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Components of Auslander–Reiten Quivers with Only Preprojective Modules

FLÁVIO ULHOA COELHO

Departamento de Matemática, IME, Universidade de São Paulo, CP 20570 São Paulo, São Paulo, CEP 01498, Brazil

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Let A be an Artin algebra and ${}_A\Gamma$ its Auslander–Reiten quiver. Our main aim is to characterize the components of ${}_A\Gamma$ which contain only preprojective A-modules using intrinsic properties of translation quivers

1. INTRODUCTION

In this article A will denote a connected basic Artin algebra, that is, an artinian ring with 1 such that its centre contains an artinian subring over which A is a finitely generated module. Denote by A-mod the category of all finitely generated left A-modules and by A-ind the subcategory of A-mod with one representative of each isomorphism class of indecomposable A-modules. All modules and maps are in A-mod.

We use here the notions of sink and source maps, irreducible maps, Auslander–Reiten sequences, and quivers. For their definition and basic properties we refer to [Ri] (see also [AR3, 6]). The Auslander–Reiten quiver of an algebra A is denoted by ${}_A\Gamma$. All components of ${}_A\Gamma$ are connected. We denote the Auslander–Reiten translate (respectively the translate inverse) of X by τX (respectively by $\tau^- X$). The modules in the *τ-orbit* of a given indecomposable module X are those of the form $\tau^n X$, for $n \in \mathbb{Z}$. An oriented cycle is a chain of irreducible maps through indecomposable modules $X_0 \to X_1 \to \cdots \to X_t$ where $X_0 = X_t$.

A component $\mathbf{C}$ of ${}_A\Gamma$ is called a *τ-preprojective component* provided that (i) all modules in $\mathbf{C}$ lie in a τ-orbit of projectives, and (ii) there are no oriented cycles in $\mathbf{C}$. In the literature τ-preprojective components are usually referred as preprojective components. We choose to use the term τ-preprojective components mainly to avoid misunderstanding. In this

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article the term preprojective modules is always used in the following sense, as introduced by Auslander and Smalø [AS].

According to [AS] A-ind has a unique partition $\mathbf{P}_i(A)$, $i \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, called the *preprojective partition*, satisfying the following:

(a) A-ind is equal to the disjoint union of $\mathbf{P}_i(A)$, $i \in \mathbb{N}_\infty$.

(b) For each $j < \infty$, $\mathbf{P}_j(A)$ has the following property: for all $X \in \mathbf{P}_i(A)$ with $i \geqslant j$ there are an $Y \in \operatorname{add} \mathbf{P}_j(A)$ and an epimorphism from Y to X. Moreover $\mathbf{P}_j(A)$ is minimal with respect to this property and it is finite.

An A-module M is called *preprojective* if all indecomposable summands of M belong to the union of $\mathbf{P}_i(A)$, $i < \infty$. Given an i, $0 \leqslant i \leqslant \infty$, and $X \in A$-mod we denote by $t_i X$ the submodule of X generated by the image of all morphisms from all indecomposable modules in the union of $\mathbf{P}_j(A)$, $j \geqslant i$.

An indecomposable A-module Y is called an *irreducible predecessor* (respectively an *irreducible successor*) of $X \in A$-ind if there is a chain of irreducible maps $X_0 \to X_1 \to \cdots \to X_t$, $t \geqslant 0$, with $X_0 = Y$ and $X_t = X$ (respectively with $X_0 = X$ and $X_t = Y$). We denote by $\operatorname{Pr}(X)$ (respectively by $\operatorname{Sc}(X)$) the set of all nonisomorphic indecomposable irreducible predecessors (respectively successors) of X. Note that the module X is included in both sets $\operatorname{Pr}(X)$ and $\operatorname{Sc}(X)$.

1.1. DEFINITION. A component $\mathbf{C}$ of ${}_A\Gamma$ containing a preprojective A-module is called a *π-component* if for every preprojective module $X \in \mathbf{C}$, $\operatorname{Pr}(X)$ contains only preprojectives.

Our main aim in this article is to give other descriptions of π-components. Some of them have already appeared implicitly in the work of Auslander and Smalø [AS]. Now we state our main result. We say that some property holds for *almost all modules* if it holds for all but a finite number of the nonisomorphic indecomposable modules.

1.2. THEOREM. *Let A be an algebra. The following are equivalent for a component $\mathbf{C}$ of ${}_A\Gamma$ containing a preprojective A-module.*

(a) $\mathbf{C}$ *is a π-component.*

(b) $t_\infty X = 0$ *for every preprojective module X in* $\mathbf{C}$.

(c) $\mathbf{C}$ *contains only preprojective modules.*

(d) (i) *Almost all modules in* $\mathbf{C}$ *lie in τ-orbits of projectives.*
(ii) *Only a finite number of modules in* $\mathbf{C}$ *belong to oriented cycles.*

(e) *For every module $X \in \mathbf{C}$, the set* $\operatorname{Pr}(X)$ *is finite.*

Note that condition (d) above generalises that defining τ-preprojective components.

If A is an algebra of finite representation type then all A-modules are preprojectives. Therefore all five conditions of Theorem 1.2 are clearly satisfied and there is nothing to prove. The proof of Theorem 1.2 for algebras of infinite representation type will be given in Sections 3 to 8. In Section 2 we recall some basic results and established some notations.

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2. Preliminaries

In this section we recall some results and establish some further notations. We keep those already established in the Introduction.

If $X \in A$-ind, σX (respectively, $\sigma^- X$) denotes the domain of a sink map of X (respectively, source map of X). In particular an Auslander–Reiten sequence starting at $X \in A$-ind, if defined, has the form $0 \to X \to \sigma^- X \to \tau^- X \to 0$, while one ending at $X \in A$-ind, if defined, has the form $0 \to \tau X \to \sigma X \to X \to 0$. A finite chain of irreducible maps is also referred as a path.

We do not distinguish between a component of ${}_A\Gamma$, the module class defined by this component, and the full subcategory of A-mod defined by this module class.

Given a functor F from A-mod to Ab, the category of all abelian groups, the *length of* F, $l(F)$, is defined to be the sum $\sum l(F(X))$ taken over all non-isomorphic indecomposable A-modules X with $F(X) \neq 0$ and where $l(F(X))$ is the length of $F(X)$ as an End X-module. Of special importance in our work is the length of the functor $(-, X)$ for $X \in A$-ind. The length $l((-, X))$ is finite for a module $X \in A$-ind if and only if there are at most finitely many indecomposable modules Y with $(Y, X) \neq 0$. We need the following result due to Auslander–Reiten.

