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Normal Forms for Nonautonomous Difference Equations

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Abstract—We extend Henry Poincaré's normal form theory for autonomous difference equations $x_{k+1} = f(x_k)$ to nonautonomous difference equations $x_{k+1} = f_k(x_k)$. Poincaré's nonresonance condition $\lambda_j - \prod_{i=1}^n \lambda_i^{q_i} \neq 0$ for eigenvalues is generalized to the new nonresonance condition $\lambda_j \cap \prod_{i=1}^n \lambda_i^{q_i} = \emptyset$ for spectral intervals.

Keywords—Poincaré normal form theory, Nonautonomous normal forms, Resonance, Nonautonomous difference equations.

1. INTRODUCTION

The famous French mathematician Henry Poincaré founded the normal form theory for autonomous differential equations $\dot{x} = f(x)$ near a rest point in his thesis in 1879. Soon a parallel theory for autonomous difference equations $x_{k+1} = f(x_k)$ was developed. If the eigenvalues $\lambda_1, \dots, \lambda_n$ of the linearization $x_{k+1} = Df(x^0)x_k$ at the rest point x^0 satisfy the *nonresonance condition*

$$\lambda_j \neq \lambda_1^{q_1} \cdots \lambda_n^{q_n}, \tag{1}$$

$j \in \{1, \dots, n\}$, $q_i \in \mathbb{N}_0 = \{0, 1, \dots\}$, $\sum_{i=1}^n q_i \geq 2$, then the difference equation can be formally linearized.

As an example, we consider the following planar autonomous system:

$$x_{k+1} = 2x_k,$$
$$y_{k+1} = \lambda y_k + x_k^2,$$

with $\lambda \in (0, \infty)$. We are looking for a near-identity transformation

$$H(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} + h_2(x,y),$$

which eliminates the second order nonlinearity $\binom{0}{x^2}$ and we choose $h_2 \in \text{span}\{\binom{x^2}{0}, \binom{0}{x^2}, \binom{xy}{0}, \binom{0}{xy}, \binom{y^2}{0}, \binom{0}{y^2}\}$. It is not difficult to show that the transformed equation has no second-order nonlinearity if and only if the so-called *homological equation*

$$Ah_2(x,y) - h_2\left(A\begin{pmatrix} x \\ y \end{pmatrix}\right) = f_2(x,y)$$

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is satisfied with

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad f_2(x,y) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}.$$

It is solvable if and only if $\lambda \neq 4$ and with its unique solution we get

$$H(x,y) = \begin{pmatrix} x \\ y + \dfrac{1}{\lambda - 4} x^2 \end{pmatrix}.$$

In this simple example, the transformed equation $x_{k+1} = 2x_k$, $y_{k+1} = \lambda y_k$ is linear. In general, the elimination of second-order nonlinearities produces higher-order nonlinearities, and the process has to be iterated. The resulting transformation is the composition of the transformations of each elimination step, and it is *nonlinear* but is constructed by solving a *sequence of linear* equations.

In this article, we consider nonautonomous invertible difference equations

$$x_{k+1} = f_k(x_k) \tag{2}$$

not in the vicinity of a rest point as Poincaré did it in the autonomous case, but in the vicinity of an arbitrary reference solution $v^0 : \mathbb{Z} \to \mathbb{R}^N$. For some $p \geq 2$ we assume $f_k : D_{f_k} \subset \mathbb{R}^N \to f_k(D_{f_k}) \subset \mathbb{R}^N$ to be a C^p diffeomorphism for every $k \in \mathbb{Z} = \{0, \pm 1, \dots\}$. We will extend Poincaré's normal form theory by showing that if the linearization $x_{k+1} = Df_k(v_k^0)x_k$ of (2) along the reference solution v^0 has invertible coefficient matrices $Df_k(v_k^0) \in \mathbb{R}^{N \times N}$, $k \in \mathbb{Z}$, and satisfies a *nonresonance condition*, then system (2) is locally C^p equivalent to a system $x_{k+1} = g_k(x_k)$ in normal form; i.e., with zero reference solution, block diagonal linear part $x_{k+1} = Dg_k(0)x_k$ and all nonresonant Taylor coefficients of g up to order p are zero.

We therefore have to use a proper replacement of the "linear algebra" for autonomous systems (i.e., eigenvalues and eigenspaces) in our nonautonomous situation. A spectral theory for nonautonomous difference equations is developed in [1]. The dichotomy spectrum of the linearized difference equation $x_{k+1} = Df_k(v_k^0)x_k$ consists of at most N closed intervals of the positive real line $\mathbb{R}^+ = (0, \infty)$; in general, the spectrum may be empty or unbounded. It is nonempty and compact, i.e., consists of n compact intervals with $1 \leq n \leq N$, if the system has *bounded growth*. A linear system $x_{k+1} = A_k x_k$ has bounded growth if its evolution operator Φ satisfies the estimate $\|\Phi(k,\ell)\| \leq K a^{|k-\ell|}$ for $k, \ell \in \mathbb{Z}$ with constants $K, a \geq 1$. Bounded growth is equivalent to the boundedness of the coefficients and their inverses [1, Lemma 2.3], and hence, the linearized difference equation $x_{k+1} = Df_k(v_k^0)x_k$ has bounded growth if $\|A_k\| \leq M$ and $\|A_k^{-1}\| \leq M$ for $k \in \mathbb{Z}$ with some constant $M \geq 0$ and $A_k = Df_k(v_k^0)$.

For simplicity, we assume in the following that the linearized equation has bounded growth, although the theory could also be developed in the general case.

2. PRELIMINARIES

Let $\phi(\cdot\,; \ell, \xi) : I_{\ell,\xi} \to \mathbb{R}^N$ denote the unique maximal solution of the initial value problem (2), $x_\ell = \xi$ for $\xi \in D_{f_\ell}$ where $I_{\ell,\xi}$ is a $\mathbb{Z}$-interval (i.e., an intersection of an interval with $\mathbb{Z}$) containing ℓ such that the solution identity

$$\phi(k+1; \ell, \xi) = f_k(\phi(k; \ell, \xi)), \qquad \text{for } k, k+1 \in I_{\ell,\xi},$$

holds. We have $\phi(\ell; \ell, \xi) = \xi$ and $\phi(k; \phi(m; \ell, \xi)) = \phi(k+m; \ell, \xi)$ for $m, k+m \in I_{\ell,\xi}$.

There is no straightforward way to define a notion of conjugacy for nonautonomous difference equations. What do we mean by this? Two autonomous difference equations $x_{k+1} = f^1(x_k)$ and $x_{k+1} = f^2(x_k)$ in $\mathbb{R}^N$ are said to be conjugate if there exists a homeomorphism $H : \mathbb{R}^N \to \mathbb{R}^N$ such that the flows $\phi_1(\cdot\,; \xi)$, respectively, $\phi_2(\cdot\,; \eta)$ satisfy the conjugacy relation $H(\phi_1(k; \xi)) = \phi_2(k; H(\xi))$ for all $\xi \in \mathbb{R}^N$, $k \in I_\xi$; i.e., H maps solutions of the first equation onto solutions of the second equation and vice versa for H^{-1}. Now if we would define a conjugacy between two nonautonomous difference equations $x_{k+1} = f_k^1(x_k)$ and $x_{k+1} = f_k^2(x_k)$ by the same property, but now with a k-dependent H, then for every $\ell \in I_{k,x}$

$$H_k(x) := \phi_2(k; \ell, \phi_1(\ell; k, x))$$

would establish a conjugacy; i.e., H maps solutions of the first equation onto solutions of the second equation and vice versa with

$$H_k^{-1}(x) := \phi_1(k; \ell, \phi_2(\ell; k, x)).$$

In the nonautonomous situation, we need, therefore, some additional conditions which ensure that qualitative behaviour—at least for a single reference solution—is preserved under the transformation.

