A New Class of Projection and Contraction Methods for Solving Variational Inequality Problems

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Abstract—This paper presents a new class of projection and contraction methods for solving monotone variational inequality problems. The methods can be viewed as combinations of some existing projection and contraction methods and the method of shortest residuals, a special case of conjugate gradient methods for solving unconstrained nonlinear programming problems. Under mild assumptions, we show the global convergence of the methods. Some preliminary computational results are reported to show the efficiency of the methods. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Variational inequality problems, Projection and contraction methods, Conjugate gradient methods, Global convergence.

1. INTRODUCTION

Let $\Omega$ be a closed convex subset of $\mathbb{R}^n$ and let $F$ be a continuous mapping from $\mathbb{R}^n$ into itself. The variational inequality problem, which we denote by VIP($F, \Omega$), is the problem of finding a vector $x^* \in \Omega$, such that

$$\langle x - x^* \rangle^T F(x^*) \geq 0, \quad \forall x \in \Omega. \quad (1)$$

When $\Omega = \mathbb{R}^n$, VIP($F, \Omega$) reduces to the system of nonlinear equations,

$$F(x) = 0. \quad (2)$$

Furthermore, if $F = \nabla f$, where $f: \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, then VIP($F, \Omega$) is equivalent to the unconstrained minimization problem,

$$\min_{x \in \mathbb{R}^n} f(x). \quad (3)$$

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Variational inequality problems find many important applications in elasticity, structural analysis, economics, transportation research, see, e.g., [1]. For solving variational inequality problems, many iterative methods have been established, see the excellent monographs [2,3] and the survey papers [4,5].

Among these iterative methods, the simplest one is the projection type methods. This type of methods, starting with any $x \in \mathbb{R}^n$, generates a sequence $\{x^k\}$ according to the following recursion,

$$x^{k+1} = P_{\Omega} \left[ x^k - \alpha (x^k) g(x^k) \right],$$

(4)

where $P_{\Omega}[\cdot]$ denotes the orthogonal projection from $\mathbb{R}^n$ onto $\Omega$, $-g(x)$ is a descent direction of the function $(1/2)\|x - x^*\|^2$ $(x^* \in \Omega^*$, the solution set of VIP($F, \Omega$)) at $x^k$ satisfying

$$(x^k - x^*)^T g(x^k) \geq \varphi(x^k) \geq 0,$$

(5)

$\varphi(x)$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}_+$ and

$$\varphi(x) = 0 \Leftrightarrow x \text{ is a solution of VIP($F, \Omega$)}.$$

In many papers [6–9], the authors referred to $g(x)$ as a profitable direction and $\varphi(x)$ as an error measure function, which measures how much $x$ fails to be a solution of VIP($F, \Omega$). Based on $g(x)$ and $\varphi(x)$, a class of projection and contraction methods were proposed [10–22]. The stepsize $\alpha(x)$ were generally taken as

$$\alpha(x) = \frac{\varphi(x)}{\|g(x)\|^2}.$$

(6)

The projection methods are attractive for large scale problems, especially when the projection to $\Omega$ is easy to implement. It uses little storage and can readily exploit any sparsity or separable structure of $F$ and $\Omega$.

It follows from (4)–(6) that the efficiency of the projection and contraction methods depends heavily on the profitable direction $g(x)$ and the corresponding error measure function $\varphi(x)$. Many efforts have been devoted to constructing new profitable functions satisfying

$$\|g^{\text{new}}(x)\| \leq \|g(x)\|$$

(7)

and

$$(x - x^*)^T g^{\text{new}}(x) \geq (x - x^*)^T g(x).$$

(8)

For example, see [9,15]. The basic idea is simple and can be summarized as follows. Let $x^*$ be any element of the solution set of VIP($F, \Omega$), we further assume that we have some $t(x) : \Omega \rightarrow \mathbb{R}^n$, such that

$$(x - x^*)^T t(x) \geq 0, \quad \forall x \in \Omega.$$

(9)

