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A semiclassical string description of Wilson loop with local operators

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Abstract

We discuss a semiclassical string description to circular Wilson loops without/with local operator insertions. Type IIB string theory on $AdS_5 \times S^5$ is expanded around the corresponding classical solutions with respect to fluctuations and semiclassical quadratic actions are computed. Then the dual corresponding operators describing the fluctuations are discussed from the point of view of a small deformation of the Wilson loops. The result gives new evidence for AdS/CFT correspondence.

1. Introduction

Almost a decade has passed from a discovery of AdS/CFT correspondence [1,2]. Now it is firmly supported by enough evidences, but there is no proof of it now. Hence it is still important to continue to seek further, new confirmation to support it.

One of the difficulties is to analyze type IIB string on $AdS_5 \times S^5$. The action is constructed in [3] and its classical integrability is shown in [4]. However, it still seems difficult to quantize the theory manifestly, simply because the action is quite non-linear. A sensible way is to find a solvable subsector such as the BMN sector [5]. The BMN sector is pulled out by taking a Penrose limit [6]. Then the simplified string theory is exactly solvable [7,8], and hence one can test the duality at stringy level though the argument is restricted to a certain region.

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It is pointed out in [9] that a non-relativistic limit of type IIB string on $AdS_5 \times S^5$ gives a new arena to test the AdS/CFT. It is shown in [10] that the limit is regarded as a semiclassical approximation around a static AdS_2 solution [11] like as the Penrose limit is around a BPS particle [12]. This equivalence holds even for AdS-branes [10,13]. With this semiclassical interpretation it has been shown that the corresponding operator in the gauge theory is nothing but a small deformation of straight Wilson line [10].

The purpose of this paper is to generalize the result for the straight line to circular Wilson loops without/with local operator insertions. An AdS_2 solution corresponds to a $1/2$ BPS circular Wilson loop *without* the insertions [14]. A semiclassical approximation around the solution has already been studied in [15]. We newly compute a quadratic action around the solution corresponding to a circular Wilson loop *with* the local operators, Z^J and its complex conjugate. Here Z is a complex scalar composed of the two real scalar fields in $\mathcal{N} = 4$ SYM like $Z \equiv \phi_1 + i\phi_2$. The resulting quadratic fluctuations describe the action in [15] around $\sigma = 0$ while those behave as a pp-wave string at $\sigma = \infty$.

Then we clarify that the fluctuations correspond to a small deformation of the circular Wilson loops without/with the insertions. In particular, the dictionary of impurity insertion is derived. With no local operator the dictionary is the same as in the case of the straight line [10]. This result is not so surprising since the difference between a straight Wilson line and a circular Wilson loop is the behavior at infinity and it only gives an anomalous contribution to the expectation value (and the value of classical action of the corresponding string solution). However the local behavior around a finite point should not be different. With the local operators, it is the same as the case without them apart from the insertion points while it is nothing but the BMN dictionary [5] on the inserted local operators. This result nicely agrees with the behavior of the semiclassical action.

This paper is organized as follows. In Section 2 we reproduce a semiclassical action around an AdS_2 solution whose boundary is a circular Wilson loop with no local operator. Then, in Section 3, we discuss the corresponding operators in the gauge theory from a small deformation of the circular Wilson loop. In Section 4, as a further generalization, we consider a semiclassical action around the Miwa–Yoneya solution [16], which is a generalization of the solution constructed by Drukker–Kawamoto [17] in the Lorentzian case. This solution corresponds to a circular Wilson loop with local operator insertions. The resulting action interpolates the pp-wave string action and the semiclassical action around the AdS_2 as expected. In Section 5 we consider a small deformation of the Wilson loop corresponding to the semiclassical action obtained in Section 4. The configuration of the Wilson loop is more involved. Section 6 is devoted to a summary and discussions.

2. Semiclassical limit around a circular solution

In this section, as a warming up, let us consider a classical string solution whose boundary describes a circular Wilson loop [14] *without* local operator insertions. Note that the quadratic string action with respect to the fluctuations has already been computed by Drukker–Gross–Tseytlin [15]. To make the present paper self-contained, however, we shall rederive the result of [15] here. Then we show the agreement between the fluctuations around the classical solution in the string side and those around the circular Wilson loop in the gauge theory.

2.1. Classical solution for a circular Wilson loop

First let us discuss a classical solution describing a circular Wilson loop. We begin with the string action in the Polyakov formulation and the bosonic part is given by

$$S_{\mathrm{B}} = \frac{\sqrt{\lambda}}{4\pi}\int d^2\xi\,\sqrt{\gamma}\gamma^{ij}\partial_i X^M\partial_j X^N G_{MN}, \tag{2.1}$$

where γ_{ij} is an auxiliary world-sheet metric and we work in conformal gauge, $\sqrt{\gamma}\gamma^{ij} = \delta^{ij}$. The spacetime metric G_{MN} describes $\mathrm{AdS}_5 \times \mathrm{S}^5$ and it is given by

$$ds^2 = \frac{1}{z^2}\left(r^2\,d\theta^2 + dr^2 + dx_2^2 + dx_3^2 + dz^2\right) + d\Omega_5^2.$$

Hereafter we will work in Euclidean signature and Poincaré coordinates. See Appendix A for the detail expressions of vielbeins and spin connections.

The equation of motion reads

$$0 = -\partial_i\left(\delta^{ij}\partial_j X^N G_{MN}\right) + \frac{1}{2}\partial_M G_{PQ}\delta^{ij}\partial_i X^P\partial_j X^Q. \tag{2.2}$$

The Virasoro constraints to be imposed are

$$0 = G_{MN}\left(-\dot{X}^M\dot{X}^N + X'^M X'^N\right), \qquad 0 = G_{MN}\left(\dot{X}^M X'^N\right). \tag{2.3}$$

It is easy to see that

$$z = \tanh\sigma, \qquad r = \frac{1}{\cosh\sigma}, \qquad \theta = \tau \tag{2.4}$$

solves the equation of motion (2.2) and the Virasoro conditions (2.3). The classical solution (2.4) at $\sigma = 0$ describes a circular Wilson loop with a unit radius on the boundary

$$z = 0, \qquad r = 1, \qquad \theta = \tau.$$

Here it is valuable to comment on the relation between a circular loop and a straight line. First we move to Cartesian coordinates,

$$z = R\tanh\sigma, \qquad x_0 = R\frac{\sin\tau}{\cosh\sigma}, \qquad x_1 = R\left(\frac{\cos\tau}{\cosh\sigma} - 1\right),$$

where the radius of the circle R has been recovered. Then let us rescale τ and σ as

$$\tau \to \frac{\tau}{R}, \qquad \sigma \to \frac{\sigma}{R}$$

and take the large R limit. As a result, (2.4) is reduced to a static AdS_2 solution

$$z = \sigma, \qquad x_0 = \tau, \qquad x_1 = 0.$$

At the boundary $\sigma = 0$ this solution describes a straight Wilson line.

2.2. *Semiclassical limit*

Next we consider a semiclassical approximation of the full type IIB string on $\mathrm{AdS}_5 \times \mathrm{S}^5$ around the classical solution (2.4).