It is easy to see in the autonomous situation that for a conjugacy, periodic solutions, limit sets, and invariant sets of the first equation are bijectively mapped onto periodic solutions, limit sets, and invariant sets, respectively, of the second equation and that (asymptotic) stability, attractivity, and instability of bounded solutions are preserved under the conjugacy. In most cases this is enough, but note that the assumption of boundedness of solutions is essential for the preservation of stability. For example: the two linear systems $x_{k+1} = x_k + 1$, $y_{k+1} = (1/2)y_k$ and $x_{k+1} = x_k + 1$, $y_{k+1} = 2y_k$ are conjugate via $H(x, y) = (x, y2^{2x})$, but the first system is stable and the second is unstable. To preserve the stability of an unbounded solution v_k, it would be necessary to pose some uniformity condition on H, e.g., $\lim_{x\to 0} H(v_k + x) = H(v_k)$ uniformly in $k \in \mathbb{Z}$. Such a uniformity condition is exactly what we need in the nonautonomous situation to define a meaningful notion of C^p equivalence.

Consider difference equations together with reference solutions

$$x_{k+1} = f_k(x_k), \qquad v^0 : \mathbb{Z} \to \mathbb{R}^N, \tag{3}$$

$$x_{k+1} = g_k(x_k), \qquad w^0 : \mathbb{Z} \to \mathbb{R}^N, \tag{4}$$

where f_k and g_k are C^p diffeomorphisms, i.e., $f_k \in \mathrm{Diff}^p(D_{f_k}, f_k(D_{f_k}))$, $g_k \in \mathrm{Diff}^p(D_{g_k}, g_k(D_{g_k}))$, $p \geq 0$. We assume that uniform neighbourhoods of the reference solutions are contained in the corresponding sets of definition, i.e., there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$B_\varepsilon\left(v_k^0\right) \subset D_{f_k} \quad \text{and} \quad B_\delta\left(w_k^0\right) \subset D_{g_k}, \qquad \text{for } k \in \mathbb{Z},$$

where $B_\varepsilon(x^0) := \{x \in \mathbb{R}^N : \|x - x^0\| < \varepsilon\}$. Define $U_\varepsilon(v^0) := \{(k, x) \in \mathbb{Z} \times \mathbb{R}^N : x \in B_\varepsilon(v_k^0)\}$.

DEFINITION 1. *Consider systems (3) and (4). If there exist* ε', δ' *with* $0 < \varepsilon' \leq \varepsilon$ *and* $0 < \delta' \leq \delta$ *together with functions*

$$H : U_{\varepsilon'}(v^0) \to \mathbb{R}^N, \qquad H^{-1} : U_{\delta'}(w^0) \to \mathbb{R}^N,$$

then H *is called a local* C^p *equivalence between system (3) with solution* v^0 *and system (4) with solution* w^0*, if the following statements are valid.*

(A) *For each* $k \in \mathbb{Z}$*, the mappings*

$$H_k : B_{\varepsilon'}\left(v_k^0\right) \to H_k\left(B_{\varepsilon'}\left(v_k^0\right)\right) \subset B_\delta\left(w_k^0\right),$$
$$H_k^{-1} : B_{\delta'}\left(w_k^0\right) \to H_k^{-1}\left(B_{\delta'}\left(w_k^0\right)\right) \subset B_\varepsilon\left(v_k^0\right)$$

are C^p *diffeomorphisms (or homeomorphisms if* $p = 0$*) with*

$$H_k(H_k^{-1}(x)) = x \qquad \text{and} \qquad H_k^{-1}(H_k(x)) = x$$

for all x *for which the compositions are defined.*

(B) *If v_k is a solution of (3) in $U_{\varepsilon'}(v^0)$, then $H_k(v_k)$ is a solution of (4). If w_k is a solution of (4) in $U_{\delta'}(w^0)$, then $H_k^{-1}(w_k)$ is a solution of (3).*

(C) *The reference solutions are mapped uniformly onto each other,*

$$\lim_{x\to 0} H_k\left(v_k^0 + x\right) = w_k^0, \qquad \text{uniformly in } k \in \mathbb{Z},$$
$$\lim_{x\to 0} H_k^{-1}\left(w_k^0 + x\right) = v_k^0, \qquad \text{uniformly in } k \in \mathbb{Z}.$$

LEMMA 2. *Consider systems (3) and (4) together with a solution $v^0 : I \to \mathbb{R}^N$ of (3) which is defined on some $\mathbb{Z}$-interval I. Then a mapping $w : J \to \mathbb{R}^N$ defined on a $\mathbb{Z}$-interval $J \subset I$ is a solution of the difference equation*

$$x_{k+1} = g_k\left(x_k + v_k^0\right) - f_k\left(v_k^0\right) \tag{5}$$

if and only if $w + v^0 : J \to \mathbb{R}^N$ is a solution of the difference equation (4).

PROOF. Since v^0 is a solution of (3) one has for $k, k+1 \in J$

$$w_{k+1} = g_k\left(w_k + v_k^0\right) - f_k\left(v_k^0\right) \Leftrightarrow w_{k+1} + v_{k+1}^0 = g_k\left(w_k + v_k^0\right),$$

and the claim is proved.

3. NORMAL FORMS

We consider a difference equation together with a reference solution

$$x_{k+1} = f_k(x_k), \qquad v^0 : \mathbb{Z} \to \mathbb{R}^N, \tag{6}$$

which satisfy the following conditions.

- Smoothness: $f_k \in \mathrm{Diff}^p(D_{f_k}, f_k(D_{f_k}))$ for a $p \geq 2$.
- Set of definition: $\varepsilon := \inf\{\varepsilon' \geq 0 : B_{\varepsilon'}(v_k^0) \subset D_{f_k} \text{ for all } k \in \mathbb{Z}\} > 0$.
- Linearity: $x_{k+1} = Df_k(v_k^0)x_k$ has bounded growth.
- Nonlinearity: $\|D^j f_k(v_k^0)\| \leq M$ for $2 \leq j \leq p$ and all $k \in \mathbb{Z}$.

We will simplify system (6) in three steps.

STEP 1: TRIVIALIZATION OF THE REFERENCE SOLUTION.

Recall Lemma 2. If $f \equiv g$, then system (5) reduces to

$$x_{k+1} = f_k\left(x_k + v_k^0\right) - f_k\left(v_k^0\right), \tag{7}$$

which is usually called the difference equation of perturbed motion of (6) w.r.t. the solution v^0. Obviously, (7) has the zero solution, and because of Lemma 2, $w : J \subset I \to \mathbb{R}^N$ is a solution of (7) if and only if $w + v^0$ is a solution of (6), and hence, the mappings

$$R_k : B_\varepsilon\left(v_k^0\right) \to \mathbb{R}^N, \qquad x \mapsto x - v_k^0,$$
$$R_k^{-1} : B_\varepsilon(0) \to \mathbb{R}^N, \qquad x \mapsto x + v_k^0,$$

define a C^∞ equivalence between (6) with reference solution v^0 and system (7) with zero reference solution. We rewrite (7) as

$$x_{k+1} = A_k^* x_k + F_k^*(x_k), \tag{8}$$

where $A_k^* = Df_k(v_k^0)$ is the linear part and $F_k^*(x_k) = f_k(x_k + v_k^0) - f_k(v_k^0) - Df_k(v_k^0)x_k$ is the nonlinearity. Obviously, $U_\varepsilon(0) = \mathbb{R} \times B_\varepsilon(0)$ is contained in the set of definition of the right-hand side of (8). Note that this simple transformation is a powerful nonautonomous tool. It is of no use in a purely autonomous framework, since (8) in general is nonautonomous.