Thus, for any nonnegative constant $\beta$, we have

$$(x - x^*)^T (g(x) + \beta t(x)) \geq \varphi(x) \geq 0, \quad \forall x \in \Omega,$$

and $g(x) + \beta t(x)$ is also a profitable direction. The new profitable direction thus can be defined as

$$g^{\text{new}}(x) = g(x) + \beta t(x)$$

(10)

and, according to (7), the parameter $\beta$ should be

$$\beta(x) = \arg \min\{\|g(x) + \eta t(x)\| \mid \eta \geq 0\}.$$  

(11)
In other words,
\[ \beta(x) = \max \left\{ 0, -\frac{g(x)^T t(x)}{\|t(x)\|^2} \right\}. \] (12)

The new profitable direction \( g^{\text{new}}(x) \) satisfies (7),(8), thus it is a better profitable direction. A corresponding new stepsize thus can be calculated
\[ \alpha_{\text{new}}(x) = \frac{\varphi(x)}{\|g^{\text{new}}(x)\|^2} \geq \frac{\varphi(x)}{\|g(x)\|^2} = \alpha(x), \]
and the new iteration can be computed via
\[ x^{\text{new}} := P_\Omega \left[ x - \alpha_{\text{new}}(x) g^{\text{new}}(x) \right]. \] (13)

According to the above rule, a class of new projection and contraction methods can be constructed. However, how to choose \( t(x) \) is a main task and the method in [9,15] just for monotone complementarity problems.

AN ILLUSTRATIVE EXAMPLE. Consider the following example,
\[ F(x) = Mx = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]
and
\[ \Omega = \{ x \in \mathbb{R}^2 : \|x\| \leq 1 \}. \]

The associated VI problem has a unique solution \( x^* = (0, 0)^T \). We take the profitable direction as the extragradient
\[ g(x) = F(P_\Omega[x - F(x)]) = M \ast P_\Omega[x - Mx], \]
and since
\[ (x - x^*)^T (-F(x)) = -x^T Mx = 0, \]
we can take \( t(x) = -Mx \). Then, at \( x_0 = (1, 0) \),
\[ F(x_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]
\[ g(x_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}. \]

According to (12), we have
\[ \beta(x) = \max \left\{ 0, -\frac{g(x)^T t(x)}{\|t(x)\|^2} \right\} = \frac{\sqrt{2}}{2}, \]
and the new profitable direction is
\[ g^{\text{new}}(x) = g(x) + \beta t(x) = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}^T, \]
which is just the opposite direction from the current point to the solution \( x^* \).
On the other hand, conjugate gradient methods are very important methods for solving the unconstrained nonlinear optimization problem (3) or system of nonlinear equations (2) [23-25], which have similar attractive properties to projection and contraction methods for variational inequality problems. The search direction $d_k$ is a combination of the current steepest descent direction $-\nabla f(x^k)$ and the last search direction, i.e.,

$$d_k = \begin{cases} 
-\nabla f(x^k), & \text{if } k = 1, \\
-\nabla f(x^k) + \beta_k d_{k-1}, & \text{if } k \geq 2.
\end{cases}$$ (14)

An important special case of the conjugate gradient method is the method of shortest residuals, which was presented by Hestenes [26] and studied by Pytlak [27]. More recently, Dai and Yuan [28] proved the global convergence of the method and the computational results reported there are encouraging.

Note that the 'better' conditions in projection methods are similar to those of the method of shortest residuals. Inspired by these, in this paper, we propose a new class of projection and contraction methods. The search directions can be viewed as combinations of the search directions of the existing projection and contraction methods and the search direction used in the last iteration. Thus the new methods have all the favorable properties of projection methods and conjugate gradient methods.

This paper is organized as follows. In Section 2, we summarize some definitions and some basic properties of the projection operator. Section 3 describes the new class of methods formally and establish the global convergence of these methods. In Section 4, we summarize some existing profitable directions, which can be used in our new class of methods. We report some computational results in Section 5 and give some concluding remarks in the last section.

Throughout this paper, we assume that the solution set of VIP($F, \Omega$), denoted by $\Omega^*$, is nonempty and the projection on $\Omega$, denoted by $P_{\Omega}[]$, is simple to carry out.

2. PRELIMINARIES

In this section, we summarize some definitions about the underlying mapping $F$ and some properties of the projection operator $P_{\Omega}[]$. 

