\|_{L^p} + \big\|\mathcal{G}(t-\tau)\Delta_j G(\tau)\big\|_{L^p}\big)\,\mathrm{d}\tau,$$

$$\mathcal{II} = C\int_{t/2}^{t} 2^{j}\big(\big\|\mathcal{G}(t-\tau)D_x^{\alpha-1}\Delta_j F(\tau)\big\|_{L^p} + \big\|\mathcal{G}(t-\tau)D_x^{\alpha-1}\Delta_j H(\tau)\big\|_{L^p}\big)\,\mathrm{d}\tau.$$

Noting the inequality

$$e^{-ct2^{2j}}2^{j|\beta|} \leqslant e^{-\tilde c t2^{2j}}t^{-\frac{|\beta|}{2}}, \quad |\beta| \geqslant 0, \tag{6.2}$$

and Proposition 3.5, we get that

$$\mathcal{I} \leqslant C\int_0^{t/2} e^{-\tilde c 2^{2j}(t-\tau)}(t-\tau)^{-\frac{|\alpha|}{2}}\big(\big\|\Delta_j F(\tau)\big\|_{L^p} + \big\|\Delta_j H(\tau)\big\|_{L^p}\big)\,\mathrm{d}\tau,$$

$$\mathcal{II} \leqslant C\int_{t/2}^{t} e^{-\tilde c 2^{2j}(t-\tau)}(t-\tau)^{-\frac{1}{2}}\big(\big\|D_x^{\alpha-1}\Delta_j F(\tau)\big\|_{L^p} + \big\|D_x^{\alpha-1}\Delta_j H(\tau)\big\|_{L^p}\big)\,\mathrm{d}\tau.$$

In the following we denote by $c_j (j\in\mathbb{Z})$ a sequence in ℓ^1 with the norm $\|\{c_j\}\|_{\ell^1} \leqslant 1$. In light of (4.5) and interpolation (4.7), a straightforward calculation shows that

$$\begin{aligned}
\mathcal{I} &\leqslant Ct^{-\frac{|\alpha|}{2}} \int_0^{t/2} e^{-\tilde{c}2^{2j}(t-\tau)} \big(\|\Delta_j F(\tau)\|_{L^p} + \|\Delta_j H(\tau)\|_{L^p} \big)\, \mathrm{d}\tau \\
&\leqslant Cc_j 2^{-j(\frac{3}{p}-1)} t^{-\frac{|\alpha|}{2}} \big(\|F\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} + \|H\|_{\widetilde{L}^1_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \big) \\
&\leqslant Cc_j 2^{-j(\frac{3}{p}-1)} t^{-\frac{|\alpha|}{2}} \|(u,\omega)\|^2_{E^p_T} \\
&\leqslant Cc_j 2^{-j(\frac{3}{p}-1)} t^{-\frac{|\alpha|}{2}} \|(u_0,\omega_0)\|^2_{E^p_0}.
\end{aligned} \tag{6.3}$$

Thanks to the Hölder inequality, we have

$$\begin{aligned}
\mathcal{II} &\leqslant C \|e^{-c2^{2j}t}\|_{L^4_T} \bigg(\int_{t/2}^{t} (t-\tau)^{-\frac{2}{3}}\, \mathrm{d}\tau \bigg)^{\frac{3}{4}} \big(\|D_x^{\alpha-1}\Delta_j F\|_{L^\infty_T L^p} + \|D_x^{\alpha-1}\Delta_j H\|_{L^\infty_T L^p} \big) \\
&\leqslant C2^{-\frac{j}{2}} t^{\frac{1}{4}} \big(\|D_x^{\alpha-1}\Delta_j F\|_{L^\infty_T L^p} + \|D_x^{\alpha-1}\Delta_j H\|_{L^\infty_T L^p} \big).
\end{aligned}$$