2.1. Proposition (Corollary 1.8 in [AR6]). *Let X and Y be in A-ind and assume $l((-, X)) < \infty$.*

(a) *If* $\mathrm{Hom}_A(Y, X) \neq 0$ *then there is a path through indecomposable A-modules from Y to X with nonzero composition.*

(b) *There is a path through indecomposable A-modules from a projective to X with nonzero composition.*

We now establish a result that, though straightforward, will be useful in the proof of Theorem 1.2.

2.2. PROPOSITION. *Let Y and X in A-ind and $Y \notin \mathrm{Pr}(X)$. If $\mathrm{Hom}_A(Y, X) \neq 0$ then*

(a) *for each $n \geqslant 1$ there are indecomposable A-modules $Y = Y_0, ..., Y_n$ with irreducible maps α_i: $Y_{i-1} \to Y_i$ for $i = 1, ..., n$ and $f \in \mathrm{Hom}_A(Y_n, X)$ with $f\alpha_n \cdots \alpha_1 \neq 0$.*

(b) *for each $n \geqslant 1$ there are indecomposable A-modules $X_n, ..., X_0 = X$ with irreducible maps β_i: $X_i \to X_{i-1}$ for $i = 1, ..., n$ and $g \in \mathrm{Hom}_A(Y, X_n)$ with $\beta_1 \cdots \beta_n g \neq 0$.*

Proof. We prove part (a) by induction on $n \geqslant 1$. Part (b) follows easily by dualizing the argument. Suppose $n = 1$ and consider $g \in \mathrm{Hom}_A(Y, X)$. Since $Y \neq X$, g is not a split monomorphism; in particular, Y is not a simple injective. If $(h_1, ..., h_r)^t : Y \to \bigoplus E_i$ is a source map for Y then there is a morphism $(f_1, ..., f_r) : \bigoplus E_i \to X$ such that $(f_1, ..., f_r)(h_1, ..., h_r)^t = g$. In particular there is a summand of $\bigoplus E_i$, say E_1, such that $f_1 h_1 \neq 0$ and the first step of the induction is proved since h_1 is an irreducible map. For the induction step the argument is rather similar and we leave it for the reader. ∎

Now let $\{\mathbf{P}_i(A),\ i \in \mathbb{N}_\infty\}$ be the preprojective partition of A. It is clear that $\mathbf{P}_0(A)$ is the set of the indecomposable projectives. If $X \in \mathbf{P}_n(A)$, $0 \leqslant n \leqslant \infty$, then we define $\pi(X)$ to be n. In the case $n < \infty$, we say that X is *preprojective of level n* and $\mathscr{P}(A)$ denotes $\{X \in A\text{-ind: } 0 \leqslant \pi(X) < \infty\}$. Following [AS], $\mathbf{P}^m(A)$ denotes $\{X \in A\text{-ind}: \pi(X) \leqslant m\}$. When it is clear which algebra A we are working with, we also use the simpler notations $\mathbf{P}_n$, $\mathscr{P}$, and $\mathbf{P}^m$ instead of $\mathbf{P}_n(A)$, $\mathscr{P}(A)$, and $\mathbf{P}^m(A)$.

We need the following results throughout the article. Note that $t_\infty X$ was originally denoted by X_0 in [AS].

2.3. PROPOSITION (Theorem 5.1 and Corollary 9.3 in [AS]). *Let $X \in A$-ind.*

(1) *$X \in \mathscr{P}(A)$ if and only if $t_\infty X \neq X$.*

(2) *If $X \in \mathscr{P}(A)$ then there are at most finitely many indecomposable A-modules, all of them preprojectives, admitting morphisms into X with image not contained in $t_\infty X$.*

(3) *$t_\infty X = 0$ if and only if $l((-, X)) < \infty$.*

2.4. PROPOSITION (Proposition 1.2 in [Co]). *Let* $X \in \mathscr{P}(A)$ *and* $Y \in A$-ind. *If* $\operatorname{rad}(Y, X)$ *contains a morphism with image not contained in* $t_\infty X$ *then there exists a path from* Y *to* X *with nonzero composite.*

Now let $X \in \mathscr{P}(A)$ and consider its projective cover $(p_1, ..., p_t)$: $\bigoplus P_i \to X$. Since this is an epimorphism there is an i such that Im p_i is not contained in $t_\infty X$. According to Proposition 2.4 there is a path from P_i to X. In particular all components of ${}_A\Gamma$ containing a preprojective contain also a projective.

We start now the proof of our main result. Unless otherwise stated the algebra A is assumed to be of infinite representation type.

3. EQUIVALENCE OF CONDITIONS (a), (b), AND (c) OF THEOREM 1.2

In this section we establish the equivalence of conditions (a), (b), and (c) of Theorem 1.2. The equivalence of conditions (b) and (c) were proved by Auslander and Smalø for the case where $\mathbf{C}$ is the union of all components containing preprojective modules. In fact their proof can be adapted to components. For the convenience of the reader we give a simplified version of the equivalence of these conditions and (a) since this contains useful informations needed on π-components. We will use the following lemma.

3.1. LEMMA (Lemma 2.1 in [Co]). *If there is an irreducible map* $X \to Y$ *with* $\pi(X) < \pi(Y) - 1$ *then* $\pi(\tau Y) < \pi(X)$.

3.2. THEOREM. *Let* $\mathbf{C}$ *be a connected component of* ${}_A\Gamma$ *containing a preprojective module. The following are equivalent*:

(a) $\mathbf{C}$ *is a* π*-component.*

(b) $t_\infty X = 0$ *for each preprojective* A*-module* X *in* $\mathbf{C}$.

(c) $\mathbf{C}$ *contains only preprojective modules.*

Proof. (a) $\Rightarrow$ (b) Assume $\mathbf{C}$ is a π-component and let $X \in \mathbf{C} \cap \mathscr{P}(A)$ with $t_\infty X \neq 0$. Let $M \in \mathbf{P}_\infty$ be an indecomposable summand of $t_\infty X$. Consider a source map of M, $g: M \to \sigma^- M$. Note that the natural inclusion $i: M \to X$ lifts through g; that is, there is an $f: \sigma^- M \to X$ such that $fg = i$. We claim that Imf is not contained in $t_\infty X$. Otherwise, since the inclusion $i': M \to t_\infty X$ is a split monomorphism, we have that g is also a split monomorphism, which contradicts the fact that g is a source map, and the claim is proved. Moreover, there exists an indecomposable summand Y of $\sigma^- M$ and $f': Y \to X$ with Imf' not contained in $t_\infty X$. If $f' \in \operatorname{rad}(Y, X)$ then according to Proposition 2.4 there is a path from Y to X and so $M \in \Pr(X)$. On the other hand, if f' is an isomorphism it is clear that

$M \in \Pr(X)$. Therefore $\Pr(X)$ contains a nonpreprojective module M, which contradicts our hypothesis on **C**.