**Definitions.** Let $F$ be a mapping from a set $\Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then,

(a) $F$ is said to be monotone on $\Omega$, if

$$((u - v)^\top (F(u) - F(v)) \geq 0, \quad \forall u, v \in \Omega;$$
(b) $F$ is said to be strictly monotone on $\Omega$, if the above inequality holds strictly, i.e.,

$$(u-v)^T(F(u)-F(v)) > 0, \quad \forall u, v \in \Omega, \quad u \neq v;$$

(c) $F$ is said to be strongly monotone on $\Omega$ with modulus $\gamma > 0$, if

$$(u-v)^T(F(u)-F(v)) \geq \gamma \|u-v\|^2, \quad \forall u, v \in \Omega;$$

(d) $F$ is said to be pseudo-monotone on $\Omega$, if

$$(u-v)^TF(u) > 0 \Rightarrow (u-v)^TF(v) \geq 0, \quad \forall u, v \in \Omega;$$

(e) $F$ is said to be Lipschitz continuous on $\Omega$ with constant $L > 0$, if

$$\|F(u)-F(v)\| \leq L\|u-v\|, \quad \forall u, v \in \Omega.$$
3. METHODS AND CONVERGENCE

In this section, we will describe the methods and establish their global convergence.

Suppose we have a profitable direction $g(x)$ and the corresponding error measure function $\varphi(x)$. Our motivation in this paper is to present a new class of projection and contraction methods. To this end, we use (10) and (12) to construct a new search direction. The framework of this class of methods can be described as follows.

**Algorithm 3.1. A New Class of Projection and Contraction Methods.**

Step 0. Let $g$ and $\varphi$ be profitable function and error measure function, respectively.

Choose a starting point $x_1 \in \mathbb{R}^n$, $d_0 = 0$ and a number $\epsilon \geq 0$.

Step 1. Compute $\varphi(x_1)$. If $\varphi(x_1) \leq \epsilon$, stop; otherwise, let $k = 1$ and compute $d_k = g(x^k)$.

Step 2. If $k > 1$, compute $\beta_k$ via (12) and

$$d_k = g(x^k) + \beta_k d_{k-1}.$$  \hspace{1cm} (15)

Step 3. Compute the stepsize $\alpha_k$

$$\alpha_k = \frac{\varphi(x^k)^2}{\|d_k\|^2}.$$ \hspace{1cm} (16)

Step 4. Find the new iterate

$$x^{k+1} = P_\Omega [x^k - \alpha_k d_k].$$ \hspace{1cm} (17)

Step 5. If $\varphi(x^{k+1}) \leq \epsilon$, then stop; Otherwise, set $k := k + 1$ and goto Step 2.

**Remarks.** Note that the search direction $d_k$ is constructed in a similar way as the method of shortest residuals. That is, the new direction is a combination of the existed direction (profitable direction for projection methods and steep descent direction for shortest residuals) and the direction used in the last iteration. Note also that, if the parameters $\beta_k \equiv 0$, than the new method is just the old one and to get the new direction, we just need little more work than the original projection methods (vector product) per iteration.

We now begin the convergence analysis. To show the global convergence of the methods, we first show that $d_{k-1}$ is a nondecreasing direction for $(1/2)\|x - x^\ast\|^2$ at $x^k$.

**Lemma 3.1.** For any $k \geq 1$, we have

$$d_{k-1}^T (x^k - x^\ast) \geq 0.$$ \hspace{1cm} (18)

**Proof.** Note that this is trivially true for $k = 1$ since $d_0 = 0$. By induction, consider any $k \geq 2$ and assume that

$$d_{k-2}^T (x^{k-1} - x^\ast) \geq 0.$$

Using the definition of $d_{k-1}$, we have

$$d_{k-1}^T (x^k - x^\ast) = d_{k-1}^T (x^{k-1} - x^\ast) + d_{k-1}^T (x^k - x^{k-1})$$

$$= g(x^{k-1})^T (x^{k-1} - x^\ast) + \beta_{k-1}d_{k-2}^T (x^{k-1} - x^\ast) + d_{k-1}^T (x^k - x^{k-1})$$

$$\geq \varphi(x^{k-1}) - \|d_{k-1}\| \|P_\Omega [x^{k-1} - \alpha_{k-1} d_{k-1}] - x^{k-1}\|$$

$$\geq \varphi(x^{k-1}) - \|d_{k-1}\| \|\alpha_{k-1} d_{k-1}\|$$

$$= 0,$$ \hspace{1cm} (19)
where the first inequality follows from the induction hypothesis, the second one follows from
the definition of \( g(x) \) and the Cauchy-Schwarz inequality and the third one follows from
the nonexpansivity of the projection operator. This completes the proof.