STEP 2: BLOCK DIAGONALIZATION OF THE LINEAR PART.

In [2, Reduction Theorem] it is shown that there exists a kinematic similarity $S:\mathbb{Z}\to\mathbb{R}^{N\times N}$ between the linearization $x_{k+1}=A_k^* x_k$ of (8) and a linear system

$$x_{k+1}=A_k x_k \tag{9}$$

such that $A:\mathbb{Z}\to\mathbb{R}^{N\times N}$ is in block diagonal form

$$A_k=\begin{pmatrix} A_k^1 & & \\ & \ddots & \\ & & A_k^n \end{pmatrix}$$

and each block $A^i:\mathbb{Z}\to\mathbb{R}^{N_i\times N_i}$, $i=1,\dots,n$, corresponds to a spectral interval λ_i. System (9) also has bounded growth, the dichotomy spectra $\Sigma(A^*)$ and $\Sigma(A)$ are the same, and they equal $\lambda_1\cup\dots\cup\lambda_n$.

LEMMA 3. *There exist ε', σ, σ' with $0<\varepsilon'\le\varepsilon$ and $0<\sigma'\le\sigma$ such that for $k\in\mathbb{Z}$, the mappings*

$$\begin{aligned} B_{\varepsilon'}(0)&\to B_\sigma(0), &\quad x&\mapsto S_k^{-1}x,\\ B_{\sigma'}(0)&\to B_\varepsilon(0), &\quad x&\mapsto S_k x, \end{aligned}$$

define a C^∞ equivalence between (8) and the difference equation

$$x_{k+1}=A_k x_k+F_k(x_k) \tag{10}$$

with $F_k(x)=S_{k+1}^{-1}F_k^*(S_k x)$ and $[A_k+F_k]\in \operatorname{Diff}^p(B_\sigma(0),[A_k+F_k](B_\sigma(0)))$. Moreover, $\|D^jF_k(0)\|\le M'$ for all $k\in\mathbb{Z}$ and all $j\in\{2,\dots,p\}$ with some $M'\ge 0$.

PROOF. Due to [2, Corollary 2.1], the kinematic similarity satisfies $S_{k+1}=A_k^*S_kA_k^{-1}$. Let v_k be a solution of (8). Then $w_k:=S_k^{-1}v_k$ satisfies for all $k\in\mathbb{Z}$

$$w_{k+1}=S_{k+1}^{-1}\left[A_k^*v_k+F_k^*(v_k)\right]=A_kw_k+S_{k+1}^{-1}F_k^*(S_kw_k);$$

i.e., w_k is a solution of (10) with $F_k(x_k)=S_{k+1}^{-1}F_k^*(S_kx_k)$. The remaining claims of the lemma follow with the definitions $\sigma:=\sigma':=|S|^{-1}\varepsilon$, $\varepsilon':=\min\{\varepsilon,|S|^{-1}\sigma\}$, and $M':=|S|^{p+1}M$ where

$$|S|:=\max\left\{\sup_{k\in\mathbb{Z}}\|S_k\|,\ \sup_{k\in\mathbb{Z}}\left\|S_k^{-1}\right\|\right\}<\infty,$$

due to the boundedness of a kinematic similarity.

STEP 3: ELIMINATION OF NONRESONANT TAYLOR COMPONENTS.

This is the crucial step. We will eliminate Taylor components of the nonlinearity which correspond to the blocks $A_k^i\in\mathbb{R}^{N_i\times N_i}$, $i=1,\dots,n$, of the linear part A_k. Therefore, define $E^i:=\mathbb{R}^{N_i}$, $i=1,\dots,n$, and write $F=(F^1,\dots,F^n)$ with the component functions $F^i:D_F\to E^i$. Let Φ^i denote the evolution operator of the linear block system $x_{k+1}^i=A_k^ix_k^i$.

In order to present the ideas, we first motivate the construction of the transformation and the nonresonance condition. For simplicity assume, therefore, that system (10) is globally defined, i.e., $D_F=\mathbb{Z}\times\mathbb{R}^N$ and that each solution exists on $\mathbb{Z}$; this can be achieved by cutting of F outside the neighbourhood $U_\delta(0)$ of the zero solution. Now for all $k\in\mathbb{Z}$ we can expand F_k into a Taylor series at $x=0$,

$$F_k(x)=\sum_{q\in\mathbb{N}_0^n:2\le|q|\le p}\frac{1}{q!}D^qF_k(0)\cdot x^q+o\left(\|x\|^p\right),$$

$q = (q_1, \dots, q_n) \in \mathbb{N}_0^n$ a multi-index, $q! = q_1! \cdots q_n!$, $x^q = (x^1, \dots, x^n)^q = (x^1)^{q_1} \cdots (x^n)^{q_n}$, $|q| = q_1 + \cdots + q_n$. Now we are looking for a condition under which a C^p transformation exists which eliminates the j^{th} component $(1/q!) D^q F_k^j(0) \cdot x^q$ of a summand in the Taylor expansion. Therefore, choose and fix a $j \in \{1, \dots, n\}$ and a multi-index $q \in \mathbb{N}_0^n$ with $2 \leq |q| \leq p$. For simplicity we assume that the Taylor coefficients of F at $x = 0$ up to order $|q| - 1$ are already eliminated; i.e.,

$$D^{\hat{q}} F_k(0) = 0, \qquad \text{for all } k \in \mathbb{Z}, \quad \text{and all } \hat{q} \in \mathbb{N}_0^n, \quad \text{with } |\hat{q}| \leq |q| - 1. \tag{11}$$

We define a new function $G : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$G_k(x) := F_k(x) - \left(0, \dots, 0, \frac{1}{q!} D^q F_k^j(0) \cdot x^q, 0, \dots, 0\right).$$

To derive some necessary conditions for the existence of an equivalence we assume now that a near identity C^p equivalence $H_k(x) = x + h_k(x)$ between (10) with zero reference solution and the difference equation

$$x_{k+1} = A_k x_k + G_k(x_k) \tag{12}$$

with zero reference solution exists, where $h : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is a mapping with $h_k(0) = 0$ and $Dh_k(0) = 0$ for $k \in \mathbb{Z}$. We will make some observations which will help us to construct an explicit candidate for a C^p equivalence.

First, we will assign a difference equation to the values of the transformation H along a fixed solution $\phi(\cdot\,; m, \xi)$ of (10).

OBSERVATION 1. *For each initial condition $(m, \xi) \in \mathbb{Z} \times \mathbb{R}^N$ the mapping $h_k(\phi(k; m, \xi))$ is a solution of*

$$x_{k+1} = A_k x_k + G_k(x_k + \phi(k; m, \xi)) - F_k(\phi(k; m, \xi)). \tag{13}$$

Observation 1 is a simple but powerful consequence of Lemma 2. Next, we expose a connection between $D^q h_k(0)$ and $D_\xi^q [h_k(\phi(k; m, \xi))]|_{\xi=0}$.