(b) $\Rightarrow$ (c) Let **C** be a connected component containing a preprojective and satisfying condition (b). To prove that **C** has only preprojective modules we prove that for each irreducible map $X \to Y$ with X and Y in **C**, $X \in \mathscr{P}(A)$ if and only if $Y \in \mathscr{P}(A)$. Suppose first that $g: X \to Y$ is an irreducible map with X and Y in **C** and $Y \in \mathscr{P}(A)$. If g is an epimorphism then clearly $\pi(X) < \pi(Y)$. Otherwise g is a monomorphism and since $t_\infty Y = 0$, by hypothesis $X \in \mathscr{P}(A)$. In both cases $X \in \mathscr{P}(A)$ and this proves the part "if." We now show that all successors of a preprojective in **C** are also preprojectives. Suppose this is not the case. Then there exists an irreducible map $f: X \to Y$ with $Y \in \mathbf{P}_\infty$ and $X \in \mathbf{P}_n$ for some n. Assume in addition that n is minimal with respect to this property. Let $0 \to \tau Y \to \sigma Y \to Y \to 0$ be an Auslander–Reiten sequence ending at Y. According to Lemma 3.1, $\pi(\tau Y) < n$. Therefore by the minimality property on n all summands of σY are preprojectives. Moreover, by hypothesis on the preprojectives in **C**, $t_\infty \sigma Y = 0$ or, equivalently since σY is artinian, $t_k \sigma Y = 0$ for some k. Consider now an epimorphism $h: Z \to Y$ with $Z \in \operatorname{add} \mathbf{P}_k$ (this is possible because $Y \in \mathbf{P}_\infty$). Note that h lifts through the sink map $\sigma Y \to Y$. In particular $\operatorname{Hom}_A(Z, \sigma Y) \neq 0$ which contradicts the fact that $t_k \sigma Y = 0$.

(c) $\Rightarrow$ (a) Trivial. ∎

We freely use the equivalence of conditions (a), (b), and (c) for the rest of this article. Amongst them, we will see that the most useful formulation is (b).

4. Proof of (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (b)

In this section we prove the implications (e) $\Rightarrow$ (b) and (d) $\Rightarrow$ (e) of Theorem 1.2. We assume throughout this section that **C** is a component of ${}_A\Gamma$.

4.1. Proposition ((e) $\Rightarrow$ (b)). *Let* $X \in A$-ind *such that* $\Pr(X)$ *is a finite set. Then*

(i) $\operatorname{Supp}((-, X)) \subseteq \Pr(X)$.

(ii) $t_\infty X = 0$.

Proof. Let $X \in A$-ind be such that $\Pr(X)$ is finite. Suppose first that $\operatorname{Supp}((-, X))$ is not contained in $\Pr(X)$; that is, suppose there is a $Y \in A\text{-ind} \setminus \Pr(X)$ such that $\operatorname{Hom}_A(Y, X) \neq 0$. According to Proposition 2.2 there are indecomposable A-modules $X = X_0, X_1, ..., X_n, ...,$ and irreducible maps $\beta_i: X_i \to X_{i-1}$, for $i \geqslant 1$, such that for all $n \geqslant 1$ $\beta_1, ..., \beta_n \neq 0$. Note that $\{X_i : i \in \mathbb{N}\} \subseteq \Pr(X)$ and thus it is finite. Therefore the radical of the

Artin algebra $\mathrm{End}(\bigoplus X_i)$ is nilpotent and this contradicts the fact that $\beta_n \cdots \beta_1 \neq 0$ for all $n \geqslant 1$. Hence $\mathrm{Supp}((-, X)) \subseteq \mathrm{Pr}(X)$ and this proves (i). To get (ii) it is enough to note that according to Proposition 2.3 to show that $t_\infty X = 0$ is equivalent to prove that $l((-, X)) < \infty$. However this is an easy consequence of part (i) since $\mathrm{Pr}(X)$ is finite and the result is proved. ∎

The proof of the implication (d) ⇒ (e) is an adaptation of the Happel–Ringel proof that if $\mathbf{C}$ is a τ-preprojective component then $\mathrm{Pr}(X)$ is finite for $X \in \mathbf{C}$ (see (1.3) in [HR]).

4.2. Proposition ((d) ⇒ (e)). *Let* $\mathbf{C}$ *be a component of* ${}_A\Gamma$ *satisfying*:

(d1) *Almost all modules in* $\mathbf{C}$ *lie in* τ*-orbits of projectives*; *and*

(d2) *Only a finite number of modules in* $\mathbf{C}$ *belong to oriented cycles.*

Then for every module $X \in \mathbf{C}$ *the set* $\mathrm{Pr}(X)$ *is finite.*

Proof. Suppose there is an indecomposable module $X \in \mathbf{C}$ such that $\mathrm{Pr}(X)$ is an infinite set. Then there is an infinite chain of irreducible maps to X

$$\cdots X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X \qquad (*)$$

in A-ind such that $X_i \neq X_j$ for $i \neq j$. By hypothesis $\mathbf{C}$ satisfies conditions (d1) and (d2). Therefore almost all modules in $\mathbf{C}$ lie in τ-orbits of projectives and do not belong to any oriented cycle. Hence there is an m such that (i) for any $j \geqslant m$ there exists some $t_j \geqslant 0$ such that $\tau^{t_j} X_j \in \mathbf{P}_0(A)$ and (ii) there is no oriented cycle passing through X_i for $i \geqslant m$. Now since there are only finitely many τ-orbits of projectives there exists $P \in \mathbf{P}_0(A)$ such that the set $I = \{j \geqslant m : \tau^{t_j} X_j = P\}$ is infinite. The function from I to $\mathbb{N}$ which assigns t_i to $i \in I$ cannot be strictly decreasing because I is an infinite set. Therefore there are two elements $i < j$ in I such that $t_i < t_j$. (Note that if $t_i = t_j$ then $X_i = X_j$ and so $i = j$.) Therefore $X_i = \tau^{(t_j - t_i)} X_j$ and thus there is a path from X_i to X_j. Since there is also a path from X_j to X_i we obtain a cycle containing X_i, which is not possible. Therefore $\mathrm{Pr}(X)$ is finite for $X \in \mathbf{C}$. ∎

5. Some General Results

To complete the proof of Theorem 1.2 it remains to prove that π-components satisfy condition (d). The strategy to prove this is the following. First we consider π-components containing no injectives and prove that they are τ-preprojective components (Theorem 6.7). The next step is to give a structural description of algebras with π-components. We show in section 7 that the Auslander–Reiten quiver of such an algebra is closely related with that of an algebra with π-components containing no injectives. Finally in Section 8, using the results proved in Sections 6 and 7, we give the proof that π-components satisfy condition (d) (Theorem 8.1). This completes the proof of Theorem 1.2.