The following theorem shows that the sequence \( \{x^k\} \) generated by the new projection and
contraction methods is bounded.

**THEOREM 3.2.** The sequence \( \{x^k\} \) generated by the projection and contraction methods for
variational inequality problems (1) satisfies

\[
\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \alpha_k \varphi(x^k),
\]

and the sequence \( \{x^k\} \) is bounded.

**PROOF.** It follows from (17) that

\[
\|x^{k+1} - x^*\|^2 = \|P_\Omega [x^k - \alpha_k d_k] - x^*\|^2.
\]

Since \( x^* \in \Omega \), it follows from the nonexpansivity of the projection operator that

\[
\|x^{k+1} - x^*\|^2 \leq \|x^k - x^* - \alpha_k d_k\|^2.
\]

Using the definition of \( d_k \), we get

\[
\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - 2\alpha_k \varphi(x^k) + \alpha_k^2 \|d_k\|^2
\]

\[
= \|x^k - x^*\|^2 - 2\alpha_k \varphi(x^k) + \alpha_k^2 \|d_k\|^2
\]

\[
\leq \|x^k - x^*\|^2 - 2\alpha_k \varphi(x^k) + \alpha_k \varphi(x^k)
\]

\[
= \|x^k - x^*\|^2 - \alpha_k \varphi(x^k),
\]

where the second inequality follows from (5) and the definition of \( \alpha_k \). Since \( \alpha_k \geq 0 \) and \( \varphi(x^k) \geq 0 \), we have

\[
\|x^{k+1} - x^*\|^2 \leq \cdots \leq \|x^1 - x^*\|^2.
\]

The assertion then follows immediately.

We are now in the position to present our global convergence results.

**THEOREM 3.3.** Suppose that \( g(x) \) and \( \varphi(x) \) are the profitable direction and the error measure
function, respectively. Furthermore, suppose that \( g(x) \) and \( \varphi(x) \) are continuous. Then, the
generated sequence \( \{x^k\} \) according to (17) converges to a solution of (1) globally.

**PROOF.** Since \( g(x) \) is continuous, and by Theorem 3.2, \( \{x^k\} \) is bounded, there is a constant
\( M > 0 \), such that

\[
\|g(x^k)\|^2 \leq M, \quad \forall k \geq 1.
\]

If \( \beta_k = 0 \), then \( \|d_k\|^2 = \|g(x^k)\|^2 \leq M \); otherwise,

\[
\|d_k\|^2 = \|g(x^k) + \beta_k d_{k-1}\|^2
\]

\[
= \|g(x^k)\|^2 + \left( \frac{(g(x^k)^T d_{k-1})^2}{\|d_{k-1}\|^2} \right)
\]

\[
\leq M,
\]
where the second inequality follows from the definition of $\beta_k$ (12). Thus, it follows from (20) that
\[
\|x^{k+1} - x^*\|^2 \leq \|x^{k} - x^*\|^2 - \frac{\varphi(x^k)^2}{M},
\]
which means
\[
\sum_{k=0}^{\infty} \varphi(x^k)^2 < \infty.
\]

Then, we have
\[
\lim_{k \to \infty} \varphi(x^k) = 0.
\]

Since $\{x^k\}$ is bounded, it has at least one cluster point. Let $\bar{x}$ be a cluster point of $\{x^k\}$ and $\{x^{k_j}\}$ be the subsequence converging to $\bar{x}$. Since $\varphi(x)$ is continuous, we have
\[
\varphi(\bar{x}) = \lim_{j \to \infty} \varphi(x^{k_j}) = 0.
\]

Since $\varphi$ is a measure function, $\bar{x}$ is a solution of the problem (1). We can take $x^* = \bar{x}$ in (21) and
\[
\|x^{k+1} - \bar{x}\| \leq \|x^k - \bar{x}\|, \quad \forall k \geq 1.
\]

The whole sequence $\{x^k\}$ thus converges to $\bar{x}$. This completes the proof.