OBSERVATION 2. *For all $k, m \in \mathbb{Z}$ and $\eta = (\eta^1, \dots, \eta^n) \in E^1 \times \cdots \times E^n = \mathbb{R}^N$ we have*

$$D_\xi^q [h_k(\phi(k; m, \xi))]|_{\xi=0} \cdot \eta^q = D^q h_k(0) \cdot \left[\Phi^1(k, m)\eta^1\right]^{q_1} \cdots \left[\Phi^n(k, m)\eta^n\right]^{q_n}.$$

This can be seen by calculating the partial derivatives which is easily possible since, by (11), the Taylor coefficients of F and G are zero up to order $|q| - 1$. The evolution operators Φ^i of the linear block systems $x_{k+1}^i = A_k^i x_k^i$ come into play because of

$$D_{\xi^i} \phi(k; m, \xi)|_{\xi=0} = \left(0, \dots, 0, \Phi^i(k, m), 0, \dots, 0\right) \in L(E^i; \mathbb{R}^N).$$

Now, we replace the η^i in Observation 2 by $\Phi^i(m, k)\zeta^i$, and with the identity $[\Phi^i(k, m)]^{-1} = \Phi^i(m, k)$, we get the following proposition.

OBSERVATION 3. *For all $k, m \in \mathbb{Z}$ and $\zeta \in \mathbb{R}^N$ we have*

$$D^q h_k(0) \cdot \zeta^q = D_\xi^q [h_k(\phi(k; m, \xi))]|_{\xi=0} \cdot \left[\Phi^1(m, k)\zeta^1\right]^{q_1} \cdots \left[\Phi^n(m, k)\zeta^n\right]^{q_n}.$$

Now, we have a relationship between the Taylor coefficient $D^q h_k(0)$ of h and the partial derivative $D_\xi^q [h_k(\phi(k; m, \xi))]|_{\xi=0}$. Observation 1 implies that $h_k(\phi(k; m, \xi))$ is a solution of (13). Then, by differentiation, one can show the following proposition.

OBSERVATION 4. *The function $D^q_\xi[h_k(\phi(k;m,\xi))]_{\xi=0}$ is a solution of*

$$x_{k+1} = A_k x_k + c_k, \tag{14}$$

the variational equation of (13) in $L^q(E^1,\ldots,E^n;\mathbb{R}^N)$, where

$$c_k = -\left(0,\ldots,0,D^q F^j_k(0),0,\ldots,0\right)\cdot\left[\Phi^1(k,m)\right]^{q_1}\cdots\left[\Phi^n(k,m)\right]^{q_n}.$$

So far, we have assumed that a C^p equivalence $H_k(x) = x + h_k(x)$ between (10) and (12) exists and by Observation 3, the Taylor coefficient $(1/q!)D^q h_k(0)\cdot x^q$ is a function of a special solution $D^q_\xi[h_k(\phi(k;m,\xi))]|_{\xi=0}$ of the difference equation (14) and the known evolution operators Φ^i.

From now on, we want to use this information to construct a candidate for a C^p equivalence between (10) and (12). We make the ansatz

$$H_k(x) = x + \frac{1}{q!}D^q h_k(0)\cdot x^q;$$

i.e., $h_k(x) = (1/q!)D^q h_k(0)\cdot x^q$ has only one nontrivial Taylor coefficient.

We make use of Observations 3 and 4 in the way that we choose a special solution z of (14) and interpret z_k as $D^q_\xi[h_k(\phi(k;m,\xi))]|_{\xi=0}$ for an arbitrary but fixed $m\in\mathbb{Z}$. With Observation 3 and our ansatz, this yields

$$H_k(x) = x + \frac{1}{q!}z_k\cdot\left[\Phi^1(m,k)x^1\right]^{q_1}\cdots\left[\Phi^n(m,k)x^n\right]^{q_n}. \tag{15}$$

Which solution z of (14) should we choose? To satisfy the condition (C) of Definition 1, it is necessary that $\lim_{x\to 0} H_k(x) = 0$ uniformly in $k\in\mathbb{Z}$, and this is satisfied if $z_k\cdot[\Phi^1(m,k)\cdot]^{q_1}\cdots[\Phi^n(m,k)\cdot]^{q_n}$ is bounded for $k\in\mathbb{Z}$. Using [2], one knows that the borders of the spectral intervals $\lambda_i = [a_i,b_i]$ yield the exponential growth rates of the evolution operators Φ^i of $x^i_{k+1} = A^i_k x^i_k$. Now it is the exponential growth rate of z we have to take care of. Here a key lemma comes in play.

LEMMA 8. *Consider the j^{th} component of (14)*

$$x^j_{k+1} = A^j_k x^j_k + c^j_k. \tag{16}$$

(A) *Assume that the spectral intervals $\lambda_i = [a_i,b_i]$ satisfy the condition*

$$a_j > b_1^{q_1}\cdots b_n^{q_n}. \tag{17}$$

Choose a $\gamma\in(b_1^{q_1}\cdots b_n^{q_n},a_j)$. Then $z^j_k := -\sum_{\ell=k}^{\infty}\Phi^j(k,\ell+1)c^j_\ell$ is the unique solution of (16) with the exponential growth rate γ^k for $k\to\infty$, i.e., $\|z^j_k\|\le C'\gamma^k$ for all $k\ge 0$ with some $C'\ge 0$.

(B) *Assume the condition*

$$b_j < a_1^{q_1}\cdots a_n^{q_n}. \tag{18}$$

Choose a $\gamma\in(b_j,a_1^{q_1}\cdots a_n^{q_n})$. Then $z^j_k := \sum_{\ell=-\infty}^{k-1}\Phi^j(k,\ell+1)c^j_\ell$ is the unique solution of (16) with the exponential growth rate γ^k for $k\to-\infty$.

PROOF. We prove only (A). For every $\varepsilon > 1$ we get with [2, Corollary 3.1] a constant $K\ge 1$ with

$$\begin{aligned}\left\|c^j_\ell\right\| &\le \left\|D^q F^j_\ell(0)\right\|\cdot\left\|\Phi^1(\ell,m)\right\|^{q_1}\cdots\left\|\Phi^n(\ell,m)\right\|^{q_n}\\ &\le MK^{|q|}\begin{cases}\left(b_1^{q_1}\cdots b_n^{q_n}\varepsilon^{|q|}\right)^{\ell-m}, & \text{for } \ell\ge m,\\ \left(a_1^{q_1}\cdots a_n^{q_n}\varepsilon^{-|q|}\right)^{\ell-m}, & \text{for } \ell\le m.\end{cases}\end{aligned}$$

The rest follows from [3, Lemma 3.4].