The divergence free condition on u, Lemma 2.4 and Lemma 2.7 give that

$$\begin{aligned}
\|\Delta_j D_x^{\alpha-1} H\|_{L^\infty_T L^p} &\leqslant C2^j \big\|\Delta_j\big((D_x^{\alpha-1}u)\omega + u(D_x^{\alpha-1}\omega) \big)\big\|_{L^\infty_T L^p} \\
&\leqslant Cc_j 2^{-j(\frac{3}{p}-\frac{3}{2})} \big\|(D_x^{\alpha-1}u)\omega + u(D_x^{\alpha-1}\omega)\big\|_{L^\infty_T \dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,\infty}} \\
&\leqslant Cc_j 2^{-j(\frac{3}{p}-\frac{3}{2})} \big(\|D_x^{\alpha-1}u\|_{L^\infty_T \dot{B}^{\frac{3}{p}-\frac{1}{4}}_{p,\infty}} \|\omega\|_{L^\infty_T \dot{B}^{\frac{3}{p}-\frac{1}{4}}_{p,\infty}} + \|D_x^{\alpha-1}\omega\|_{L^\infty_T \dot{B}^{\frac{3}{p}-\frac{1}{4}}_{p,\infty}} \|u\|_{L^\infty_T \dot{B}^{\frac{3}{p}-\frac{1}{4}}_{p,\infty}} \big).
\end{aligned}$$

By means of interpolation and Lemma 2.3(ii), we have

$$\begin{aligned}
\|D_x^{\alpha-1}u\|_{L^\infty_T \dot{B}^{\frac{3}{p}-\frac{1}{4}}_{p,\infty}} &\leqslant C\|u\|^{1-\frac{1}{4\alpha}}_{L^\infty_T \dot{B}^{\frac{3}{p}-1+\alpha}_{p,\infty}} \|u\|^{\frac{1}{4\alpha}}_{L^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \leqslant C\|D_x^{\alpha}u\|^{1-\frac{1}{4\alpha}}_{L^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \|u\|^{\frac{1}{4\alpha}}_{L^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \\
&\leqslant Ct^{-\frac{|\alpha|}{2}+\frac{1}{8}} W(t)^{1-\frac{1}{4\alpha}} \|(u_0,\omega_0)\|^{\frac{1}{4\alpha}}_{E^p_0},
\end{aligned}$$

and

$$\begin{aligned}
\|\omega\|_{L^\infty_T \dot{B}^{\frac{3}{p}-\frac{1}{4}}_{p,\infty}} &\leqslant C\|\omega\|^{\frac{1}{4}}_{L^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \|\omega\|^{\frac{3}{4}}_{L^\infty_T \dot{B}^{\frac{3}{p}}_{p,\infty}} \leqslant C\|\omega\|^{\frac{1}{4}}_{L^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \|D_x\omega\|^{\frac{3}{4}}_{L^\infty_T \dot{B}^{\frac{3}{p}-1}_{p,\infty}} \\
&\leqslant Ct^{-\frac{3}{8}} W(t)^{\frac{3}{4}} \|(u_0,\omega_0)\|^{\frac{1}{4}}_{E^p_0}.
\end{aligned}$$

The term of F is done in the same way. Thus

$$\sup_{j\in\mathbb{Z}} 2^{j(\frac{3}{p}-1)} t^{\frac{|\alpha|}{2}} \mathcal{II} \leqslant CW(t)^{\frac{7}{4}-\frac{1}{4\alpha}} \|(u_0,\omega_0)\|^{\frac{1}{4}+\frac{1}{4\alpha}}_{E^p_0}. \tag{6.4}$$

For the estimate of ω_d, we localize the third equation of (1.12), then taking D_x^α on the localized equation yields

$$\partial_t \Delta_j D_x^\alpha \omega_d - 2\Delta\Delta_j D_x^\alpha \omega_d + 2\Delta_j D_x^\alpha \omega_d = -\Lambda^{-1} \operatorname{div} D_x^\alpha \Delta_j (u \cdot \nabla\omega).$$