4. SOME PROFITABLE DIRECTIONS

In the last several years, many profitable directions have been developed. The first projection method for solving nonlinear variational inequality problems is the Goldstein-Levitin-Polyak method [14,23], which starts with any $x^0 \in \Omega$, generates a sequence $\{x^k\}$ according to the recursion,
\[
x^{k+1} = P_\Omega \left[ x^k - \beta_k F(x^k) \right].
\]

When $F$ is Lipschitz continuous (with constant $L > 0$) and strongly monotone (with modulus $\tau > 0$) (See the definitions in Section 2), the sequence $\{x^k\}$ converges to the solution of VIP($F,\Omega$) globally for some suitable stepsize $\beta_k$. In fact, under the Lipschitz continuity and the strong monotonicity, there exists a constant $\delta > 0$, such that
\[
\|x - x^*\| \geq \delta \|e(x)\|^2. \tag{22}
\]

Thus, for any $x \in \Omega$,
\[
F(x)^\top (x - x^*) \geq F(x^*)^\top (x - x^*) + \tau \|x - x^*\|^2 \\
\geq \tau \delta \|e(x)\|^2 \\
\geq 0,
\]
and, $F(x)$ can be viewed as a profitable direction. Goldstein-Levitin-Polyak method was then extended to select $\beta_k$ self-adaptively [17,30].

If there exists a sequence $\{\beta_k\}$ and two numbers $\beta_{\text{min}} > 0$, $L \in (0, 1)$, such that $\beta_k \geq \beta_{\text{min}}$ and
\[
\|\beta_k (F(x^k) - F(P_\Omega [x^k - \beta_k F(x^k)])\| \leq L \|e(x^k, \beta_k)\|, \quad \forall k \geq 1, \tag{23}
\]
then we can prove [8] that
\[
(x^k - x^*)^\top (e(x^k, \beta_k) - \beta_k (F(x^k) - F(P_\Omega [x^k - \beta_k F(x^k)]))) \\
\geq e(x^k, \beta_k)^\top (e(x^k, \beta_k) - \beta_k (F(x^k) - F(P_\Omega [x^k - \beta_k F(x^k)]))) \\
\geq (1 - L) \|e(x^k, \beta_k)\|^2,
\]
which implies that we can use
\[ g(x^k) = e(x^k, \beta_k) - \beta_k (F(x^k) - F(P_{\Omega}[x^k - \beta_k F(x^k)])) \] (24)
as the profitable direction at the \( k \)th iteration.

When \( F \) is pseudo-monotone, then, from
\[ F(x^*)^T(x - x^*) \geq 0, \quad \forall x \in \Omega, \]
we have
\[ F(x)^T(x - x^*) \geq 0, \quad \forall x \in \Omega. \]
Thus, if (23) holds, we can get a new profitable direction, by combining (24) and \( F \):
\[ \hat{g}(x) = e(x^k, \beta_k) + \beta_k F(P_{\Omega}[x^k - \beta_k F(x^k)]), \]
which is used in [31].

Some profitable directions are developed for solving linear variational inequality problems, that is
\[ F(x) = Mx + q, \]
where \( q \in \mathbb{R}^n \) is a given vector and \( M \in \mathbb{R}^{n \times n} \) is a given positive semidefinite matrix. In [15], based on the inequalities,
\[ (x - x^*)^T(MT e(x) + (Mx + q)) \geq (Mx + q)^T e(x) \] (25)
and
\[ (Mx + q)^T e(x) \geq \| e(x) \|^2, \quad \forall x \in \Omega, \] (26)
He developed the profitable direction,
\[ g_1(x) = M^T e(x) + Mx + q. \] (27)
In [6,7], based on the inequalities,
\[ (x - x^*)^T(I + MT) e(x) \geq \| e(x) \|^2 + (x - x^*)^T M(x - x^*) \geq \| e(x) \|^2, \]
He developed another profitable direction,
\[ g_2(x) = (I + M^T) e(x). \] (28)

When \( \Omega = \mathbb{R}_+^n \), the nonnegative orthant of \( \mathbb{R}^n \), VIP(\( F, \Omega \)) reduces to nonlinear complementarity problem NCP(\( F \)) of finding \( x \in \mathbb{R}^n \), such that
\[ x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0. \] (29)
A strategy to construct ‘better’ profitable directions for NCP(\( F \)) is to define the index set
\[ N(x) = \{ i \mid x_i = 0 \text{ and } g_i(x) \geq 0 \} \]
and
\[ B(x) = \{ 1, 2, \ldots, n \} \setminus N(x), \]
then, denote

\[ x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \]

\[ g(x) = \begin{pmatrix} g_B(x) \\ g_N(x) \end{pmatrix}. \]