Now, we assume that (17) or (18) holds (both together cannot hold) and choose the following special solution $z = (z^1, \dots, z^n)$ of (14):

$$z_k^i := \begin{cases} 0, & \text{if } i \neq j, \\ -\sum\limits_{\ell=k}^{\infty} \Phi^j(k, \ell+1) c_\ell^j, & \text{if } i = j \text{ and (17) holds}, \\ \sum\limits_{\ell=-\infty}^{k-1} \Phi^j(k, \ell+1) c_\ell^j, & \text{if } i = j \text{ and (18) holds}. \end{cases}$$

Using (15) and the identity $\Phi^j(\ell, m) \cdot \Phi^j(m, k) = \Phi^j(\ell, k)$, our explicit candidate $H_k(x) = x + h_k(x)$ for a C^p equivalence is defined by

$$h_k^i(x) = \begin{cases} 0, & \text{if } i \neq j, \\ \sum\limits_{\ell=k}^{\infty} \Phi^j(k, \ell+1) \frac{1}{q!} D^q F_\ell^j(0) \cdot \left[\Phi^1(\ell, k) x^1\right]^{q_1} \cdots \left[\Phi^n(\ell, k) x^n\right]^{q_n}, & \\ & \text{if } i = j \text{ and (17) holds}, \\ -\sum\limits_{\ell=-\infty}^{k-1} \Phi^j(k, \ell+1) \frac{1}{q!} D^q F_\ell^j(0) \cdot \left[\Phi^1(\ell, k) x^1\right]^{q_1} \cdots \left[\Phi^n(\ell, k) x^n\right]^{q_n}, & \\ & \text{if } i = j \text{ and (18) holds}. \end{cases} \tag{19}$$

Let us have a closer look at conditions (17) and (18). For two compact intervals $[a, b]$ and $[c, d]$ in $\mathbb{R}^+$, we introduce a multiplication $[a, b] \cdot [c, d] := [ac, bd]$ and $[a, b]^r := [a^r, b^r]$ for $r \in \mathbb{N}$; furthermore, we will use the relations $[a, b] < [c, d] :\Leftrightarrow b < c$ and $[a, b] > [c, d] :\Leftrightarrow a > d$. With this notation, conditions (17) and (18) for the spectral intervals $\lambda_i = [a_i, b_i]$ are equivalent to

$$\lambda_j > \lambda_1^{q_1} \cdots \lambda_n^{q_n}, \quad \text{respectively,} \quad \lambda_j < \lambda_1^{q_1} \cdots \lambda_n^{q_n}.$$

So for our explicit candidate of H to be well defined, we have to assume that one of these two conditions is satisfied and this is equivalent to the so-called *nonresonance condition*

$$\lambda_j \cap \prod_{i=1}^{n} \lambda_i^{q_i} = \emptyset. \tag{20}$$

If condition (20) does not hold, then we have a *resonance of order* $|q|$ and the term $(0, \dots, 0, (1/q!) D^q F_k^j(0) \cdot x^q, 0, \dots, 0)$ is called *resonant.*

If the linear part A of system (10) does not depend on k, then the dichotomy spectrum $\Sigma(A)$ consists of the absolute values $\lambda_1, \dots, \lambda_n$ of the eigenvalues of A, and the nonresonance condition (20) for the spectral intervals reduces to Poincaré's discrete nonresonance condition $\lambda_j \neq \lambda_1^{q_1} \cdots \lambda_n^{q_n}$ for real eigenvalues.

Now, we prove that H is indeed a C^p equivalence, which eliminates a nonresonant term. In contrast to condition (11) in the motivation, we now allow the right-hand side of the difference equation to have nontrivial Taylor coefficients of arbitrary order.

THEOREM 9. *Consider the difference equation (10). Let $j \in \{1, \dots, n\}$ be an index and $q \in \mathbb{N}_0^n$, $2 \leq |q| \leq p$, a multi-index. Assume that the nonresonance condition (20) holds for the spectral intervals $\lambda_1, \dots, \lambda_n$. Then a local C^p equivalence H exists, which eliminates the j^{th} Taylor component $(1/q!) D^q F_k^j(0) \cdot x^q$ belonging to the multi-index q and leaves fixed all other Taylor coefficients up to order $|q|$.*

That is, equation (10) is locally C^p equivalent to an invertible difference equation

$$x_{k+1} = A_k x_k + G_k(x_k), \tag{21}$$

with zero reference solution, $[A_k + G_k] \in \mathrm{Diff}^p(B_\delta(0), [A_k+G_k](B_\delta(0)))$ *with a* $\delta = \delta(j,q,A) > 0$ *and for all* $\hat{q} \in \mathbb{N}_0^n$ *with* $1 \le |\hat{q}| \le |q|$ *and all* $i \in \{1,\dots,n\}$, $k \in \mathbb{Z}$, *the identity*

$$D^{\hat{q}}G_k^i(0) \equiv \begin{cases} D^{\hat{q}}F_k^i(0), & \text{for } \hat{q} \neq q \text{ or } i \neq j, \\ 0, & \text{for } \hat{q} = q \text{ and } i = j, \end{cases}$$

holds. There exist σ', δ' *with* $0 < \sigma' \le \sigma$, $0 < \delta' \le \delta$ *and the local near-identity* C^p *equivalence* $H : \mathbb{Z} \times B_{\sigma'}(0) \to B_\delta(0)$, $(k,x) \mapsto x + h_k(x)$ *between (10) and (21) with respect to the zero solutions is defined through (19). The inverse transformation* $H^{-1} : \mathbb{Z} \times B_{\delta'}(0) \to B_\sigma(0)$ *has the form*

$$H_k^{-1}(x) = x - h_k(x) + \psi_k(x)$$

with a continuous mapping $\psi : \mathbb{Z} \times B_{\delta'}(0) \to \mathbb{R}^N$, *which satisfies the limiting relation* $\lim_{x\to 0} \times (\psi_k(x)/\|x\|^{|q|^2-1}) = 0$ *uniformly in* $k \in \mathbb{Z}$. *Moreover, for every* $k \in \mathbb{Z}$ *one has the estimates*

$$\|H_k(x) - H_k(\bar{x})\| \le 2\|x - \bar{x}\|, \qquad \text{for all } x, \bar{x} \in B_{\sigma'}(0),$$
$$\left\|H_k^{-1}(x) - H_k^{-1}(\bar{x})\right\| \le 2\|x - \bar{x}\|, \qquad \text{for all } x, \bar{x} \in B_{\delta'}(0).$$

PROOF. The proof is divided into seven steps. In the first step, we show that h is well defined. The smoothness of H is examined in the second step. In the third step, we construct the inverse transformation H^{-1}, and in the following step the Lipschitz estimates for H and H^{-1} are shown. The explicit form of H^{-1} is elaborated in Step 5. The difference equation $x_{k+1} = A_k x_k + G_k(x_k)$ is constructed in the sixth step and it is shown that G_k coincides up to order $|q|$ with F_k except for the j^{th} component of the Taylor component belonging to the multi-index q; this component is eliminated in G_k. In the final step, it is proved that H is a local C^p equivalence between (10) and (21) with respect to the zero reference solutions.

STEP 1. The mapping $h : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is well defined and the estimate

$$\|h_k(x)\| \le C\|x\|^{|q|} \tag{22}$$

holds with a constant $C = C(j,q,A) \ge 0$.