To prove Theorem 6.7 we need two results which will be established in the remaining of this section. The first result records informations in the connection between τ-periodic preprojective modules and injective modules lying in the same component. We recall that $X \in A$-ind is τ-periodic if there is an n such that $\tau^n X = X$. We also need the following result due to Auslander–Reiten.

5.1. LEMMA (Proposition 6.2 in [AR5]). *Let $Y \to X$ be an irreducible map between indecomposables. If X is τ-periodic then either Y is τ-periodic or there are n, $m \geq 0$ such that $\tau^n Y$ is a projective and $\tau^{-m} Y$ is an injective.*

5.2. PROPOSITION. *If X is τ-periodic in $\mathscr{P}(A)$ then X is a predecessor of an injective.*

Proof. Suppose $X \in \mathscr{P}(A)$ is a τ-periodic. We shall prove that $X \in \Pr(I)$ for some injective I. Suppose this is not the case and assume $\pi(X)$ minimal with respect to this property. Note that $\pi(X) \geq 1$. Therefore there exists an irreducible map $Y \to X$ such that $\pi(Y) < \pi(X)$. The minimality of $\pi(X)$ implies that either (a) Y is not a τ-periodic module or (b) Y is a predecessor of an injective. If (a) holds then according to Lemma 5.1 there exists an $n \geq 0$ such that $\tau^{-n} Y$ is an injective. Therefore X is clearly a predecessor of an injective, which is a contradiction. Suppose now condition (b) holds; that is, $Y \in \Pr(I)$ for an injective I. Hence clearly $X \in \Pr(I)$ since X is a τ-periodic module. This is, however, a contradiction, which finishes the proof. ∎

The next result concerns the existence of oriented cycles. To prove it we use the following result established by Bautista and Smalø in [BS]. We recall that a path $X_0 \to X_1 \to \cdots \to X_t$, $t \geq 1$, is called *sectional* if for $2 \leq j \leq t$, τX_j is not isomorphic to X_{j-2}.

5.3. PROPOSITION [BS]. *There is no sectional path in ${}_A\Gamma$ which is a cycle.*

5.4. PROPOSITION. *Suppose $X \in A$-ind belongs to an oriented cycle. If* $\mathrm{Sc}(X)$ *has no injective modules then for all $n > 0$ $\tau^{-n} X \in \Pr(X)$.*

Proof. We prove this result by induction on $n \geq 1$. Let $X = X_0 \to X_1 \to \cdots \to X_t = X$ be an oriented cycle through X. By Proposition 5.3 this is not sectional and then there exists a (minimum) $j \geq 1$ such that $\tau^- X_j = X_{j+2}$. Since Sc(X) has no injectives we can apply τ^- to any of the indecomposable modules X_i, for $i = 1, ..., t$. Consider now the path

$$\tau^- X_0 \to \tau^- X_1 \to \cdots \to \tau^- X_j \to X_{j+3} \to \cdots \to X_t = X.$$

Therefore $\tau^- X \in \Pr(X)$. Suppose now that for all i, $0 \leq i < n$, $\tau^{-i} X \in \Pr(X)$. Let us prove that $\tau^{-n} X \in \Pr(X)$. Since by induction $\tau^{-(n-1)} X \in \Pr(X)$ there is a path $(*)\, \tau^{-(n-1)} X \to \cdots \to X$. Note that all modules in $(*)$ are in Sc(X) and then we can apply τ^- to them. Therefore we will end up with a path from $\tau^{-n} X$ to $\tau^- X$, which is a predecessor of X and so $\tau^{-n} X \in \Pr(X)$, which proves the result. ∎

6. π-Components Containing no Injective Modules

We show in this section that π-components containing no injective modules are τ-preprojective components (Theorem 6.7). Note that since we have already established the equivalence of the conditions (a), (b), and (c) of Theorem 1.2 we freely use any of them as a defining property of π-components.

Our very first goal, however, is to prove that if **C** is a π-component containing no injectives then the set $\Pr(X)$ is finite for every (preprojective) module $X \in \mathbf{C}$. Before we prove it we establish some general results.

Let A be any Artin algebra, **C** be a component of ${}_A\Gamma$, and $X \in \mathbf{C}$. Define $\mathcal{S}_X \subseteq \mathbf{C}$ to be the union of $\mathcal{S}_X^{(n)}$, $n \geq 0$, where $\mathcal{S}_X^{(n)}$ is defined inductively as follows:

(a) $\mathcal{S}_X^{(-1)} = \varnothing$ and $\mathcal{S}_X^{(0)} = \{X\}$.

(b) Assume $\mathcal{S}_X^{(j)}$ are defined for $0 \leq j < n$. $Y \in A$-ind is in $\mathcal{S}_X^{(n)}$ if and only if

(b1) Y is in $\mathcal{S}_X^{(n-1)}$ or

(b2) Y is a summand of σS for an indecomposable $S \in \mathcal{S}_X^{(n-1)} \backslash S_X^{(n-2)}$ and $\tau^- Y$, if defined, is not in $\mathcal{S}_X^{(n-2)}$.

The first result we need is that given any module in $\mathcal{S}_X$ there is a nonzero morphism from it to X. To prove this we need the following lemma.