Multiplying by $p|\Delta_j D_x^\alpha \omega_d|^{p-2}\Delta_j D_x^\alpha \omega_d$ and integrating with respect to x yield that

$$\begin{aligned}
&\frac{d}{dt}\|\Delta_j D_x^\alpha \omega_d\|_{L^p}^p + 2p\int_{\mathbb{R}^3} (-\Delta)\Delta_j D_x^\alpha \omega_d \big|\Delta_j D_x^\alpha \omega_d\big|^{p-2}\Delta_j D_x^\alpha \omega_d \,\mathrm{d}x + 2p\int_{\mathbb{R}^3} \big|\Delta_j D_x^\alpha \omega_d\big|^p \,\mathrm{d}x \\
&\quad = -p\int_{\mathbb{R}^3} \Lambda^{-1} \operatorname{div} D_x^\alpha \Delta_j (u\cdot\nabla\omega)\big|\Delta_j D_x^\alpha \omega_d\big|^{p-2}\Delta_j D_x^\alpha \omega_d \,\mathrm{d}x.
\end{aligned}$$

Using Lemma 2.5, we have

$$\frac{d}{dt}\|\Delta_j D_x^\alpha \omega_d\|_{L^p}^p + c_p\big(2^{2j}+1\big)\|\Delta_j D_x^\alpha \omega_d\|_{L^p}^p \leqslant C\|\Delta_j D_x^\alpha (u\cdot\nabla\omega)\|_{L^p}\|\Delta_j D_x^\alpha \omega_d\|_{L^p}^{p-1}.$$

This yields that

$$\frac{d}{dt}\|\Delta_j D_x^\alpha \omega_d\|_{L^p} + c_p\big(2^{2j}+1\big)\|\Delta_j D_x^\alpha \omega_d\|_{L^p} \leqslant C\|\Delta_j D_x^\alpha (u\cdot\nabla\omega)\|_{L^p}.$$

This together with Gronwall's inequality implies that

$$\|\Delta_j D_x^\alpha \omega_d\|_{L^p} \leqslant e^{-c_p t(2^{2j}+1)}\|\Delta_j D_x^\alpha \omega_{0,d}\|_{L^p} + \mathcal{III},$$

where

$$\mathcal{III} = C\int_0^t e^{-c_p(t-\tau)2^{2j}} e^{-(t-\tau)}\|D_x^\alpha \Delta_j (u\cdot\nabla\omega)(\tau)\|_{L^p}\,\mathrm{d}\tau.$$

Using Lemma 2.4 and (6.2), we obtain

$$\begin{aligned}
\mathcal{III} \leqslant{}& C\int_0^{t/2} e^{-\tilde{c}_p(t-\tau)2^{2j}} e^{-(t-\tau)}(t-\tau)^{-|\alpha|/2}\|\Delta_j (u\cdot\nabla\omega)(\tau)\|_{L^p}\,\mathrm{d}\tau \\
&+ C\int_{t/2}^{t} e^{-\tilde{c}_p(t-\tau)2^{2j}} e^{-(t-\tau)}(t-\tau)^{-1/2}\|D_x^{\alpha-1}\Delta_j (u\cdot\nabla\omega)(\tau)\|_{L^p}\,\mathrm{d}\tau.
\end{aligned}$$

The first term is treated as $\mathcal{I}$, the second term is treated as $\mathcal{II}$, then

$$\sup_{j\in\mathbb{Z}} 2^{j(\frac{3}{p}-1)} t^{\frac{|\alpha|}{2}}\mathcal{III} \leqslant CW(t)^{\frac{7}{4}-\frac{1}{4\alpha}}\|(u_0,\omega_0)\|_{E_0^p}^{\frac{1}{4}+\frac{1}{4\alpha}}. \tag{6.5}$$

Combining (6.1) with (6.3)–(6.5), we have

$$W(t) \leqslant \left\|(u_0,\omega_0)\right\|_{E_0^p} + C\left\|(u_0,\omega_0)\right\|_{E_0^p}^2 + CW(t)^{\frac{7}{4}-\frac{1}{4\alpha}}\left\|(u_0,\omega_0)\right\|_{E_0^p}^{\frac{1}{4}+\frac{1}{4\alpha}}.$$

Then by the continuous induction, we have $W(t) \leqslant 2CE$. This completes the proof of Theorem 1.6.

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