Let us expand the string action (2.1) about the classical solution (2.4)

$$z = \tanh\sigma + \tilde{z}, \qquad r = \frac{1}{\cosh\sigma} + \tilde{r}, \qquad \theta = \tau + \tilde{\theta},$$
$$x_{2,3} = 0 + \tilde{x}_{2,3}, \qquad \varphi_i = 0 + \tilde{\varphi}_i \quad (i = 1, \ldots, 5),$$

where quantum fluctuations are denoted as the symbols with tilde like $\tilde{X}$. Hereafter the overall factor of the action (2.1), $\sqrt{\lambda}$ is absorbed into the definition of fluctuations by rescaling the variables as $\tilde{X} \to \lambda^{-1/4}\tilde{X}$. Then the value of λ should be taken to be large in order for the semiclassical (quadratic) approximation to be valid.

An additional redefinition of the variables is performed as

$$\bar{r} = \tilde{r}\coth\sigma, \qquad \bar{x} = \tilde{x}\coth\sigma, \qquad \bar{z} = \tilde{z}\coth\sigma, \qquad \bar{\theta} = \tilde{\theta}\frac{1}{\sinh\sigma},$$

and the following quadratic action is obtained,

$$S_{2\mathrm{B}} = \frac{1}{4\pi}\int d^2\xi\bigg[(\partial\bar{\theta})^2 + (\partial\bar{r})^2 + (\partial\bar{x}_2)^2 + (\partial\bar{x}_3)^2 + (\partial\bar{z})^2 + (\partial\tilde{\varphi}_i)^2 + \frac{2}{\cosh^2\sigma}\bar{z}^2 - 4\frac{\sinh\sigma}{\cosh^2\sigma}\bar{z}\bar{r} + \bar{r}^2 + \frac{4}{\cosh\sigma}\bar{z}\bar{r}' + 4\bar{r}\dot{\bar{\theta}} - \frac{4}{\sinh\sigma}\bar{z}\dot{\bar{\theta}}\bigg]. \tag{2.5}$$

Although one should impose the Virasoro constraints to eliminate the longitudinal modes, it is not an easy task for small fluctuations. Thus we shall take another course following [15] instead. Note that the action (2.5) can be rewritten with conformal gauge as follows:

$$S_{2\mathrm{B}} = \frac{1}{4\pi}\int d^2\xi\,\big[\delta^{ij}D_i\zeta^A D_j\zeta^B\delta_{AB} + X_{AB}\zeta^A\zeta^B\big],$$
$$\zeta^A = (\bar{\theta}, \bar{r}, \bar{x}_2, \bar{x}_3, \bar{z}; \bar{\varphi}_i). \tag{2.6}$$

Here the following quantities have been introduced:

$$D_i\zeta^A = \partial_i\zeta^A + \Omega_i^A{}_B\zeta^B, \quad \Omega_i^A{}_B = \partial_i X^M\Omega_M^A{}_B, \qquad E_i^A = \partial_i X^M E_M^A,$$
$$X^{ab} = \delta^{ab}\delta^{ij}E_i^c E_j^d\delta_{cd} - \delta^{ij}E_i^a E_j^b,$$
$$X^{a'b'} = -\delta^{a'b'}\delta^{ij}E_i^{c'}E_j^{d'}\delta_{c'd'} + \delta^{ij}E_i^{a'}E_j^{b'},$$

where E^A and $\Omega^A{}_B$ are vielbein and spin connection of $\mathrm{AdS}_5 \times \mathrm{S}^5$ evaluated with the classical solution (2.4). In the present case we obtain

$$D_i\zeta^0 = \left(\partial_\tau\zeta^0 - \frac{1}{s}\zeta^4 + \zeta^1, \partial_\sigma\zeta^0\right), \qquad D_i\zeta^1 = \left(\partial_\tau\zeta^1 - \zeta^0, \partial_\sigma\zeta^1 + \frac{1}{c}\zeta^4\right),$$
$$D_i\zeta^4 = \left(\partial_\tau\zeta^4 + \frac{1}{s}\zeta^0, \partial_\sigma\zeta^4 - \frac{1}{c}\zeta^1\right), \qquad D_i\zeta^a = \partial_i\zeta^a \quad (a = 2, 3, 5, \ldots, 9),$$
$$X^{AB} = \mathrm{diag}\left(\frac{1}{s^2}, \frac{2}{s^2} - \frac{1}{c^2}, \frac{2}{s^2}, \frac{2}{s^2}, \frac{2}{s^2} - \frac{1}{c^2 s^2}; 0, 0, 0, 0, 0\right) + \frac{2}{c^2 s}\delta_1^{(A}\delta_4^{B)}.$$

Here the following abbreviations have been introduced:

$$s \equiv \sinh\sigma, \qquad c \equiv \cosh\sigma. \tag{2.7}$$

Then the mass term for ζ^1 and ζ^4 can be expressed as

$$(\zeta^1, \zeta^4)\begin{pmatrix} \frac{2}{s^2} - \frac{1}{c^2} & \frac{1}{c^2 s} \\ \frac{1}{c^2 s} & \frac{2}{s^2} - \frac{1}{c^2 s^2} \end{pmatrix}\begin{pmatrix} \zeta^1 \\ \zeta^4 \end{pmatrix},$$

and it is diagonalizable by taking the following linear combination

$$\begin{pmatrix} \tilde{\zeta}^1 \\ \tilde{\zeta}^4 \end{pmatrix} = \begin{pmatrix} \frac{s}{c} & -\frac{1}{c} \\ \frac{1}{c} & \frac{s}{c} \end{pmatrix}\begin{pmatrix} \zeta^1 \\ \zeta^4 \end{pmatrix}.$$

The mass eigenvalues are given by

$$\tilde{X}^{AB} = \mathrm{diag}\left(\frac{1}{s^2}, \frac{1}{s^2}, \frac{2}{s^2}, \frac{2}{s^2}, \frac{2}{s^2}; 0, 0, 0, 0, 0\right).$$

The new linear combination has been introduced and then the covariant derivative accordingly turn to be

$$D\zeta^0 = \nabla\zeta^0, \qquad D\zeta^1 = \left(\frac{s}{c}\nabla_\tau\tilde{\zeta}^1 + \frac{1}{c}\partial_\tau\tilde{\zeta}^4, \frac{s}{c}\nabla_\sigma\tilde{\zeta}^1 + \frac{1}{c}\partial_\sigma\tilde{\zeta}^4\right),$$
$$D\zeta^4 = \left(-\frac{1}{c}\nabla_\tau\tilde{\zeta}^1 + \frac{s}{c}\partial_\tau\tilde{\zeta}^4, -\frac{1}{c}\nabla_\sigma\tilde{\zeta}^1 + \frac{s}{c}\partial_\sigma\tilde{\zeta}^4\right),$$

where we have defined

$$\nabla_i\zeta^0 \equiv \left(\partial_\tau\zeta^0 - \frac{c}{s}\tilde{\zeta}^1, \partial_\sigma\zeta^0\right), \qquad \nabla_i\tilde{\zeta}^1 \equiv \left(\partial_\tau\tilde{\zeta}^1 + \frac{c}{s}\zeta^0, \partial_\sigma\tilde{\zeta}^1\right).$$

Substituting the above quantities into the action, the resulting action is

$$S_{2\mathrm{B}} = \frac{1}{4\pi}\int d^2\xi\,\left[\delta^{ij}D_i\hat{\zeta}^A D_j\hat{\zeta}^A + \tilde{X}_{AB}\hat{\zeta}^A\hat{\zeta}^B\right],$$
$$\hat{\zeta}^A = \left(\zeta^0, \tilde{\zeta}^1, \zeta^2, \zeta^3, \tilde{\zeta}^4, \zeta^5, \ldots, \zeta^9\right),\ D\tilde{\zeta}^1 = \nabla\tilde{\zeta}^1,\ D\tilde{\zeta}^4 = \partial\tilde{\zeta}^4.$$