6.1. Lemma. *If $Y \in \mathcal{S}_X^{(k)} \backslash \mathcal{S}_X^{(k-1)}$ for some $k \geq 1$ then there exists a sectional path from Y to X passing through only modules in $\mathcal{S}_X^{(k-1)}$.*

Proof. We use induction on $k \geq 1$. For $k = 1$ there is nothing to prove since by definition of $\mathcal{S}_X^{(1)}$ there is an irreducible map, thus a sectional path, $Y \to X$ as required. Suppose now that $k > 1$ and that the result is true for modules belonging to $\mathscr{S}_X^{(j)} \backslash \mathscr{S}_X^{(j-1)}$ with $1 \leqslant j < k$. Consider $Y \in \mathscr{S}_X^{(k)} \backslash \mathscr{S}_X^{(k-1)}$. By definition there is an irreducible map $Y \to S$ with $S \in \mathscr{S}_X^{(k-1)} \backslash \mathscr{S}_X^{(k-2)}$ and $\tau^- Y$, if defined, is not in $\mathscr{S}_X^{(k-2)}$. Now by the inductive hypothesis there is a sectional path from S to X passing through modules in $\mathscr{S}_X^{(k-2)}$. Since $\tau^- Y \notin \mathscr{S}_X^{(k-2)}$ the path resulting from the composition of the irreducible map $Y \to S$ and the sectional path from S to X is a sectional path from Y to X as required. ▮

6.2. Lemma (e.g., Corollary 13.4 in [IT]). *The composition of irreducible maps on a sectional path is nonzero.*

6.3. Corollary. *If* $Y \in \mathscr{S}_X$ *then* $\operatorname{Hom}_A(Y, X) \neq 0$.

Proof. Let $Y \in \mathscr{S}_X$. There is nothing to prove if $X = Y$. Assume $Y \in \mathscr{S}_X^{(k)} \backslash \mathscr{S}_X^{(k-1)}$ for some $k \geqslant 1$. By Lemma 6.1 there is a sectional path from Y to X whose composition is nonzero according to Lemma 6.2. Therefore $\operatorname{Hom}_A(Y, X) \neq 0$ as required. ▮

Our next step is to define a distance from $\mathscr{S}_X$ of a module in $\operatorname{Pr}(X)$. For $Y \in \operatorname{Pr}(X)$ let dY denote the length of a shortest path from Y to a module in $\mathscr{S}_X$. We show that, under suitable conditions, this distance does not increase along paths in $\operatorname{Pr}(X)$.

6.4. Lemma. *Let* X, $Y \in A$-ind. *Suppose* $\operatorname{Pr}(X)$ *contains no injective* A-*modules*, $Y \in \operatorname{Pr}(X)$, *and* $dY \geqslant 1$. *Then*

(i) $\tau^- Y \in \operatorname{Pr}(X)$ *and* $d\tau^- Y \leqslant dY - 1$.

(ii) *If* E *is a summand of* $\sigma^- Y$ *then* $dY - 1 \leqslant dE \leqslant dY$.

Proof. We will prove part (i) by induction on $d = dY \geqslant 1$. Suppose $d = 1$. Then there are a module $S \in \mathscr{S}_X$ and an irreducible map $Y \to S$. Since $Y \notin \mathscr{S}_X$ it follows that $\tau^- Y \in \mathscr{S}_X$. Hence $\tau^- Y \in \operatorname{Pr}(X)$ and $d\tau^- Y = 0 = dY - 1$ as required. Suppose now $d > 1$. Then there is an irreducible map $Y \to Z$ with $dZ = d - 1$ and $Z \in \operatorname{Pr}(X)$. By induction $\tau^- Z \in \operatorname{Pr}(X)$ and $d\tau^- Z \leqslant dZ - 1$.

Since there is an irreducible map $\tau^- Y \to \tau^- Z$, we have $\tau^- Y \in \operatorname{Pr}(X)$ and $d\tau^- Y \leqslant d\tau^- Z + 1 \leqslant d - 1$ as required. Part (ii) follows from the definition of d and (i). ▮

Assume now and for the rest of this section that $\mathbf{C}$ is a π-component containing no injective modules. For $i \geqslant 0$ let $D_i(X)$ denote the set $\{Y \in \operatorname{Pr}(X) : dY = i\}$.

6.5. Lemma. *Let* $X \in \mathbf{C}$. *Then for all* $i \geqslant 0$ $D_i(X)$ *is a finite subset of* $\operatorname{Pr}(X)$.

Proof. We prove this by induction on $i \geqslant 0$. For $i = 0$ the statement is equivalent to $\mathcal{S}_X$ being a finite set. Since $\mathbf{C}$ is a π-component, $l((-, X)) < \infty$ for each $X \in \mathbf{C}$. Using Corollary 6.3 it follows that $\mathcal{S}_X$ is a finite set. Assume now that $D_j(X)$ is finite for all $0 \leqslant j < i$. Note that if $Y \in D_i(X)$ then there is an irreducible map $Y \to Z$ with $Z \in D_{i-1}(X)$. Since ${}_A\Gamma$ is a locally finite quiver it follows that $D_i(X)$ is finite and the result is proved. ∎

6.6. PROPOSITION. *Let* $\mathbf{C}$ *be a* π*-component containing no injective modules and* $X \in \mathbf{C}$.

(a) *If* $Y \in \operatorname{Pr}(X) \setminus \mathcal{S}_X$ *then there exists a path from* Y *to a module in* $\mathcal{S}_X$ *with nonzero composite.*

(b) $\operatorname{Pr}(X)$ *is a finite set.*

Proof. (a) Let $Y \in \operatorname{Pr}(X) \setminus \mathcal{S}_X$ and consider its injective envelope $i: Y \to I_0(Y)$. Since there are no injectives in $\mathbf{C}$ the summands of $I_0(Y)$ do not belong to $\operatorname{Sc}(Y)$. Hence according to Proposition 2.2 there are indecomposable modules Y_i, $i \in \mathbb{N}$, with $Y_0 = Y$ and irreducible maps $\alpha_i: Y_{i-1} \to Y_i$ for $i \geqslant 1$ such that $\alpha_n \cdots \alpha_1 \neq 0$ for all n. Suppose now that $Y_i \notin \mathcal{S}_X$ for all $i \geqslant 0$. We claim that $Y_i \in \operatorname{Pr}(X) \setminus \mathcal{S}_X$ for all $i \geqslant 0$. Otherwise there is a j such that $Y_j \notin \operatorname{Pr}(X)$ and we can choose such j minimal, ie with $Y_{j-1} \in \operatorname{Pr}(X) \setminus \mathcal{S}_X$. But now $dY_{j-1} > 0$ and by Lemma 6.4 $dY_j \geqslant 0$, which is a contradiction. Therefore $dY_i > 0$ for all $i \geqslant 0$. Moreover by Lemma 6.4 $dY \geqslant dY_i$ for all $i \geqslant 0$. Now according to Lemma 6.5 for all $i \geqslant 0$ $D_i(X)$ is finite and this implies that $\{Y_j : j \in \mathbb{N}\}$ has also only finitely many nonisomorphic modules, say $Z_1, ..., Z_t$. Hence for all $n \geqslant 0$, $\alpha_1 \cdots \alpha_n$ is in fact a nonzero morphism in $\operatorname{rad}^n(\operatorname{End} \oplus Z_k)$, which is not possible since $\operatorname{rad}(\operatorname{End} \oplus Z_k)$ is nilpotent. Therefore there is a $j > 0$ such that $Y_j \in \mathcal{S}_X$ and (a) is proved since $\alpha_j \cdots \alpha_1$ is a nonzero path from Y to $Y_j \in \mathcal{S}_X$.