PROOF OF STEP 1. The nonresonance condition (20) implies one of the two estimates (17) or (18) for the spectral intervals $\lambda_i = [a_i, b_i]$, $i = 1,\dots,n$. For every spectral interval $\lambda_i = [a_i,b_i]$ choose two numbers α_i and β_i with $0 < \alpha_i < a_i$ and $b_i < \beta_i$ such that $\alpha_j > \beta_1^{q_1}\cdots\beta_n^{q_n}$ if (17) holds, and $\beta_j < \alpha_1^{q_1}\dots\alpha_n^{q_n}$ if (18) holds. Then as a consequence of [2, Corollary 3.1], there exists a $K = K(j,q,A) \ge 1$ with

$$\left\|\Phi^i(k,\ell)\right\| \le K\beta_i^{k-\ell}, \qquad \text{for } k \ge \ell,$$
$$\left\|\Phi^i(k,\ell)\right\| \le K\alpha_i^{k-\ell}, \qquad \text{for } k \le \ell,$$

for $i = 1,\dots,n$. For all $k,\ell \in \mathbb{Z}$, $x \in \mathbb{R}^N$ one has

$$\begin{aligned} &\left\|\Phi^j(k,\ell+1)\frac{1}{q!}D^qF_k^j(0)\cdot[\Phi^1(\ell,k)x^1]^{q_1}\cdots[\Phi^n(\ell,k)x^n]^{q_n}\right\| \\ &\le \left\|\Phi^j(k,\ell+1)\right\|\cdot\frac{1}{q!}M\cdot\left\|\Phi^1(\ell,k)\right\|^{q_1}\cdots\left\|\Phi^n(\ell,k)\right\|^{q_n}\cdot\left\|x^1\right\|^{q_1}\cdots\left\|x^n\right\|^{q_n} \\ &\le \frac{1}{q!}MK^{|q|+1}\left\|x^1\right\|^{q_1}\cdots\left\|x^n\right\|^{q_n}\begin{cases} \beta_j^{-1}\left(\beta_j\alpha_1^{-q_1}\cdots\alpha_n^{-q_n}\right)^{k-\ell}, & \text{if } k \ge \ell, \\ \alpha_j^{-1}\left(\alpha_j\beta_1^{-q_1}\cdots\beta_n^{-q_n}\right)^{k-\ell}, & \text{if } k \le \ell, \end{cases} \end{aligned}$$

and the claim follows.

STEP 2. For every $k \in \mathbb{Z}$ the mapping $H_k : \mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto x + h_k(x)$ is C^∞.

PROOF OF STEP 2. Arguing for each component separately, the proof of this claim reduces to the verification that $h_k^j : \mathbb{R}^N \to E^j$ is C^∞ for every $k \in \mathbb{Z}$. Assume that (17) holds. Similarly to Step 1, one can show that h_k^j is differentiable and one gets for all $k \in \mathbb{Z}$ and $x, \xi \in \mathbb{R}^N$ the derivative

$$\begin{aligned} Dh_k^j(x) \cdot \xi &= \sum_{\ell=k}^{\infty} D_x \left[\Phi^j(k, \ell+1) \frac{1}{q!} D^q F_\ell^j(0) \cdot \left[\Phi^1(\ell,k)x^1\right]^{q_1} \cdots \left[\Phi^n(\ell,k)x^n\right]^{q_n} \right] \cdot \xi \\ &= \sum_{i=1,\dots,n:q_i \geq 1} q_i \sum_{\ell=k}^{\infty} \Phi^j(k,\ell+1) \frac{1}{q!} D^q F_\ell^j(0) \\ &\quad \times \left[\Phi^1(\ell,k)x^1\right]^{q_1} \cdots \left[\Phi^i(\ell,k)\xi^i\right] \cdot \left[\Phi^i(\ell,k)x^i\right]^{q_i-1} \cdots \left[\Phi^n(\ell,k)x^n\right]^{q_n} \, ds, \end{aligned}$$

and the estimate

$$\left\| Dh_k^j(x) \right\| \leq |q| C \|x\|^{|q|-1}. \tag{23}$$

Now, $Dh_k^j(x)$ is again differentiable and the second derivative operates on $\xi, \eta \in \mathbb{R}^N$ through

$$\begin{aligned} D^2 h_k^j(x) \cdot \xi \cdot \eta = &\sum_{i,m=1,\dots,n:q_i,q_m \geq 1,\, i \neq m} q_i q_m \sum_{\ell=k}^{\infty} \Phi^j(k,\ell+1) \frac{1}{q!} D^q F_\ell^j(0) \\ &\times \left[\Phi^1(\ell,k)x^1\right]^{q_1} \cdots \left[\Phi^i(\ell,k)\xi^i\right] \cdot \left[\Phi^i(\ell,k)x^i\right]^{q_i-1} \\ &\cdots \left[\Phi^m(\ell,k)\eta^m\right] \cdot \left[\Phi^m(\ell,k)x^m\right]^{q_m-1} \cdots \left[\Phi^n(\ell,k)x^n\right]^{q_n} \\ &+ \sum_{i=1,\dots,n:q_i \geq 2} q_i(q_i-1) \sum_{\ell=k}^{\infty} \Phi^j(k,\ell+1) \frac{1}{q!} D^q F_\ell^j(0) \cdot \left[\Phi^1(\ell,k)x^1\right]^{q_1} \\ &\cdots \left[\Phi^i(\ell,k)\xi^i\right] \cdot \left[\Phi^i(\ell,k)\eta^i\right] \cdot \left[\Phi^i(\ell,k)x^i\right]^{q_i-2} \cdots \left[\Phi^n(\ell,k)x^n\right]^{q_n}, \end{aligned}$$

and implies the estimate

$$\left\| D^2 h_k^j(x) \right\| \leq |q|^2 C \|x\|^{|q|-2}. \tag{24}$$

Mathematical induction yields the existence of the derivatives $D^m h_k^j : \mathbb{R}^N \to L^m(\mathbb{R}^N; E^j)$ for $k \in \mathbb{Z}$ and $m = 1, \dots, p$. For $m > p$, the mapping $D^m h_k^j$ is zero, and for this reason h_k^j and, therefore also, H_k is C^∞.

STEP 3. There exist σ_0', δ_0', δ_0 with $0 < \sigma_0' \leq \sigma_0 := \min\{\sigma, (2|q|C)^{-1/(|q|-1)}\}$, $0 < \delta_0' \leq \delta_0$, such that for arbitrary $k \in \mathbb{Z}$,

$$H_k : B_{\sigma_0'}(0) \to H_k\left(B_{\sigma_0'}(0)\right) \subset B_{\delta_0}(0)$$

is a C^p diffeomorphism and such that a C^p diffeomorphism

$$H_k^{-1} : B_{\delta_0'}(0) \to H_k^{-1}\left(B_{\delta_0'}(0)\right) \subset B_{\sigma_0}(0)$$

exists and the identities

$$H_k^{-1}(H_k(x)) = x \qquad \text{and} \qquad H_k(H_k^{-1}(x)) = x$$

are valid for all $x \in B_{\sigma_0'}(0)$, respectively, $x \in B_{\delta_0'}(0)$.

PROOF OF STEP 3. With (24), [4, Proposition 2.5.6, pp. 119–121] implies the claim.

STEP 4. For every $k \in \mathbb{Z}$ we have

$$\|H_k(x) - H_k(\bar{x})\| \leq 2\|x - \bar{x}\|, \qquad \text{for } x, \bar{x} \in B_{\sigma_0'}(0), \tag{25}$$

$$\left\|H_k^{-1}(x) - H_k^{-1}(\bar{x})\right\| \leq 2\|x - \bar{x}\|, \qquad \text{for } x, \bar{x} \in B_{\delta_0'}(0). \tag{26}$$

PROOF OF STEP 4. First we prove the Lipschitz continuity of $h_k : B_{\sigma'_0}(0) \to \mathbb{R}^N$. Estimate (23) for the derivative of h_k^j implies for all $k \in \mathbb{Z}$ and $x \in B_{\sigma'_0}(0)$ the inequality $\|Dh_k(x)\| \le 1/2$, and hence,

$$\|h_k(x) - h_k(\bar{x})\| \le \frac{1}{2}\|x - \bar{x}\|, \tag{27}$$

which implies (25). To prove the Lipschitz estimate for H^{-1} we use (27) to show for $k \in \mathbb{Z}$ and $y, \bar{y} \in B_{\sigma'_0}(0)$ the estimate

$$\|y - \bar{y}\| - \frac{1}{2}\|y - \bar{y}\| \le \|y - \bar{y}\| - \|h_k(y) - h_k(\bar{y})\|,$$

and it follows that

$$\frac{1}{2}\|y - \bar{y}\| \le \|H_k(y) - H_k(\bar{y})\|.$$

Step 3 implies for $x, \bar{x} \in B_{\delta'_0}(0)$ the identities $H_k(H_k^{-1}(x)) = x$ and $H_k(H_k^{-1}(\bar{x})) = \bar{x}$ and with $y := H_k^{-1}(x)$, $\bar{y} := H_k^{-1}(\bar{x})$ and one gets estimate (26).