Let us write it as the action on the two-dimensional induced metric

$$g_{ij} = \frac{1}{s^2}\delta_{ij}, \qquad R^{(2)} = -2, \tag{2.8}$$

so that

$$S_{2\mathrm{B}} = \frac{1}{4\pi}\int d^2\xi\,\sqrt{g}\left[g^{ij}D_i\hat{\zeta}^A D_j\hat{\zeta}^A + s^2\tilde{X}_{AB}\hat{\zeta}^A\hat{\zeta}^B\right].$$

Imposing Virasoro constraints is equivalent to removing the longitudinal modes by adding the ghost action

$$S_{\mathrm{gh}} = \frac{1}{2}\int d^2\xi\,\sqrt{g}\left[g^{ij}\nabla_i\epsilon^\alpha\nabla_j\epsilon^\alpha - \frac{1}{2}R^{(2)}\epsilon^\alpha\epsilon^\alpha\right]. \tag{2.9}$$

We choose g_{ij} as the induced metric (2.8). The covariant derivative is defined as

$$\nabla_i\epsilon^\alpha \equiv \partial_i\epsilon^\alpha + \omega_i^\alpha{}_\beta\epsilon^\beta,$$

where ω is the two-dimensional spin connection: $\omega^0{}_1 = -\frac{c}{s}\,d\tau$, then

$$\nabla_i\epsilon^0 = \left(\partial_\tau\epsilon^0 - \frac{c}{s}\epsilon^1, \partial_\sigma\epsilon^0\right), \qquad \nabla_i\epsilon^1 = \left(\partial_\tau\epsilon^1 + \frac{c}{s}\epsilon^0, \partial_\sigma\epsilon^1\right).$$

The ghost action is the same as that of ζ^0 and $\tilde{\zeta}^1$. Thus these modes may be eliminated by the constraints. The final gauge-fixed action is

$$S_{2\mathrm{B}} = \frac{1}{4\pi}\int d^2\xi\,\sqrt{g}\left[g^{ij}\partial_i\hat{\zeta}^A\partial_j\hat{\zeta}^A + 2\left(\zeta^2\right)^2 + 2\left(\zeta^3\right)^2 + 2\left(\tilde{\zeta}^4\right)^2\right],$$
$$\hat{\zeta}^A = \left(\zeta^2, \zeta^3, \tilde{\zeta}^4, \zeta^5, \ldots, \zeta^9\right). \tag{2.10}$$

This action contains three massive bosons with $m^2 = 2$ and five massless bosons propagating on EAdS_2. The fluctuations respect an $SO(3) \times SO(5)$ symmetry, which is also preserved by the circular Wilson loop.

The resulting action (2.10) is the same as the straight Wilson line case. This result is not so surprising since the difference between a straight Wilson line and a circular Wilson loop is the behavior at infinity and it only gives an anomalous contribution to the value of the action. However the local behavior around a finite point should not be different.

Finally, let us comment on the fermionic fluctuations. The quadratic action is

$$S_{2\mathrm{F}} = \frac{i}{2\pi} \int d^2\xi \left[\sqrt{g} g^{ij} \delta^{IJ} - \epsilon^{ij} \sigma_3^{IJ}\right] \bar{\theta}^I \rho_i D_j \theta^J,$$

$$D_i \theta^I = \partial_i \theta^I + \frac{1}{4} \Omega_i^{AB} \Gamma_{AB} \theta^I - \frac{i}{2} \epsilon^{IJ} \left(E_i^a \Gamma_a + i E_i^{a'} \Gamma_{a'}\right) \theta^J, \quad \rho_i = E_i^A \Gamma_A,$$

where E and Ω are evaluated with the classical solution. After rotating the spinor basis so that a two-dimensional spinor covariant derivative is manifest, and fixing κ-symmetry appropriately, the mass squared for the fermions is $m^2 = 1$ and the mass term is proportional to $\bar{\vartheta} \Gamma_{01} \vartheta$ [15] (where ϑ is the rotated spinor). Thus the $SO(3) \times SO(5)$ symmetry is not broken by the fermions.

3. Small deformations of circular Wilson loop

From now on let us discuss the corresponding gauge-theory operator describing the fluctuations obtained in the previous section by following [10,18].

Let us consider a Wilson loop

$$W(C) = \mathrm{Tr}\, P \mathcal{W}, \quad \mathcal{W} \equiv \exp\left(\oint ds \left(i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i\right)\right) = \exp\left(\oint ds \left(i A_M \dot{Y}^M\right)\right), \tag{3.1}$$

where $\dot{Y}^M = (\dot{x}^\mu, -i \dot{y}^i)$. The supersymmetry transformation

$$\delta_\epsilon A_\mu = i \bar{\Psi} \Gamma_\mu \epsilon, \qquad \delta_\epsilon \phi_i = i \bar{\Psi} \Gamma_i \epsilon,$$

gives the following expression

$$\delta_\epsilon W(C) = \mathrm{Tr}\, P\left[i \delta_\epsilon A_M \dot{Y}^M \mathcal{W}\right] = \mathrm{Tr}\, P\left[-\dot{Y}^M \bar{\Psi} \Gamma^M \epsilon \mathcal{W}\right].$$

Thus the Wilson loop (3.1) is invariant under supersymmetry transformation if $\dot{Y}^M \Gamma_M \epsilon = 0$. The locally supersymmetry condition is derived as the integrability

$$\left(\dot{Y}^M \Gamma_M\right)^2 = \dot{Y}^M \dot{Y}^N \eta_{MN} = \left(\dot{x}^\mu\right)^2 - \left(\dot{y}^i\right)^2 = 0.$$

We are interested in a circular Wilson loop configuration C_0,

$$x_{C_0}^\mu = \left(R \sin s, R(\cos\tau - 1), 0, 0\right), \qquad \dot{y}_{C_0}^i = (0, 0, 0, 0, 0, R)$$

where R is the radius of the loop. Because $A_0\, dx^0 + A_1\, dx^1 = A_r\, dr + A_\theta r\, d\theta$, we have

$$\left[i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i\right]\Big|_{C_0} = R(i A_\theta + \phi_6).$$

Hence the Wilson loop $W(C_0)$ is

$$W(C_0) = \mathrm{Tr}\, P[\mathcal{W}_{C_0}], \qquad \mathcal{W}_{C_0} = \exp\left(\oint ds\, R(i A_\theta + \phi_6)\right),$$

which satisfies the locally supersymmetric condition. We identify $W(C_0)$ as the vacuum operator.

Let us consider a small deformation of C_0: $C = C_0 + \delta C$,

$$x^\mu(C) = x^\mu(C_0) + \delta x^\mu, \qquad \dot{y}^i(C) = \dot{y}^i(C_0) + \delta \dot{y}^i.$$

The Wilson loop can be expanded as

$$\begin{aligned} W(C) &= W(C_0) + \oint ds\, \delta x^\mu(s) \frac{\delta W(C)}{\delta x^\mu(s)}\bigg|_{C_0} + \oint ds\, \delta \dot{y}^i(s) \frac{\delta W(C)}{\delta \dot{y}^i(s)}\bigg|_{C_0} + \cdots \\ &= W(C_0) + \oint ds \big(\delta x^\mu \operatorname{Tr} P\big[i\big(F_{\mu N}\dot{Y}^N\big)\mathcal{W}\big]\big|_{C_0} + \delta \dot{y}^i \operatorname{Tr} P[\phi_i \mathcal{W}]\big|_{C_0}\big) + \cdots, \end{aligned}$$

where ellipsis implies higher order fluctuations. Note that

$$\delta x^\mu(s) \frac{\delta W(C)}{\delta x^\mu}\bigg|_{C_0} = \delta\theta(s) \operatorname{Tr} P R (D_\theta \phi_6)_s \mathcal{W}_{C_0} + \delta x^a(s) \operatorname{Tr} P (i F_{a\theta} + D_a \phi_6)_s \mathcal{W}_{C_0}, \tag{3.2}$$

$$\delta \dot{y}^i \frac{\delta W(C)}{\delta \dot{y}^i}\bigg|_{C_0} = \delta \dot{y}^i \operatorname{Tr} P(\phi_i)_s \mathcal{W}_{C_0}, \tag{3.3}$$

where $a = r, 2, 3$.