From the definition of \( N(x) \), we can see that for the NCP (29),

\[ -\begin{pmatrix} 0 \\ g_N(x) \end{pmatrix}^T (x - x^*) = -g_N(x)^T (x_N - x_N^*) \geq 0. \]

Thus, we can take

\[ t(x) = -\begin{pmatrix} 0 \\ g_N(x) \end{pmatrix} \]

and from (11), \( \beta \equiv 1 \), and

\[ g^{\text{new}}(x) := g(x) + \beta t(x) = \begin{pmatrix} g_B(x) \\ 0 \end{pmatrix}. \]

It is easy to verify that the new profitable direction \( g^{\text{new}} \) satisfies (7) and (8), and thus can be viewed as better direction than the original one. This strategy has been used in [9,15].

5. NUMERICAL RESULTS

In order to give some insight to the behavior of the new class of projection and contraction methods, we implement it in MATLAB to solve some variational inequality problems.

Note that by combining the profitable directions in Section 4 and the framework of Algorithm 3.1, many new profitable directions and thus some new projection and contraction methods can be derived. In the following, we first pay our attention to the profitable directions \( g_1 \) and \( g_2 \) defined by (27) and (28), respectively. Based on these directions, we construct two new projection and contraction methods. For comparison, we also code the original projection and contraction methods [5,15] based on \( g_1 \) and \( g_2 \) and denote these methods by PC1 and PC2, respectively. For simplicity, we denote our methods by NPC1 and NPC2.

The first problem under consideration is the linear complementarity problem with

\[ F(x) = Mx + q, \]

where

\[ M = \begin{bmatrix} 1 & 2 & \cdots & \cdots & 2 \\ 0 & 1 & 2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad q = (-1, -1, \cdots, -1)^T. \]

This problem is a standard test problem and was used in many papers [20,21,32], for which Lemke’s method is known to run in exponential time. The unique solution is \((0, \ldots, 0, 1)^T\). We tested these methods with the dimension varying from 8 to 2000. The results are reported in Table 1. The column ‘N’ denotes the dimension of the problem. The numerical results are given in the form \( I/C \), where \( I \) is numbers of iterations and \( C \) is cpu time. The initial point \( x^1 \) is \((10, \cdots, 10)\) and the stopping criterion is

\[ \| e(x^k) \| \leq \epsilon \]
Table 1. Numerical results.

<table>
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<tr>
<th>N</th>
<th>PC1</th>
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<th>NPC2</th>
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<td>8/0.03</td>
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<td>13/30.76</td>
<td>2973/1103.86</td>
<td>1473/976.78</td>
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</table>

Table 2. Numerical results with different initial points.

<table>
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<th>Trial</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC1</td>
<td>16</td>
<td>18</td>
<td>21</td>
<td>16</td>
<td>25</td>
<td>27</td>
<td>29</td>
<td>26</td>
<td>46</td>
<td>31</td>
</tr>
<tr>
<td>NPC1</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>25</td>
<td>28</td>
<td>23</td>
<td>21</td>
<td>12</td>
<td>16</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 3. Number of $\beta_k > 0$.

<table>
<thead>
<tr>
<th>Trial</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPC1</td>
<td>6</td>
<td>8</td>
<td>11</td>
<td>4</td>
<td>11</td>
<td>11</td>
<td>13</td>
<td>12</td>
<td>13</td>
<td>12</td>
</tr>
</tbody>
</table>

and $\epsilon$ is set to $10^{-6}$. Note also that since $\Omega = R^n_+$, the projection in the sense of the Euclidean norm is very easy to carry out. For any $y \in R^n$, $P_R[y]$ is defined as component-wise

$$(P_R[y])_j = \begin{cases} y_j, & \text{if } y_j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

From Table 1, we found that NPC1 performs well comparable to PC1 and NPC2 performs much better than PC2, especially when $N$ is large. Therefore, the new class of projection and contraction methods will be a promising alternative of the existing methods.