STEP 5. For $k \in \mathbb{Z}$, the mapping $H_k^{-1} : B_{\delta'_0}(0) \to \mathbb{R}^N$ is of the form

$$H_k^{-1}(x) = x - h_k(x) + \psi_k(x)$$

with a continuous mapping $\psi_k : B_{\delta'_0}(0) \to \mathbb{R}^N$, which satisfies

$$\lim_{x\to 0} \frac{\|\psi_k(x)\|}{\|x\|^{|q|^2-1}} = 0, \qquad \text{uniformly in } k \in \mathbb{Z}. \tag{28}$$

PROOF OF STEP 5. For $k \in \mathbb{Z}$ the inverse of H_k can be given explicitely with the Neumann-series (see, e.g., [4, p. 117])

$$H_k^{-1}(x) := \sum_{i=0}^{\infty}(-h_k)^i(x), \qquad \text{for } x \in B_{\delta'_0}(0).$$

With the mapping $\psi_k : B_{\delta'_0}(0) \to \mathbb{R}^N$, $x \mapsto \sum_{i=2}^{\infty}(-h_k)^i(x)$ one has for arbitrary $x \in B_{\delta'_0}(0)$ the identity

$$H_k^{-1}(x) = x - h_k(x) + \psi_k(x).$$

To show the limiting relation (28) one has to apply twice estimate (22) together with (27) to get for all $x \in B_{\delta'_0}(0)$ and $i \ge 2$ the following inequalities:

$$\begin{aligned}\left\|(-h_k)^i(x)\right\| &\le C\left\|(-h_k)^{i-1}(x)\right\|^{|q|} \le C^2\left\|(-h_k)^{i-2}(x)\right\|^{|q|^2}\\ &\le C^2\left(\frac{1}{2}\right)^{(i-2)|q|^2}\|x\|^{|q|^2},\end{aligned}$$

and this implies

$$\|\psi_k(x)\| \le \frac{C^2}{1-(1/2)^{|q|^2}}\|x\|^{|q|^2},$$

and therefore, the limiting relation (28).

STEP 6. Define $\delta := \min\{\delta'_0, \sigma/2, \sigma'_0/2\tilde{M}\}$, $\delta' := \delta$, and $\sigma' := \min\{\sigma'_0, \delta/2\}$ where $\tilde{M} > 0$ is a constant with

$$\|A_k + DF_k(x)\| \le \tilde{M}, \qquad \text{for } x \in B_\sigma(0).$$

If v is a solution of (10) in $B_{\sigma'}(0)$, then $H_k(v_k)$ is a solution of $x_{k+1} = \tilde{G}_k(x_k)$ with $\tilde{G}_k \in \mathrm{Diff}^p(B_\delta(0), \tilde{G}_k(B_\delta(0)))$ and

$$\tilde{G}_k(x_k) = H_{k+1}\left(A_kH_k^{-1}(x_k) + F_k\left(H_k^{-1}(x_k)\right)\right).$$

If w is a solution of $x_{k+1} = \tilde{G}_k(x_k)$ in $B_{\delta'}(0)$, then $H_k^{-1}(w_k)$ is a solution of (10). Moreover, $\tilde{G}_k$ has the form $\tilde{G}_k(x_k) = A_k x_k + G_k(x_k)$ and for the components of the Taylor coefficients of $G_k : B_\delta(0) \to E^1 \times \cdots \times E^n = \mathbb{R}^N$ the following identities hold:

$$D^{\hat{q}} G_k^i(0) \equiv \begin{cases} D^{\hat{q}} F_k^i(0), & \text{for } \hat{q} \neq q \text{ or } i \neq j, \\ 0, & \text{for } \hat{q} = q \text{ and } i = j, \end{cases}$$

for all $\hat{q} \in \mathbb{N}_0^n$ with $|\hat{q}| \leq |q|$ and all $i \in \{1, \dots, n\}$.

PROOF OF STEP 6. The Lipschitz estimates (25),(26) and $\|[A_k x + F_k(x)] - [A_k \bar{x} + F_k(\bar{x})]\| \leq \tilde{M}\|x - \bar{x}\|$ for $x, \bar{x} \in B_\sigma(0)$ imply that $\tilde{G}_k$ is a composition of C^p diffeomorphisms on $B_\delta(0)$.

Let $v_k \in B_{\sigma'}(0)$ be a solution of (10). Then $H_k(v_k) \in B_\delta(0)$ is a solution of $x_{k+1} = \tilde{G}_k(x)$, and similarly if $w_k \in B_{\delta'}(0)$ is a solution of $x_{k+1} = \tilde{G}_k(x)$, then $H_k^{-1}(w_k) \in B_\sigma(0)$ is a solution of (10).

Now, we write the components $\tilde{G}^i : \mathbb{Z} \times B_\delta(0) \to E^i$, $i = 1, \dots, n$, of the right-hand side of the transformed difference equation as a sum of terms up to order $|q|$ and terms of higher order. The most important relation to do this is the following connection between h_{k+1} and h_k. For all $k \in \mathbb{Z}$ and $x \in B_\sigma(0)$, one has the identity

$$h_{k+1}(A_k x) = A_k h_k(x) - \left(0, \dots, 0, \frac{1}{q!} D^q F_k^j(0) \cdot x^q, 0, \dots, 0\right).$$

Taylor-expanding $\tilde{G}_k$ near $x = 0$, one gets the identity

$$\begin{aligned} \tilde{G}_k(x) &= A_k x - A_k h_k(x) + F_k(x) + h_{k+1}(A_k x) + o\left(\|x\|^{|q|}\right) \\ &= A_k x + F_k(x) - \left(0, \dots, 0, \frac{1}{q!} D^q F_k^j(0) \cdot x^q, 0, \dots, 0\right) + o\left(\|x\|^{|q|}\right), \end{aligned}$$

and the claim follows.

STEP 7. The mapping $H : \mathbb{Z} \times B_{\sigma'}(0) \to B_\delta(0)$ is a C^p equivalence between systems (10) and (21) with respect to the zero solutions with the inverse transformation $H^{-1} : \mathbb{Z} \times B_{\delta'}(0) \to B_\sigma(0)$.

PROOF OF STEP 7. We only have to verify the properties of the definition of a C^p equivalence. Use Steps 3, 4, and 6.

COROLLARY 10. *Let A_k and F_k be periodic in k with a period $\kappa \geq 1$; i.e., for all $k \in \mathbb{Z}$, the identities*

$$A_{k+\kappa} = A_k \qquad \text{and} \qquad F_{k+\kappa} = F_k$$

hold. Then H from Theorem 9 is also periodic in k with period κ. Especially, if (10) is autonomous, then H is independent of k.