We require that a small deformation should satisfy the locally supersymmetry condition, that is

$$0 = (\dot{Y} + \delta \dot{Y})^2 = \big(\dot{x}^\mu + \delta \dot{x}^\mu\big)^2 - \big(\dot{y}^i + \delta \dot{y}^i\big)^2.$$

This condition implies

$$0 = \dot{x}^\mu \delta \dot{x}^\mu - \dot{y}^i \delta \dot{y}^i = R\big(\delta\dot{\theta} - \delta \dot{y}^6\big).$$

On the other hand, by using an $SO(2)$, we can impose the condition to the fluctuations,

$$0 = \delta\dot{\theta} + \delta \dot{y}^6,$$

so that $\delta\dot{\theta} = 0$ and $\delta \dot{y}^6 = 0$. The former means $\delta\theta = 0$. As a result we are left with impurities

$$R(i F_{a\theta} + D_a \phi_6) \quad (a = r, 2, 3), \qquad \phi_{a'} \quad (a' = 1, 2, \ldots, 5). \tag{3.4}$$

Thus the resulting dictionary of impurity insertion is the same as in the case of straight line, up to the appearance of the radius parameter R. This can be absorbed into the definition of s by $s \to \frac{s}{R}$ which corresponds to $\tau \to \frac{\tau}{R}$.

This impurity insertion respects an $SO(3) \times SO(5)$ symmetry, which is also a symmetry of the quadratic action derived in Section 2. As expected, the conformal dimensions of these impurities agree with the mass dimensions of fluctuations $\Delta = \frac{1}{2}(1 + \sqrt{1 + 4m^2})$:

$$\Delta\big(\zeta^2, \zeta^3, \tilde{\zeta}^4\big) = 2, \qquad \Delta\big(\zeta^i\big) = 1 \quad (i = 1, \ldots, 5).$$

The mass dimensions of fermionic fluctuations $\Delta = \frac{1}{2}(1 + 2|m|)$ is

$$\Delta\big(\vartheta^\alpha\big) = \frac{3}{2} \quad (\alpha = 1, \ldots, 8),$$

since $m^2 = 1$. Thus we expect that the eight fermionic impurities with conformal dimension $\frac{3}{2}$ are inserted in the Wilson loop as well as bosonic impurities (3.4). This expectation is correct as we show in Appendix B.

4. Semiclassical limit around Miwa–Yoneya solution

As the second issue we consider a rotating classical solution in which the boundary is a circular Wilson loop with local operator insertions. The classical solution was constructed in [16]. This solution is a generalization of [17], which was constructed for a straight Wilson line in Lorentzian signature. We will examine a correspondence between the fluctuation about the classical solution and small deformation of the circular Wilson loop with local operator insertions.

4.1. Classical solution for a circular Wilson loop with local operator insertions

First of all, let see the classical solution found in [16]. We work in Euclidean $AdS_5 \times S^5$ with the Poincaré coordinates (see Appendix A for vielbeins and spin connections)

$$ds^2 = \frac{1}{z^2}\left(dx_i^2 + dz^2\right) + \cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\Omega_3^2. \tag{4.1}$$

The solution corresponding to a circular Wilson loop with the local operator insertions is given by [16]

$$z = \frac{\ell \sinh\sigma}{\cosh\sigma \cosh\tau \pm \alpha}, \qquad x_0 = \frac{\ell \cosh\sigma \sinh\tau}{\cosh\sigma \cosh\tau \pm \alpha}, \qquad x_1 = \frac{\pm\ell\sqrt{1-\alpha^2}}{\cosh\sigma \cosh\tau \pm \alpha} \tag{4.2}$$

for the AdS_5 part, and

$$\psi = \tau, \qquad \cos\theta = \tanh\sigma \tag{4.3}$$

for the S^5 part. The parameter α parametrizes the radius of the loop R like

$$R = \frac{\ell}{\sqrt{1-\alpha^2}}. \tag{4.4}$$

Note that we have performed a Wick rotation as $\psi \to -i\psi$ following [16,19]. That is why we can define a sensible angular momentum even in Euclidean signature. But we should keep it in mind that the signature of $d\psi^2$ in the metric (4.1) is flipped as $-d\psi^2$ due to the Wick rotation of ψ.

We can easily see that (4.2) and (4.3) solve the equation of motion (2.2), and the Virasoro constraints (2.3). The classical solution for the AdS_5 part (4.2) consists of the two patches corresponding to the upper and lower signs in (4.2). The parameter α takes the value in $0 \leqslant \alpha \leqslant 1$. When $\alpha \neq 1$, it is a circular Wilson loop, while when $\alpha = 1$, a straight Wilson loop. For $\alpha = 1$, we can see that $R = \infty$ from (4.4).

The boundary of the string worldsheet is at $z = 0$. The Wilson loop at $z = 0$ corresponds to $(\sigma, \tau) = (0, \tau)$ and $(\sigma, \tau) = (\infty, \pm\infty)$, where local operators are inserted at latter two points.

4.2. Semiclassical limit

Next lets us consider a semiclassical action around the solution (4.2) and (4.3).

The quadratic action around the classical solution (4.2) and (4.3) is basically given by (2.6), where the covariant derivatives and mass matrices are replaced by

$$
\begin{aligned}
&D_i\zeta^0=\left(\partial_\tau\zeta^0-\frac{\dot x_0}{z}\zeta^4,\partial_\sigma\zeta^0-\frac{x_0'}{z}\zeta^4\right),\qquad D_i\zeta^1=\left(\partial_\tau\zeta^1-\frac{\dot x_1}{z}\zeta^4,\partial_\sigma\zeta^1-\frac{x_1'}{z}\zeta^4\right),\\
&D_i\zeta^4=\left(\partial_\tau\zeta^4+\frac{\dot x_0}{z}\zeta^0+\frac{\dot x_1}{z}\zeta^1,\partial_\sigma\zeta^4+\frac{x_0'}{z}\zeta^0+\frac{x_1'}{z}\zeta^1\right),\\
&D_i\zeta^5=\left(\partial_\tau\zeta^5-\sin\theta\dot\psi\zeta^6,\partial_\sigma\zeta^5\right),\qquad D_i\zeta^6=\left(\partial_\tau\zeta^6+\sin\theta\dot\psi\zeta^5,\partial_\sigma\zeta^6\right),\\
&D_i\zeta^a=\partial_i\zeta^a\quad(a=2,3,7,8,9),
\end{aligned}
$$