(b) According to Lemma 6.5, $\mathcal{S}_X$ is a finite set. Let X' be the sum of all indecomposable modules of $\mathcal{S}_X$. Since $\mathcal{S}_X$ is contained in a π-component, $t_\infty X' = 0$ or equivalently $\operatorname{Supp}((-, X'))$ is finite. By part (a) $\operatorname{Pr}(X) \subseteq \operatorname{Supp}((-, X'))$. Therefore $\operatorname{Pr}(X)$ is also finite as required. ∎

We are now able to prove our main result of this section.

6.7. THEOREM. *Let* A *be an Artin algebra and let* $\mathbf{C}$ *be a* π*-component containing no injective* A*-modules. Then* $\mathbf{C}$ *is a* τ*-preprojective component.*

Proof. We prove the following two properties for $\mathbf{C}$: (i) all modules in $\mathbf{C}$ belong to τ-orbits of projectives and (ii) there are no oriented cycles in $\mathbf{C}$. Let $X \in \mathbf{C}$. By Proposition 6.6 $\operatorname{Pr}(X)$ is finite, so is its subset $\Delta(X) := \{\tau^i X : i \in \mathbb{N}\}$. Therefore either X is τ-periodic which is not possible by Proposition 5.2 since $\mathbf{C}$ contains no injectives or X is in a τ-orbit of some projective and this proves (i). Now suppose there is some $X \in \mathbf{C}$ which belongs to an oriented cycle. Since $\mathbf{C}$ contains no injectives, by Proposition 5.4 $\Delta^-(X) := \{\tau^{-i}X : i \in \mathbb{N}\} \subseteq \mathrm{Pr}(X)$, which is finite. Hence either X is τ-periodic or $\Delta^-(X)$ contains an injective leading to a contradiction in both cases. ∎

7. A Description of Algebras with π-Components

Throughout this section $\mathbf{C}$ will be a π-component of ${}_A\Gamma$. Let $1_A = \sum e_i$ be a decomposition of 1_A into a sum of primitive idempotents. This sum can be split in the following way. Consider the sum e'' of all idempotents e in the above decomposition such that $D(eA) \in \mathbf{C}$, where D denotes the usual duality. If e' denotes $1 - e''$ then $D(e'A)$ is the sum of the indecomposable injective A-modules not in $\mathbf{C}$. Since $\mathbf{C}$ is a π-component all non-isomorphisms going to a module $X \in \mathbf{C}$ are sums of composites of irreducible maps. This fact, together with the fact that the summands of $D(e'A)$ are not in $\mathbf{C}$, implies that $\mathrm{Hom}_A(D(e'A), D(e''A)) = 0$, which is equivalent to $e'Ae'' = 0$, and so $A = e'Ae' + e''Ae' + e''Ae''$. Denote the algebras $e'Ae'$ by A' and $e''Ae''$ by A'' and denote the A''–A'-bimodule $e''Ae'$ by M. Hence A is isomorphic to the matrix algebra

$$\begin{bmatrix} A' & 0 \\ M & A'' \end{bmatrix}.$$

Note that we do not exclude the possibility of A'' being zero. In this case obviously A is isomorphic to A'.

The category A-mod can be identified with the category $\mathscr{K}$ whose objects and maps are defined as follows (see [Ri]). The objects of $\mathscr{K}$ are triples (X', X'', ξ), where $X' \in A'$-mod, $X'' \in A''$-mod, and $\xi \in \mathrm{Hom}_{A'}(X', \mathrm{Hom}_{A''}(M, X''))$. A map from (X', X'', ξ) to (Y', Y'', ζ) is a pair (α, β), where $\alpha \in \mathrm{Hom}_{A'}(X', Y')$ and $\beta \in \mathrm{Hom}_{A''}(X'', Y'')$, such that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\ \xi\ } & (M, X'') \\ {\scriptstyle \alpha}\downarrow & © & \downarrow{\scriptstyle (M,\beta)} \\ Y' & \xrightarrow{\ \zeta\ } & (M, Y'') \end{array}$$

We freely use the identification between the categories A-mod and $\mathscr{K}$. Consider the following natural inclusions:

$$\iota' : A'\text{-ind} \to A\text{-ind} \qquad \text{and} \qquad \iota'' : A''\text{-ind} \to A\text{-ind}$$

$$X' \mapsto (X', 0, 0) \qquad\qquad X'' \mapsto (0, X'', 0).$$

Note that these inclusions are full.

Now let **B** denote the set of the indecomposable A-modules (X', X'', ξ) such that X'' is a nonzero A''-module.

7.1. PROPOSITION. **B** *contains only finitely many indecomposable A-modules, all of them A-preprojectives in* **C**.

Proof. Let $(X', X'', \xi) \in \mathbf{B}$. Note that the morphism $(0, \mathrm{id}): (0, X'', 0) \to (X', X'', \xi)$ is a monomorphism. Therefore the injective envelope $0 \to (0, X'', 0) \to I_0((0, X'', 0))$ of $(0, X'', 0)$ can be lifted through $(0, \mathrm{id})$. In particular, $\mathrm{Hom}_A((X', X'', \xi), I_0((0, X'', 0))) \neq 0$. Note that, by construction, $I_0((0, X'', 0))$ contains only (injective) modules in **C**. Hence all modules in **B** map to $D(e''A)$. Now since **C** is a π-component and $D(e''A)$ is an A-preprojective in **C** we have that $\mathrm{Supp}((-, D(e''A)))$ is finite and contains only (preprojective) A-modules in **C**. Therefore **B** has at most finitely many indecomposable A-modules, all of them preprojectives and in **C**, and this proves Proposition 7.1. ∎

Note that in particular we also have that A'' is an algebra of finite representation type and all indecomposable A''-modules are in **C**. This also implies the next result, which relates preprojective A-modules and preprojective A'-modules.