To give more information on the comparison of the new method NPC1 and original one (PC1), we implement them on the above example with dimension $n = 600$ and the initial point generated randomly in $(0, 100)$, each for 10 times. Table 2 gives the number of iterations of the two methods and Table 3 gives the number $k$ for $\beta_k > 0$ for each trial in NPC1.

We now consider the nonlinear complementarity problems (29). In our test problem, we take

$$F(x) = D(x) + Mx + q,$$

where $D(x)$ and $Mx + q$ are the nonlinear part and the linear part of $F(x)$, respectively. We form the linear part $Mx + q$ similarly as in [33]. The matrix $M = A^TA + B$, where $A$ is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, 5)$ and a skew-symmetric matrix $B$ is generated in the same way. The vector $q$ is generated from a uniform distribution in the interval $(-500, 500)$. In $D(x)$, the nonlinear part of $F(x)$, the components are $D_j(x) = a_j \arctan(x_j)$ and $a_j$ is a random variable in $(0, 1)$.

\footnote{In the paper by Harker and Pang [33], the matrix $M = A^TA + B + D$, where $A$ and $B$ are the same matrices as here, and $D$ is a diagonal matrix with uniformly distributed random variable $d_{jj} \in (0, 0.3)$.}
We use the modified Khobotov-Korpelevich extragradient method in [17] as the original method, which is outlined as follows.

Step 1. Select parameters $\rho \in (0, 1)$ and $\beta_0 > 0$ and $x^0 \in \Omega$ and $k = 0$.

Step 2. $\bar{x}^k = P_{\Omega} [x^k - \beta_k F(x^k)]$.

Step 3. If

$$ r_k := \beta_k^2 \left\| F(x^k) - F(\bar{x}^k) \right\|^2 $$

$$ \leq \| x^k - \bar{x}^k \|^2 \leq \rho^2 $$

then set $x^{k+1} = P_{\Omega} [x^k - \beta_k F(x^k)]$,

$$ \beta_{k+1} = \begin{cases} 
\beta_k (1 + r_k), & \text{if } r_k \leq 0.5 \rho^2, \\
\beta_k, & \text{otherwise},
\end{cases} $$

and $k = k + 1$, go to Step 2, where $r_k$ is a nonnegative adjusting factor.

Step 4. Reduce the value of $r_k$, e.g.,

$$ \beta_k := \min \left( \frac{\beta_k}{2}, \frac{1}{\sqrt{2}} \| x^k - \bar{x}^k \| \right) $$

set $\bar{x}^k = P_{\Omega} [x^k - \beta_k F(x^k)]$ and go to Step 3.

The convergence criterion utilized in the test was also

$$ \| e(x^k) \|_{\infty} \leq \varepsilon. $$

The tolerance $\varepsilon$ is set to be $10^{-6}$ and the elements of initial points $x_0$ were generated from a uniform distribution in the interval $(0, 1)$. Table 4 summarizes the computational results for 10 trails with the dimension $N$ varying from 50 to 500.

In Table 4, “IN” denotes the number of iterations, “CPU” denotes the cpu time in seconds and “NB” denotes the number that $\beta_k > 0$. From this table, we can see that both the number of iterations and the cpu time in the modified method are smaller than those in the original method, showing the ability of the new method.

<table>
<thead>
<tr>
<th>$N$</th>
<th>IN</th>
<th>CPU</th>
<th>IN</th>
<th>CPU</th>
<th>NB</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>646</td>
<td>0.190</td>
<td>157</td>
<td>0.070</td>
<td>125</td>
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<tr>
<td>100</td>
<td>359</td>
<td>0.200</td>
<td>94</td>
<td>0.090</td>
<td>40</td>
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<tr>
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<td>575</td>
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<td>0.160</td>
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<tr>
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<tr>
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<td>5412</td>
<td>37.524</td>
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<td>13.920</td>
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</table>
6. CONCLUSIONS

We have presented a new class of projection and contraction methods for solving monotone variational inequality problems. The new methods can be an improvement of the existing projection and contraction methods by the strategy of shortest residuals, a special case of conjugate gradient methods for solving unconstrained optimization problems. Under some suitable conditions, we show the global convergence of the methods. The preliminary computational results show that our strategy can introduce computational efficiency, and the new class of projection and contraction methods will be a promising alternative of the existing methods.

REFERENCES