and

$$
\begin{aligned}
X^{ab}={}&\mathrm{diag}\left(\frac{(\partial x_1)^2+(\partial z)^2}{z^2},\frac{(\partial x_0)^2+(\partial z)^2}{z^2},\frac{(\partial x_0)^2+(\partial x_1)^2+(\partial z)^2}{z^2},\right.\\
&\left.\frac{(\partial x_0)^2+(\partial x_1)^2+(\partial z)^2}{z^2},\frac{(\partial x_0)^2+(\partial x_1)^2}{z^2}\right)\\
&-\frac{2}{z^2}\partial x_0\partial x_1\delta_0^{(A}\delta_1^{B)}-\frac{2}{z^2}\partial x_0\partial z\delta_0^{(A}\delta_4^{B)}-\frac{2}{z^2}\partial x_1\partial z\delta_1^{(A}\delta_4^{B)},\\
X^{a'b'}={}&\mathrm{diag}\left(-\theta'^2,-\cos^2\theta\dot\psi^2,-\cos^2\theta\dot\psi^2-\theta'^2,-\cos^2\theta\dot\psi^2-\theta'^2,-\cos^2\theta\dot\psi^2-\theta'^2\right),
\end{aligned}
$$

respectively. Here we should note that an additional term should be added to the action because of the presence of a conserved charge associated with ψ. The only effect of adding the term is to change the sign of the kinetic term of ψ, and that is why we arrive at the same action even after the Wick rotation of ψ.

Then we shall diagonalize the mass matrix for ζ^0, ζ^1 and ζ^4,

$$
X=\begin{pmatrix}
\frac{(\partial x_1)^2+(\partial z)^2}{z^2} & -\frac{\partial x_0\partial x_1}{z^2} & -\frac{\partial x_0\partial z}{z^2}\\
-\frac{\partial x_0\partial x_1}{z^2} & \frac{(\partial x_0)^2+(\partial z)^2}{z^2} & -\frac{\partial x_1\partial z}{z^2}\\
-\frac{\partial x_0\partial z}{z^2} & -\frac{\partial x_1\partial z}{z^2} & \frac{(\partial x_0)^2+(\partial x_1)^2}{z^2}
\end{pmatrix}.
$$

For this we need to know the eigenvalues and eigenvectors

$$
Xv_{(i)}=e_{(i)}v_{(i)}\quad(i=0,1,4).
$$

It is straightforward to derive the eigenvalues

$$
e_{(0)}=\frac{1}{\sinh^2\sigma},\qquad e_{(1)}=\frac{\cosh^2\sigma}{\sinh^2\sigma},\qquad e_{(4)}=\frac{1+\cosh^2\sigma}{\sinh^2\sigma}.
$$

The corresponding eigenvectors are found to be

$$
\begin{aligned}
v_{(0)}^T&=\left(\frac{\alpha\cosh\tau+\cosh\sigma}{a},-\frac{\sqrt{1-\alpha^2}\sinh\tau}{a},-\frac{\sinh\tau\sin\sigma}{a}\right),\\
v_{(1)}^T&=\left(\frac{\alpha\sinh\sigma\sinh\tau}{a},-\frac{\sqrt{1-\alpha^2}\sinh\sigma\cosh\tau}{a},\frac{\alpha\cosh\sigma+\cosh\tau}{a}\right),\\
v_{(4)}^T&=\left(\frac{\sqrt{1-\alpha^2}\cosh\sigma\sinh\tau}{a},\frac{\alpha\cosh\sigma\sinh\tau+1}{a},\frac{\sqrt{1-\alpha^2}\sinh\sigma}{a}\right),
\end{aligned}
$$

where

$$
v_{(i)}^Tv_{(j)}=\delta_{ij},\qquad a=\cosh\sigma\cosh\tau+\alpha.
$$

Therefore we find

$$V^{-1}XV = \mathrm{diag}(e_{(0)}, e_{(1)}, e_{(4)}), \quad V = (v_{(0)}, v_{(1)}, v_{(4)}).$$

It follows that

$$\zeta^T X\zeta = \tilde{\zeta}^T V^{-1}XV\tilde{\zeta} = e_{(0)}\big(\tilde{\zeta}^0\big)^2 + e_{(1)}\big(\tilde{\zeta}^1\big)^2 + e_{(4)}\big(\tilde{\zeta}^4\big)^2, \quad \tilde{\zeta} = V^T\zeta.$$

Thus we have obtained the diagonalized mass-matrix

$$\tilde{X}^{AB} = \mathrm{diag}\bigg(\frac{1}{s^2}, \frac{c^2}{s^2}, \frac{c^2+1}{s^2}, \frac{c^2+1}{s^2}, \frac{c^2+1}{s^2}; \frac{-1}{c^2}, \frac{s^2}{c^2}, \frac{s^2-1}{c^2}, \frac{s^2-1}{c^2}, \frac{s^2-1}{c^2}\bigg).$$

Here we have used the notation (2.7).

Next let us examine the kinetic terms for ζ^0, ζ^1 and ζ^4. Then we see that

$$\begin{aligned}
D\zeta^0 &= \frac{1}{a}\big((\alpha\cosh\tau + \cosh\sigma)\nabla\tilde{\zeta}^0 + \alpha\sinh\sigma\sinh\tau\nabla\tilde{\zeta}^1 + \sqrt{1-\alpha^2}\cosh\sigma\sinh\tau\,\partial\tilde{\zeta}^4\big),\\
D\zeta^1 &= \frac{1}{a}\big(-\sqrt{1-\alpha^2}\sinh\tau\nabla\tilde{\zeta}^0 - \sqrt{1-\alpha^2}\sinh\sigma\cosh\tau\nabla\tilde{\zeta}^1\\
&\quad + (\alpha\cosh\sigma\cosh\tau + 1)\partial\tilde{\zeta}^4\big),\\
D\zeta^4 &= \frac{1}{a}\big(-\sinh\sigma\sinh\tau\nabla\tilde{\zeta}^0 + (\alpha\cosh\sigma + \cosh\tau)\nabla\tilde{\zeta}^1 + \sqrt{1-\alpha^2}\sinh\sigma\,\partial\tilde{\zeta}^4\big),
\end{aligned}$$

where the following quantities have been introduced

$$\nabla\tilde{\zeta}^0 \equiv \bigg(\partial_\tau\tilde{\zeta}^0 - \frac{1}{\sinh\sigma}\tilde{\zeta}^1, \partial_\sigma\tilde{\zeta}^0\bigg), \qquad \nabla\tilde{\zeta}^1 \equiv \bigg(\partial_\tau\tilde{\zeta}^1 + \frac{1}{\sinh\sigma}\tilde{\zeta}^0, \partial_\sigma\tilde{\zeta}^1\bigg). \tag{4.5}$$

These expressions lead us to

$$\big(D\zeta^0\big)^2 + \big(D\zeta^1\big)^2 + \big(D\zeta^4\big)^2 = \big(\nabla\tilde{\zeta}^0\big)^2 + \big(\nabla\tilde{\zeta}^1\big)^2 + \big(\partial\tilde{\zeta}^4\big)^2.$$

Thus we have obtained

$$\begin{aligned}
S_{2\mathrm{B}} &= \frac{1}{4\pi}\int d^2\xi\,\big[D\tilde{\zeta}^A D\tilde{\zeta}^A + \tilde{X}^{AB}\tilde{\zeta}^A\tilde{\zeta}^B\big], \quad D\tilde{\zeta}^{0,1} = \nabla\tilde{\zeta}^{0,1},\ D\tilde{\zeta}^4 = \partial\tilde{\zeta}^4,\\
&\tilde{\zeta}^A = \big(\tilde{\zeta}^0, \tilde{\zeta}^1, \zeta^2, \zeta^3, \tilde{\zeta}^4; \zeta^5, \dots, \zeta^9\big).
\end{aligned}$$

We have derived the eigenvalues of the mass matrix in the bosonic sector so far. It seems to be difficult to add the ghost action so that it should delete the unphysical longitudinal modes in the matrix. Actually, we have not completed this step and we will leave it as a future problem. Instead of trying to delete the unphysical modes from the full action, let us consider the Lagrangian density in the two special regions, (1) $\sigma = 0$ and (2) $\sigma = \infty$. For the former case we expect that the Lagrangian density should behave as that in Section 2. For the latter case the Lagrangian density is expected to be the one for the pp-wave string action.