7.2. LEMMA. *Let $X \in A'$-module. $X \in \mathscr{P}(A')$ if and only if $(X, 0, 0) \in \mathscr{P}(A)$.*

Proof. $(\Rightarrow)$ Suppose $X \in \mathscr{P}(A')$ but $(X, 0, 0) \notin \mathscr{P}(A)$. Then for every $i \geqslant 0$ there exist $Z_i \in \mathrm{add}(\mathbf{P}_i(A))$ and an epimorphism $f_i: Z_i \to (X, 0, 0)$. Note that for $i \neq j$, Z_i and Z_j have no common summands. According to Proposition 7.1 there are only finitely many modules in A-ind$\setminus A'$-ind and then there is an infinite subset I of $\mathbb{N}$ such that for $i \in I$, $Z_i \in A' - \mathrm{mod}$ and f_i is an A'-epimorphism. Hence $X \in \mathbf{P}_\infty(A')$, which is a contradiction.

$(\Leftarrow)$ Since A'-ind can be seen as a full subcategory of A-ind the nonpreprojective A'-modules are also nonpreprojectives as A-modules. In other words, $\imath'(\mathbf{P}_\infty(A')) \subseteq \mathbf{P}_\infty(A)$. According to Proposition 7.1 $\mathbf{B} \subseteq \mathscr{P}(A)$ and then $\imath'(\mathbf{P}_\infty(A')) = \mathbf{P}_\infty(A)$ as required. ∎

Our next step is to show how ${}_A\Gamma$ and ${}_{A'}\Gamma$ are related. Denote by $\mathbf{C}'$ the set $\{X \in A'\text{-ind}: (X, 0, 0) \in \mathbf{C}\}$. It is clear from our construction that $\mathbf{C}'$ contains no injective A'-modules. We show that $\mathbf{C}'$ is a union of π-components of ${}_{A'}\Gamma$. We first prove that irreducible maps in A'-ind can be "lifted" to paths in A-ind.

7.3. PROPOSITION. *Let $f: X \to Y$ be an irreducible map in A'*-ind.

(i) *If* $\mathrm{Hom}((X, 0, 0), \mathbf{B}) = 0$ *then* $(f, 0): (X, 0, 0) \to (Y, 0, 0)$ *is an irreducible map in* A-ind.

(ii) *There exists a path from* $(X, 0, 0)$ *to* $(Y, 0, 0)$.

Proof. (i) Suppose $\mathrm{Hom}((X, 0, 0), \mathbf{B}) = 0$ but $(f, 0) : (X, 0, 0) \to (Y, 0, 0)$ is not an irreducible map in A-ind. Then there are a morphism $g:(X, 0, 0) \to Z$ in A-mod which is not a split monomorphism and a morphism $h: Z \to (Y, 0, 0)$ in A-mod which is not a split epimorphism, such that $(f, 0) = hg$. Since $\mathrm{Hom}((X, 0, 0), \mathbf{B}) = 0$ we can assume that Z is of type $(Z', 0, 0)$. Therefore $f = hg$ is a decomposition in A'-mod, which contradicts our hypothesis that f is irreducible in A-mod. Hence $(f, 0)$ is an irreducible map in A-ind.

(ii) Suppose there is no path from $(X, 0, 0)$ to $(Y, 0, 0)$. In particular $(f, 0) \in \mathrm{rad}^{\infty}((X, 0, 0), (Y, 0, 0))$. Therefore for all $m \geq 0$, $(f, 0)$ can be written as a sum $\sum h_i g_i$, where $h_i \in \mathrm{rad}^m(A\text{-mod})$ and $g_i \in \mathrm{rad}^{\infty}(A\text{-mod})$. Consider now the algebra $\mathbf{B} = \mathrm{End}(\bigoplus M_i)$, where the sum is taken over all indecomposable modules in $\mathbf{B}$. According to Proposition 7.1 $\mathbf{B}$ is a finite set and then $\mathbf{B}$ is an Artin algebra and it has nilpotent radical. Therefore there are a sufficiently large $n > 0$ and a decomposition $\sum h_i g_i$ of f such that $h_i \in \mathrm{rad}^n(A'\text{-mod})$ and $g_i \in \mathrm{rad}^{\infty}(A'\text{-mod})$. Hence f can be decomposed as an A'-homomorphism, which contradicts our hypothesis of f being an irreducible map. Therefore there is a path from $(X, 0, 0)$ to $(Y, 0, 0)$ as required. ∎

7.4. Corollary. $\mathbf{C}'$ *is a union of* π-*components in* ${}_{A'}\Gamma$.

Proof. We first claim that if $X \in \mathbf{C}'$ then $X \in \mathscr{P}(A')$. In fact if $X \in \mathbf{C}'$ then $(X, 0, 0) \in \mathbf{C}$. Moreover $(X, 0, 0) \in \mathscr{P}(A)$ because $\mathbf{C}$ is a π-component. Now according to Lemma 7.2 $X \in \mathscr{P}(A')$ and the claim is proved. It remains to prove that $\mathbf{C}'$ is in fact a union of connected components. If this is not the case there are $Y \in A'\text{-ind} \setminus \mathbf{C}'$ and an irreducible map between Y and $X \in \mathbf{C}'$. According to Proposition 7.3 $(Y, 0, 0)$ and $(X, 0, 0)$ are in the same component, which is a contradiction because, by assumption, $(X, 0, 0) \in \mathbf{C}$ but $(Y, 0, 0) \notin \mathbf{C}$. Therefore $\mathbf{C}'$ is a union of π-components. ∎

From now on we use the following convention. When we refer to $X \in A$-ind we use the usual notation τX, σX, $\tau^- X$, $\sigma^- X$. To avoid misunderstanding, however, we use the notations $\boldsymbol{\tau} X$, $\boldsymbol{\sigma} X$, $\boldsymbol{\tau}^- X$, $\boldsymbol{\sigma}^- X$ when referring to $X \in A'$-ind.

7.5. Corollary. *For almost all* $X \in A'$-ind, $\tau(X, 0, 0)$ *is isomorphic to* $(\boldsymbol{\tau}(X), 0, 0)$.

Proof. Since $\mathbf{B}$ is finite there are only, up to isomorphism, finitely many

Auslander–Reiten sequences containing a module in **B**. Therefore almost all Auslander–Reiten sequences are in fact in A'-mod and the result follows. ∎

The results proved above give a nice relation between the Auslander–Reiten quivers of A and A'. Let **D** be a component of ${}_{A'}\Gamma$ such that $\mathrm{Hom}_A((X, 0, 0), \mathbf{B}) = 0$ for all $X \in \mathbf{D}$. Note that **D** can be any component other than those in $\mathbf{C}'$. Then **D** can be identified with a component of ${}_A\Gamma$. On the other hand, any component of ${}_A\Gamma$ other than **C** can also be identified with a component of ${}_{A'}\Gamma$.

