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# SOLUTION OF NON-INTEGER ORDER DIFFERENTIAL EQUATIONS VIA THE ADOMIAN DECOMPOSITION METHOD

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**Abstract**—This paper shows that the Adomian decomposition method can be applied to solve ordinary differential equations of non-integer order.

## 1. INTRODUCTION

The Adomian [1] method of decomposition has been successfully applied to solve large classes of both linear and nonlinear, ordinary, partial and integro-differential equations (e.g., see [2,3]). The purpose of this paper is to develop further the applicability of the decomposition method to solve extraordinary differential equations of fractional order.

The fractional order derivatives are closely connected with the Abel transform which, therefore, also arises in fields such as conduction of heat in solids or transmission of electrical signals through cables, where fractional order derivatives are encountered (see [4]). For definitions of fractional integral operators, fractional order derivatives and their properties (see [5–7]). For existence, uniqueness and stability of solution of extraordinary differential equations (see [5]).

For the sake of completeness, we give definitions and state properties which we need in this work.

The fractional integral operator $\overset{x}{\underset{a}{I}}{}^{\alpha} f$ is defined as:

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha+n)} \frac{d^n}{dx^n} \left( \int_a^x (x-t)^{\alpha+n-1} f(t)\, dt \right), \tag{1.1}$$

where $n$ is the smallest positive integer such that $\alpha + n > 0\,(n = 0$, if $\alpha > 0), \Gamma$ is the Gamma function.

The fractional derivative of $f(x)$ of order $\alpha$ is defined as:

$$f^{(\alpha)}(x) = \overset{x}{\underset{a}{I}}{}^{-\alpha} f. \tag{1.2}$$

If $\alpha > 0$, $\beta > -1$, $x > a$, then,

$$\overset{x}{\underset{a}{I}}{}^{\alpha}(t-a)^{\beta} = \begin{cases} \frac{(x-a)^{\alpha+\beta}\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}, & \text{where} \quad \alpha+\beta \neq \text{negative integer} \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

For $\alpha > 0$, $0 < \alpha < 1$, we have:

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \overset{t}{\underset{a}{I}}{}^{-\alpha} f(x) = f(x) - \frac{C(x-a)^{\alpha-1}}{\Gamma(\alpha)}, \qquad \text{for } x > a. \tag{1.4}$$

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## 2. MAIN RESULT

Consider the extraordinary differential equation:

$$\frac{d^{1/2}y}{dx^{1/2}} + y = 0. \tag{2.1}$$

This implies:

$$\overset{x}{\underset{0}{I}}{}^{-1/2}\, y = -y. \tag{2.2}$$

Now, for comparison, we first solve (2.1) (see [7]), without using the decomposition method to show the effectiveness and advantage of the decomposition method.

Using (1.1) and (1.4), equation (2.2) gives:

$$\frac{d}{dx}\left(y(x) - Cx^{-1/2}\right) = y.$$

This implies:

$$y(x) = C_1 e^x + C_2 e^x \int_0^x t^{-3/2} e^{-t}\, dt. \tag{2.3}$$

The incomplete Gamma function

$$\int_0^x t^{-3/2}\, e^{-t}\, dt = -2\sqrt{\pi}\ \mathrm{erf}\,(\sqrt{x}\,) - \frac{2e^{-x}}{\sqrt{x}}.$$

Hence, (2.3) becomes:

$$y(x) = C_1 e^x - 2\sqrt{\pi}\, C_2 e^x\ \mathrm{erf}\,(\sqrt{x}\,) - \frac{2C_2\sqrt{\pi}}{\sqrt{\pi\, x}}. \tag{2.4}$$

Since (2.4) satisfies (2.1), we get $C_1 = 2C_2\sqrt{\pi}$. Hence,

$$y(x) = C_1 e^x\ \mathrm{erf}\ c(\sqrt{x}\,) - \frac{C_1}{\sqrt{x\pi}}, \qquad \text{where erf } c = 1 - \text{erf}. \tag{2.5}$$

Thus, (2.5) is the required solution of equation (2.1).

Now, we solve (2.1) by using the decomposition method. Using (1.4), equation (2.1) becomes:

$$y(x) = Cx^{-1/2} - \overset{x}{\underset{0}{I}}{}^{1/2}\, y.$$

Let

$$y(x) = \sum_0^\infty y_n(x), \qquad \text{where } y_0(x) = Cx^{-1/2} = \frac{C\sqrt{\pi}}{\sqrt{x\pi}}$$

$$y_1(x) = -\overset{x}{\underset{0}{I}}{}^{1/2}\, y_0(x)$$

$$y_2(x) = -\overset{x}{\underset{0}{I}}{}^{1/2}\, y_1(x)$$

$$\vdots$$

$$y_{n+1}(x) = -\overset{x}{\underset{0}{I}}{}^{1/2}\, y_n(x).$$

Applying (1.3), we get:

$$\begin{aligned}
y_1(x) &= -C\sqrt{\pi} \\
y_2(x) &= \frac{C\sqrt{\pi}\,x^{1/2}}{\Gamma(3/2)} \\
y_3(x) &= -\frac{C\sqrt{\pi}\,x}{\Gamma(2)} \\
y_4(x) &= \frac{C\sqrt{\pi}\,x^{3/2}}{\Gamma(5/2)} \\
&\vdots \\
y_{2n}(x) &= \frac{C\sqrt{\pi}\,x^{(2n-1)/2}}{\Gamma((2n+1)/2)} \\
y_{2n+1}(x) &= -\frac{C\sqrt{\pi}x^n}{\Gamma(n+1)}.
\end{aligned}$$

Hence, the solution of equation (2.1) is:

$$y(x) = C_1\left[\sum_0^\infty \frac{(-1)^n\,x^{n/2}}{\Gamma(n/2+1)} - \frac{1}{\sqrt{\pi x}}\right], \qquad \text{where } C_1 = -C\sqrt{\pi}. \tag{2.6}$$

It can easily be verified that (2.6) satisfies (2.1). Also, by using series expansion:

$$\operatorname{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\left[\sqrt{x} - \frac{x^{3/2}}{3} + \frac{1}{2!}\frac{x^{5/2}}{5} - \frac{1}{3!}\frac{x^{7/2}}{7} + \cdots\right],$$

one can show that (2.5) and (2.6) are the same. This completes the proof.

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