4.3. *The fluctuations near $\sigma = 0$*

The ghost action (2.9) is introduced as before. Let us consider the two-dimensional metric

$$g_{ij} = \frac{\cosh^2\sigma}{\sinh^2\sigma}d\tau^2 + \frac{1}{\sinh^2\sigma}d\sigma^2, \qquad R^{(2)} = -2, \tag{4.6}$$

which is AdS_2 and corresponds to the induced metric of AdS_5 part only. The covariant derivative is defined by

$$\nabla_i \epsilon^\alpha \equiv \partial_i \epsilon^\alpha + \omega_i^\alpha{}_\beta \epsilon^\beta,$$

where ω is the two-dimensional spin connection: $\omega^0{}_1 = -\frac{1}{s}\, d\tau$, then

$$\nabla_i \epsilon^0 = \left(\partial_\tau \epsilon^0 - \frac{1}{\sinh\sigma}\epsilon^1, \partial_\sigma \epsilon^0\right), \qquad \nabla_i \epsilon^1 = \left(\partial_\tau \epsilon^1 + \frac{1}{\sinh\sigma}\epsilon^0, \partial_\sigma \epsilon^1\right).$$

Thus we see that the covariant derivative defined in (4.5) is nothing but two-dimensional covariant derivative.

The mass dimension of a fluctuation is derived through its behavior near the boundary. Hence we examine fluctuations near $\sigma = 0$. Let us rewrite $S_{2B} = \int d^2\xi\, \mathcal{L}_{2B}$ by using (4.6). Near $\sigma = 0$,

$$\mathcal{L}_{2B} \approx \frac{1}{4\pi}\sqrt{g}\left[g^{ij} D_i \tilde{\zeta}^A D_j \tilde{\zeta}^A + \left(\tilde{\zeta}^0\right)^2 + \left(\tilde{\zeta}^1\right)^2 + 2\sum_{i=2,3,4}\left(\tilde{\zeta}^i\right)^2\right].$$

Here we note that the contribution of fluctuations $\tilde{\zeta}^0$ and $\tilde{\zeta}^1$ is the same form as the ghost action near $\sigma = 0$, so that it is canceled out by the ghost contribution. In addition we should note that $g^{ij} D_i \tilde{\zeta}^a D_j \tilde{\zeta}^a \approx g^{ij} \partial_i \tilde{\zeta}^a \partial_j \tilde{\zeta}^a$ $(a = 5, 6)$.

As a result, we are left with three massive bosons with $m^2 = 2$ and five massless bosons propagating in AdS_2. The symmetry preserved by the fluctuations is $SO(3) \times SO(5)$. The mass dimensions $\Delta = \frac{1}{2}(1 + \sqrt{1 + 4m^2})$ corresponding to the fluctuations are

$$\Delta\left(\tilde{\zeta}^i\right) = 2 \quad (i = 2, 3, 4), \qquad \Delta\left(\tilde{\zeta}^{a'}\right) = 1 \quad (a' = 5, 6, \ldots, 9).$$

Thus we have reproduced the dictionary obtained in Section 3 and [13] as expected. As the induced metric reduces to (2.8) near $\sigma = 0$, the fermionic fluctuations are the same as those given in Section 2.2. The mass dimensions of fermionic fluctuations are $\Delta(\vartheta^\alpha) = \frac{3}{2}$ $(\alpha = 1, \ldots, 8)$.

4.4. The fluctuations near $\sigma = \infty$

In the region near $\sigma = \infty$ we should introduce the ghost action (2.9) with flat two-dimensional metric

$$g_{ij} = d\tau^2 + d\sigma^2, \qquad R^{(2)} = 0. \tag{4.7}$$

Then we can see that the longitudinal modes are canceled out with the ghost action as follows.

First let us consider the fluctuations in the neighbor of $\sigma = \infty$. Then by using (4.7) $\mathcal{L}_{2B}$ can be rewritten as

$$\mathcal{L}_{2B} \approx \frac{1}{4\pi}\sqrt{g}\left[g^{ij} D_i \tilde{\zeta}^A D_j \tilde{\zeta}^A + \sum_{i\neq 0,5}\left(\tilde{\zeta}^i\right)^2\right],$$

where $D_i \tilde{\zeta} \approx \partial_i \tilde{\zeta}$. The contribution of fluctuations $\tilde{\zeta}^0$ and $\tilde{\zeta}^5$ is the same form as the ghost action, hence it is canceled out by the ghost contribution. Thus we are left with eight massive bosons with $m^2 = 1$ propagating in the two-dimensional flat space. The fermionic fluctuations with $m^2 = 1$ break $SO(8)$ to $SO(4) \times S(4)$ (see for example [20]). This is nothing but the Lagrangian density of a pp-wave string.

5. Small deformations of circular Wilson loop

Let us consider the quadratic fluctuations discussed in the previous section as a small deformation of a circular Wilson loop with local operator insertions.

The classical solution given by (4.2) and (4.3) is attaching to the boundary. The boundary of the classical string worldsheet can be seen as a circle on $\mathbb{R}^4$. The classical solution on the boundary is described by

$$x^\mu = \left(\frac{\ell \sinh\tau}{\cosh\tau \pm \alpha}, \frac{\pm\ell\sqrt{1-\alpha^2}}{\cosh\tau \pm \alpha}, 0, 0\right),$$

and the circle is represented by

$$(x_0)^2 + (x_1 + \alpha R)^2 = R^2, \qquad R = \frac{\ell}{\sqrt{1-\alpha^2}}.$$

That is, the radius of the loop is R and its center is located at $(x_0, x_1) = (0, -\alpha R)$. It is also convenient to introduce radial coordinates r and θ

$$\tilde{x}^\mu = (x_0, x_1 + \alpha R, x_2, x_3) = (r\sin\theta, r\cos\theta, x_2, x_3).$$

Then the circle lies along $(r, \theta) = (R, s)$.

We are now considering the local operator insertions Z^J and $\bar{Z}^J$. Here we define $Z \equiv \phi_1 + i\phi_2$ and $\bar{Z} \equiv \phi_1 - i\phi_2$. As explained in [16], Z^J and $\bar{Z}^J$ are inserted at $\tau = -\infty$ and $\tau = +\infty$, respectively. That is, Z^J and $\bar{Z}^J$ are inserted at $s = s_1, s_2$ with $(x_0, x_1) = (-\ell, 0)$ and $(+\ell, 0)$, respectively.