PROOF. For $\ell \in \mathbb{Z}$ and $\xi \in \mathbb{R}^N$, the mapping $\Phi(k + \kappa, \ell + \kappa)\xi$ is the unique solution of the initial value problem $x_{k+1} = A_{k+\kappa} x_k$, $x(\ell) = \xi$ and also $\Phi(k, \ell)\xi$ is the unique solution of the same initial value problem $x_{k+1} = A_k x_k$, $x(\ell) = \xi$, and therefore, the identity $\Phi(k + \kappa, \ell + \kappa) = \Phi(k, \ell)$ holds for all $k, \ell \in \mathbb{Z}$. Moreover, the κ-periodicity of F in k implies the relation $D^q F_{k+\kappa}^j(0) = D^q F_k^j(0)$, and in case of (17) one gets the equality

$$h_{k+\kappa}^j(x) = \sum_{\ell=k+\kappa}^{\infty} \Phi^j(k + \kappa, \ell + 1) D^q F_\ell^j(0) \cdot \left[\Phi^1(\ell, k + \kappa) x^1\right]^{q_1} \cdots \left[\Phi^n(\ell, k + \kappa) x^n\right]^{q_n} = h_k^j(x),$$

and the claim follows.

Now, it is easy to get our main result on normal forms. Combining the three steps, we immediately get the following theorem.

THEOREM 11. NORMAL FORM. *Consider a difference equation*

$$x_{k+1} = f_k(x) \tag{29}$$

together with a reference solution $v^0 : \mathbb{Z} \to \mathbb{R}^N$. Assume that

(A) *a neighbourhood $U_\varepsilon(v^0)$ is contained in D_f for some $\varepsilon > 0$,*
(B) *$f_k \in \mathrm{Diff}^p(B_\varepsilon(v_k), f_k(B_\varepsilon(v_k)))$ for a $p \geq 2$,*
(C) *the linearization $x_{k+1} = Df_k(v_k^0)x_k$ of (29) along v^0 has bounded growth, and therefore, [1] the dichotomy spectrum consists of n, $1 \leq n \leq N$, compact intervals $\lambda_i = [a_i, b_i]$, $i = 1, \dots, n$, and*
(D) *higher-order terms of f in x along v^0 are uniformly bounded in k; i.e., there is an $M > 0$ such that*

$$\left\| D^j f_k\left(v_k^0\right)\right\| \leq M, \qquad \text{for all } k \in \mathbb{Z} \text{ and all } j \in \{2, \dots, p\}.$$

Then (29) is locally C^p equivalent to a difference equation

$$x_{k+1} = g_k(x_k) \tag{30}$$

with zero reference solution and (30) is in normal form; i.e., it holds that

(A′) *$g_k \in \mathrm{Diff}^p(B_\delta(0), g_k(B_\delta(0)))$ for some $\delta > 0$,*
(B′) *the linearization $x_{k+1} = Dg_k(0)x_k$ of (30) along the zero solution has the same dichotomy spectrum as the linearization of (29) along v^0 and additionally is block-diagonalized, each block corresponds to a spectral interval λ_i, and*
(C′) *all nontrivial Taylor components of g of order 2 to p are resonant; i.e., for every $j \in \{1, \dots, n\}$ and $q \in \mathbb{N}_0^n$, $2 \leq |q| \leq p$ with*

$$\lambda_j \cap \prod_{i=1}^{n} \lambda_i^{q_i} = \emptyset,$$

we have $D^q g_k^j(0) = 0$ for $k \in \mathbb{Z}$.

We apply the normal form theorem to an example. It is the same example which we used at the beginning to explain Poincaré's normal form theory. Therefore, consider again

$$\begin{aligned} x_{k+1} &= 2x_k, \\ y_{k+1} &= \lambda y_k + x_k^2, \end{aligned}$$

with $\lambda \in (0, \infty)$. The spectral intervals of the first and second equations are the one-point sets $\lambda_1 = \{2\}$ and $\lambda_2 = \{\lambda\}$, respectively, consisting of the eigenvalues of the linear part. We want to eliminate the quadratic term x_k^2 in the second component of the difference equation, i.e., $j = 2$ and $q = (2, 0)$. For $\lambda < 4$ the condition $\lambda_2 < (2\lambda_1 + 0\lambda_2)$ holds, so we have no resonance and get

$$\begin{aligned} h_k^2(x, y) &= -\sum_{\ell=-\infty}^{k-1} \Phi^2(k, \ell+1) \cdot \left[\Phi^1(\ell, k)x\right]^2 \\ &= -\sum_{\ell=-\infty}^{k-1} \lambda^{k-\ell-1} \cdot 4^{\ell-k} \cdot x^2 = \frac{1}{\lambda - 4}x^2, \end{aligned}$$

and therefore, H is (we get the same h_2 for $\lambda < 4$)

$$H_k(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ h_k^2(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y + \dfrac{1}{\lambda - 4}x^2 \end{pmatrix}.$$

This is the same result as we calculated above with Poincaré's method.

Table 1.

Autonomous Poincaré Theory	Nonautonomous Theory
$x_{k+1} = Ax_k + f(x_k)$ Linear part $A \in \mathbb{R}^{N\times N}$ in block diagonal form $A = \mathrm{diag}(A^1,\dots,A^n)$. Eigenvalues $\mu_1,\dots,\mu_N$ of A or *eigenvalue real parts* $\lambda_1,\dots,\lambda_n$ of blocks $A^1,\dots,A^n$.	$x_{k+1} = A_k x_k + f_k(x_k)$ Linear part $A_k \in \mathbb{R}^{N\times N}$ in block diagonal form $A_k = \mathrm{diag}(A_k^1,\dots,A_k^n)$. *Compact dichotomy spectrum* $\lambda_1 = [a_1,b_1],\dots,\lambda_n = [a_n,b_n]$ of blocks $A_k^1,\dots,A_k^n$.
Elimination of a Taylor coefficient: *algebraically* Solve a linear homological equation $L^{j,q}h^{j,q} = D^q F^j(0)$ with a linear operator $L^{j,q}$ on a finite-dimensional space of monomials. ⇝ Solve a linear equation.	Elimination of a Taylor coefficient: *analytically* Solve a linear difference equation $x_{k+1}^j = A_k^j x_k^j - D^q F_k^j(0)\cdot\Phi(k,m)^q$ where the solution is unique with a prescribed growth rate. ⇝ Solve a linear difference equation.
Nonresonance condition $\lambda_j - \lambda_1^{q_1}\cdots\lambda_n^{q_n} \neq 0$ $j=1,\dots,n,\ q_i \in \mathbb{N}_0,\ 2 \le \lvert q\rvert \le p.$	*Nonresonance condition (new)* $\lambda_j \cap \lambda_1^{q_1}\cdots\lambda_n^{q_n} = \emptyset$ $j=1,\dots,n,\ q_i \in \mathbb{N}_0,\ 2 \le \lvert q\rvert \le p.$

4. CONCLUSION

We extended Poincaré's normal form theory for autonomous difference equations to the class of nonautonomous differential equations in the vicinity of an arbitrary reference solution. Poincaré's nonresonance condition for the eigenvalues of the linearization is generalized to a new nonresonance condition for the spectral intervals. A comparison of the new normal form theory with Poincaré's method is contained in Table 1.

Normal forms traditionally are an important tool in bifurcation theory (see, e.g., [5–8]). We hope to stimulate the development of a nonautonomous bifurcation theory for difference equations.

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