8. Proof of (a) ⇒ (d) of Theorem 1.2

In this section we give the proof of the remaining implication of Theorem 1.2; that is, we prove the following result.

8.1. Theorem. *Let A be an algebra of infinite representation type and* **C** *be a π-component. Then*

(d1) *Almost all modules in* **C** *lie in τ-orbits of projectives.*

(d2) *Only a finite number of modules in* **C** *belong to oriented cycles.*

Suppose throughout this section that **C** is a π-component. If **C** contains no injective A-module then by Theorem 6.7 **C** is a τ-preprojective component and so conditions (d1) and (d2) are clearly satisfied for **C**. Let us now assume that **C** contains at least an injective module. Using the results of Section 7 there are algebras A' and A'' and an A''–A'-bimodule M such that:

(i) A is isomorphic to the matrix algebra

$$\begin{bmatrix} A' & 0 \\ M & A'' \end{bmatrix};$$

(ii) $\mathbf{C}' = \{X \in A'\text{-ind} : (X, 0, 0) \in \mathbf{C}\}$ is a union of π-components containing no injective modules; and

(iii) the natural inclusion $\imath' : A'\text{-ind} \to A\text{-ind}$ given by $X' \mapsto (X', 0, 0)$ is cofinite.

According to Theorem 6.7 $\mathbf{C}'$ is a union of τ-preprojective components and then (d1) and (d2) of Theorem 8.1 are true for $\mathbf{C}'$.

According to the results established in Section 7, for almost all $X \in A'$-ind, $\tau(X, 0, 0) = (\tau X, 0, 0)$. Let now **D** denote the (finite) union $\mathbf{B} \cup \{X = (X', 0, 0) \in A\text{-ind} : \tau X \neq \imath'(\tau X')\}$. For $X \in A$-ind take $\Delta(X) := \{\tau^i X : i \in \mathbb{N}\}$. Note that $\Delta(X) \subseteq \mathrm{Pr}(X)$ and since **C** is a π-component, if

$X \in \mathbf{C}$ then $\Delta(X) \subseteq \mathscr{P}(A)$. We also have the following result concerning $\Delta(X)$.

8.2. LEMMA. *The set $\Delta(X)$ is finite for every preprojective A-module $X \in \mathbf{C}$.*

Proof. Suppose first that $\Delta(X)$ has a repetition; i.e., $\tau^i X = \tau^j X$ for some $i \neq j$. Then X is a τ-periodic module and then it is clear that $\Delta(X)$ is a finite set. Suppose now that $\Delta(X)$ has no repetition. We prove that $\Delta(X) \cap \mathbf{P}_0(A) \neq \varnothing$. Since $\mathbf{D}$ is finite, there is an $n_0 \geqslant 0$ such that $\tau^i X \notin \mathbf{D}$ for all $i \geqslant n_0$. Take $\Delta'(X) := \{\tau^i X : i \geqslant n_0\}$. If $\Delta'(X) = \varnothing$ then $\tau^i X = 0$ for some $i \leqslant n_0$ and the result is proved. Suppose $\Delta'(X)$ contains a nonzero indecomposable module Y. Then Y is of the form $(Y', 0, 0)$ and $\tau Y = \iota'(\tau Y')$. Moreover, since $\Delta'(X) \subseteq \mathscr{P}(A)$, $Y' \in \mathscr{P}(A')$ by Lemma 7.2. Using the fact that $\mathbf{C}'$ is a union of τ-preprojective components and $Y' \in \mathbf{C}'$, there is a j such that $\tau^j Y' \in \mathbf{P}_0(A')$. Then $(\tau^j Y', 0, 0) \in \mathbf{P}_0(A)$. Hence $\Delta'(X) \cap \mathbf{P}_0(A) \neq \varnothing$ and so $\Delta(X)$ is finite. ∎

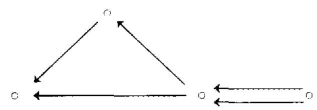
We are now able to finish the proof of Theorem 8.1 and therefore of Theorem 1.2.

Proof of Theorem 8.1. (d1) Let $X \in \mathbf{C}$ and suppose X is not in a τ-orbit of any projective, which is equivalent to $\Delta(X) \cap \mathbf{P}_0(A) = \varnothing$. Then $\Delta(X) \cap \mathbf{D} \neq \varnothing$ (same argument as in Lemma 8.2 above) and X is τ-periodic. Hence X is in a τ-orbit of some τ-periodic module in $\mathbf{D}$, which is finite. Therefore there are only finitely many indecomposable modules in $\mathbf{C}$ which are not in a τ-orbit of a projective.

(d2) Let $(*)$: $X = X_0 \to X_1 \to \cdots \to X_t = X$ be a cycle in $\mathbf{C}$. We claim that there is an i, $0 \leqslant i \leqslant t$, such that $X_i \in \mathbf{B}$. Otherwise $(*)$ is a cycle of A'-modules in $\mathbf{C}'$ which is not possible because $\mathbf{C}'$ is a union of τ-preprojective components, and the claim is proved. Suppose now there are infinitely many indecomposable modules in $\mathbf{C}$ lying in oriented cycles. Therefore since $\mathbf{B}$ is finite and ${}_A\Gamma$ is locally finite there exists $Y \in \mathbf{B}$ such that for all $m > 1$ there is a minimal oriented cycle of length greater than m (a cycle $X = X_0 \to X_1 \to \cdots \to X_t = X$ is called *minimal* if $X_i \neq X_j$ for $1 \leqslant i < j \leqslant t$). Thus using again the finiteness of $\mathbf{B}$, for all $n > 0$ there is a path in $\Pr(Y) \cap \iota'(\mathbf{C}')$ of length greater than n, which contradicts the fact that $\mathbf{C}'$ is a union of τ-preprojective components, and this finishes the proof of Theorem 8.1. ∎

All τ-preprojective components are π-components according to the results established here. Classes of algebras with τ-preprojective components include (i) hereditary algebras, (ii) concealed algebras, and (iii) tubular algebras.

Now let k be an algebraically closed field and A be the path k-algebra given by the quiver



with all composites of arrows being zero. Note that algebra A is of infinite representation type and has a π-component which is not τ-preprojective.

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