Let us consider the Wilson loop

$$W(C) = \operatorname{Tr}\mathcal{W}, \quad \mathcal{W} = P\exp\left(\oint ds\left(iA_\mu\left(x(s)\right)\dot{\tilde{x}}^\mu(s) + \phi_i\left(x(s)\right)\dot{y}^i(s)\right)\right).$$

The following expressions are useful later:

$$\frac{\delta}{\delta x^\mu(s)}\mathcal{W} = P\left(iF_{\mu\nu}\dot{x}^\nu(s) + D_\mu\phi_i\dot{y}^i(s)\right)\mathcal{W}, \qquad \frac{\delta}{\delta\dot{y}^i(s)}\mathcal{W} = P\phi_i(s)\mathcal{W}.$$

Let us take C_0 as

$$C_0: \quad \begin{cases} \tilde{x}^\mu = (R\sin s, R\cos s, 0, 0), \ \dot{y}^i = (0,0,0,0,0,R), & s \neq s_1, s_2, \\ \tilde{x}^\mu = (R\sin s_1, R\cos s_1, 0, 0), \ \dot{y}^i = (1, i, 0, 0, 0, 0), & s = s_1, \\ \tilde{x}^\mu = (R\sin s_2, R\cos s_2, 0, 0), \ \dot{y}^i = (1, -i, 0, 0, 0, 0), & s = s_2, \end{cases}$$

then

$$\begin{aligned} W(C_0) &= \operatorname{Tr}P\left[e^{\oint Z}(s_1)e^{\oint\bar{Z}}(s_2)\exp\left(\oint ds\, R\left(iA_\theta\left(\theta(s)\right) + \phi_6\left(\theta(s)\right)\right)\right)\right] \\ &= \sum_{J,J'}\frac{1}{J!J'!}\operatorname{Tr}P\left[Z(s_1)^J\bar{Z}(s_2)^{J'}\exp\left(\oint ds\, R\left(iA_\theta\left(\theta(s)\right) + \phi_6\left(\theta(s)\right)\right)\right)\right], \end{aligned}$$

which does not vanish only when $J = J'$.

We may consider three kinds of impurity insertions as depicted in Fig. 1. Let us consider each of the cases below.

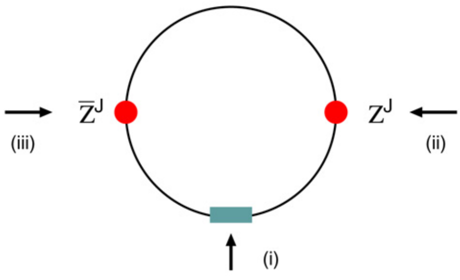

Fig. 1. The impurity insertions onto the circular Wilson loop with the local operator insertions. Three types of insertions can be considered. For (i) the insertion obeys the rule for the circular case without the local operators. For (ii) the insertion rule is given by the BMN dictionary. The case (iii) is a complex conjugate of (ii).

5.1. A small deformation at $s \neq s_1, s_2$

Let us consider a small deformation of C_0 at $s \neq s_1, s_2$. This corresponds to the case (i) in Fig. 1. The Wilson loop $W(C)$ is expanded as

$$W(C) = W(C_0) + \oint ds \left(\delta x^\mu \operatorname{Tr} P\left[\left(i F_{\mu\nu}\dot{x}^\nu + D_\mu \phi_i \dot{y}^i\right)\mathcal{W}\right]\Big|_{C_0} + \delta \dot{y}^i \operatorname{Tr} P[\phi_i \mathcal{W}]|_{C_0}\right) + \cdots$$

where ellipsis are higher order fluctuations. One derives Eqs. (3.2) and (3.3) with

$$\mathcal{W}_{C_0} = \sum_J \frac{1}{J!J!} Z(s_1)^J \bar{Z}(s_2)^J \exp\left(\oint ds\, R(iA_\theta + \phi_6)\right).$$

As in Section 3, by requiring that the small deformation should meet locally supersymmetry condition $0 = (\dot{Y} + \delta\dot{Y})^2$, and by using an $SO(2)$, we have $\delta\dot{y}^6 = 0$ and $\delta\theta = 0$. As a result we are left with impurities (3.4). As expected, the conformal dimensions of these impurities agree with the mass dimensions of fluctuations $\Delta = \frac{1}{2}(1 + \sqrt{1 + 4m^2})$:

$$\Delta\left(\tilde{\xi}^i\right) = 2 \quad (i = 2, 3, 4), \qquad \Delta\left(\tilde{\xi}^{a'}\right) = 1 \quad (a' = 5, \ldots, 9).$$

This is nothing but the dictionary for the gauge-theory operators corresponding to the fluctuations around the circular solution (non-relativistic string) [10].

5.2. A small deformation at $s = s_1, s_2$

Next let us expand $W(C)$ around C_0 at $s = s_1$. This corresponds to the cases (ii) and (iii) in Fig. 1. The argument here basically follows [18].

In this case the Wilson loop can be expanded as

$$W(C) = W(C_0) + \oint ds \left(\delta x^\mu \operatorname{Tr} P\left[D_\mu Z(s_1)\mathcal{W}\right]\Big|_{C_0} + \delta\dot{y}^i \operatorname{Tr} P\left[\phi_i(s_1)\mathcal{W}\right]\Big|_{C_0}\right) + \cdots.$$

By requiring that the small fluctuations should satisfy the locally supersymmetry condition, we obtain the following condition,

$$\delta\dot{y}^1 + i\delta\dot{y}^2 = 0.$$

In addition a reparametrization invariance allows us to fix the fluctuations as

$$\delta\dot{y}^1 - i\delta\dot{y}^2 = 0.$$

Thus we can impose the condition $\delta\dot{y}^1 = \delta\dot{y}^2 = 0$.

As a result, we are left with impurities inserted at $s = s_1$,

$$D_\mu Z \quad (\mu = 0, 1, 2, 3), \qquad \phi_i \quad (i = 3, 4, 5, 6).$$

Similarly, for a small deformation at $s = s_2$ we have the impurities inserted at $s = s_2$,

$$D_\mu \bar{Z} \quad (\mu = 0, 1, 2, 3), \qquad \phi_i \quad (i = 3, 4, 5, 6).$$

The impurities respect $SO(4) \times SO(4)$ symmetry. This dictionary is nothing but the BMN one.

In summary, the resulting dictionary says that the impurity insertion at the local operators should follow the BMN-dictionary and other than the position of local operators it should follow the dictionary we obtained in [10].

6. Summary and discussion

We have discussed a semiclassical approximation around the classical solutions corresponding to circular Wilson loops without/with local operator insertions Z^J and $\bar{Z}^J$. We have derived a quadratic action for each of the cases and identified the fluctuations with small deformations of the Wilson loop. With the local insertions the action behaves as the non-relativistic string around $\sigma = 0$, while it as the pp-wave string around $\sigma = \infty$.

The dictionary of operator insertion has been clarified from the viewpoint of a small deformation of Wilson loops without/out local operator insertions. Without the local operators it is the same as in the case of straight Wilson line. With them we have discussed it around $\sigma = 0$ and $\sigma = \infty$. Near $\sigma = 0$ it is the same as in the case of the circular Wilson loop without the local operators. Near $\sigma = \infty$ it is nothing but the BMN one as expected. This result nicely agrees with the behavior of the quadratic fluctuations.

For the case of the circular Wilson loop with local operator insertions, it remains as a problem to be solved to find a ghost terms to remove the unphysical longitudinal modes for an arbitrary σ. It would be interesting to compute a semiclassical partition function after solving these problems.

One may consider a spin chain description for the circular Wilson loop case. The gauge-theory analysis is discussed in [17]. The remaining work is to construct circular Wilson loop solutions rotating with two or more spins and compare them with the gauge-theory results. A part of this issue has already been discussed in [21].

It is also nice to consider a semiclassical analysis to a dual giant Wilson loop [22]. A rotating dual giant Wilson loop solution has been already constructed in [23]. By following this paper, one can discuss the semiclassical limit around the solution and it is possible to deduce the corresponding gauge-theory side.

We believe that our approach would give a new window to test the AdS/CFT duality.

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Appendix A. Vielbeins and spin connections for $\mathrm{AdS}_5 \times \mathrm{S}^5$

Let us summarize vielbeins and spin connections of $\mathrm{AdS}_5 \times \mathrm{S}^5$. The metric utilized in Section 2 is slightly different from the one in Section 4. In both of them we set the common radius of AdS_5 and S^5 to be 1.

A.1. The metric in Section 2

The EAdS_5 metric in the Poincaré coordinate system, which was used in Section 2, is given by

$$ds^2 = \frac{1}{z^2}\left(r^2\, d\theta^2 + dr^2 + dx_2^2 + dx_3^2 + dz^2\right).$$

Here the two-dimensional subspace is described by a polar coordinate. The vielbein and spin connection are

$$E^a = \left(\frac{r}{z}d\theta, \frac{1}{z}dr, \frac{1}{z}dx_2, \frac{1}{z}dx_3, \frac{1}{z}dz\right),$$

$$\Omega^1{}_4 = -\frac{1}{z}dr, \quad \Omega^0{}_4 = -\frac{r}{z}d\theta, \quad \Omega^0{}_1 = d\theta, \quad \Omega^2{}_4 = -\frac{1}{z}dx_2, \quad \Omega^3{}_4 = -\frac{1}{z}dx_3.$$

Then we choose the metric of S^5 as

$$ds^2 = d\varphi_1^2 + \cos^2\varphi_1\left(d\varphi_2^2 + \cos^2\varphi_2\left(d\varphi_3^2 + \cos^2\varphi_3\left(d\varphi_4^2 + \cos^2\varphi_4\, d\varphi_5^2\right)\right)\right).$$

The vielbein and spin connection are

$$\begin{aligned}
E^{a'} &= (d\varphi_1, \cos\varphi_1\, d\varphi_2, \cos\varphi_1\cos\varphi_2\, d\varphi_3, \cos\varphi_1\cos\varphi_2\cos\varphi_3\, d\varphi_4,\\
&\qquad \cos\varphi_1\cos\varphi_2\cos\varphi_3\cos\varphi_4\, d\varphi_5),\\
\Omega^6{}_5 &= -\sin\varphi_1\, d\varphi_2, \quad \Omega^7{}_5 = -\sin\varphi_1\cos\varphi_2\, d\varphi_3, \quad \Omega^7{}_6 = -\sin\varphi_2\, d\varphi_3,\\
\Omega^8{}_5 &= -\sin\varphi_1\cos\varphi_2\cos\varphi_3\, d\varphi_4, \quad \Omega^8{}_6 = -\sin\varphi_2\cos\varphi_3\, d\varphi_4, \quad \Omega^8{}_7 = -\sin\varphi_3\, d\varphi_4,\\
\Omega^9{}_5 &= -\sin\varphi_1\cos\varphi_2\cos\varphi_3\cos\varphi_4\, d\varphi_5, \quad \Omega^9{}_6 = -\sin\varphi_2\cos\varphi_3\cos\varphi_4\, d\varphi_5,\\
\Omega^9{}_7 &= -\sin\varphi_3\cos\varphi_4\, d\varphi_5, \quad \Omega^9{}_7 = -\sin\varphi_4\, d\varphi_5.
\end{aligned}$$

A.2. The metric in Section 4

The EAdS_5 metric in the Poincaré coordinate system, which was used in Section 4, is given by

$$ds^2 = \frac{1}{z^2}\left(dx_i^2 + dz^2\right),$$

where $i=0,\dots,3$ and the four-dimensional subspace is described by Cartesian coordinates. The vielbein and spin connection are

$$E^a=\left(\frac{1}{z}dx_0,\frac{1}{z}dx_1,\frac{1}{z}dx_2,\frac{1}{z}dx_3,\frac{1}{z}dz\right),$$
$$\Omega^i{}_4=-\frac{1}{z}dx_i\quad (i=0,\dots,3).$$

Then the metric of S^5 is chosen as

$$ds^2=\cos^2\theta\,d\psi^2+d\theta^2+\sin^2\theta\big(d\varphi_1^2+\cos^2\varphi_1\big(d\varphi_2^2+\cos^2\varphi_2\,d\varphi_3^2\big)\big).$$

The vielbein and spin connection are computed as follows:

$$E^{a'}=(\cos\theta\,d\psi,d\theta,\sin\theta\,d\varphi_1,\sin\theta\cos\varphi_1\,d\varphi_2,\sin\theta\cos\varphi_1\cos\varphi_2\,d\varphi_3),$$
$$\Omega^5{}_6=-\sin\theta\,d\psi,\quad \Omega^7{}_6=\cos\theta\,d\varphi_1,\quad \Omega^8{}_6=\cos\theta\cos\varphi_1\,d\varphi_2,\quad \Omega^8{}_7=-\sin\varphi_1\,d\varphi_2,$$
$$\Omega^9{}_6=\cos\theta\cos\varphi_1\cos\varphi_2\,d\varphi_3,\quad \Omega^9{}_7=-\sin\varphi_1\cos\varphi_2\,d\varphi_3,\quad \Omega^9{}_8=-\sin\varphi_2\,d\varphi_3.$$

Appendix B. Fermionic fluctuations of Wilson loop

A supersymmetrized Wilson loop, which is proposed in [24], is given by

$$W=\frac{1}{N}\operatorname{Tr}P\big[\mathrm{e}^{\int\bar\zeta(s)Q\,ds}\mathrm{e}^{\int(iA_\mu\dot x^\mu+\phi_i\dot y^i)\,ds}\mathrm{e}^{-\int\bar\zeta(s)Q\,ds}\big].$$

Here the loop includes a superpartner of $(x^\mu(s),y^i(s))$, which couples to the fermion Ψ. The supersymmetry transformation in $\mathcal{N}=4$ SYM is given by

$$[Q,A_M]=\frac{i}{2}\Gamma_M\Psi,\qquad \{Q,\Psi\}=-\frac{1}{4}\Gamma_{MN}F^{MN},$$

and the following relation may also be included:

$$[Q,\dot x_M]=\frac{i}{4}\Gamma_M\dot\zeta.$$

A small deformation for the fermionic variables may be considered. By setting that $\zeta=\bar\zeta=0$ on C_0, the only contribution is evaluated as

$$\delta W|_{\text{fermion}}=\frac{1}{N}\int ds\operatorname{Tr}P\big[\delta\bar\zeta(s)R\Gamma_\theta(1-i\Gamma_{\theta6})\Psi\mathrm{e}^{\int(iA_\mu\dot x^\mu+\phi_i\dot y^i)\,ds}\big].$$

Here note that the matrix defined as

$$\mathcal{P}\equiv\frac{1}{2}(1-i\Gamma_{\theta6})$$

is a projection operator. The original fermionic variable Ψ has 16 components but it projects out half of it. As a result, the physical eight components of the fermionic variables remain.

Thus the operator insertion for the fermionic fluctuations are described by the eight fermionic variables. We have discussed the circular case so far, but the argument for the straight line is the same. It is also the same even for the BMN case, where the eight fermions $i\Gamma_1h_+\Psi$ ($h_+=\frac{1}{2}(1+i\Gamma_{12})$) are inserted.

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