Types directed by constants

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Let $T$ be a complete, countable, first-order theory having infinite models. We introduce types directed by constants, and prove that their presence in a model of $T$ guaranties the maximal number of non-isomorphic countable models: $I(\mathbb{N}_0, T) = 2^{\mathbb{N}_0}$.

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Shelah in [6] proved that a countable theory with an infinite, definable, linear order and Skolem functions has $2^{\mathbb{N}_0}$ non-isomorphic countable models. He reduced the general case to the cases $T_1 = \text{Th}(\omega, <, \ldots)$ and $T_2 = \text{Th}(\omega + \omega^*, <, \ldots)$ (where $\omega^*$ is reversed ordered $\omega$, and is here attached on top of $\omega$). In the first case he proved that an arbitrary countable, complete linear order can be 'coded' in a model of $T_1$ by a chain of certain end extensions (using Rubin’s proof, see also [5]). In the second case, assuming that $\omega$ is not definable, Shelah first defined when two elements in a model of $T_2$ are 'near', then showed that the model can be decomposed into convex, closed under nearness components and, finally, he showed that an arbitrary linear order with successors and predecessors can be coded by the order of components. Since there are $2^{\mathbb{N}_0}$ non-isomorphic countable, complete, linear orders with successors and predecessors, the conclusion followed.

In this paper we are interested in coding linear orders in partially ordered structures $(\mathcal{M}, \leq, \ldots)$ which are somehow similar to the above. We fix an infinite subset $\mathcal{C} \subseteq \text{dcl}(\emptyset)$ (playing the role of $\omega$ in a saturated model of $T_1$ or $T_2$) and show that an arbitrary linear order can be coded in a model of (a slight modification of) $T = \text{Th}(\mathcal{M}, \leq, \ldots)$, provided that $T$ is small ($|S(T)| = \aleph_0$) and:

1. $\{ x \in \mathcal{C} \mid c \leq x \}$ is a co-finite subset of $\mathcal{C}$ for all $c \in \mathcal{C}$; and
2. $\mathcal{C}$ is an initial part for $\mathcal{M}$: $c \in \mathcal{C}$ and $m \leq c$ imply $m \in \mathcal{C}$.

The slight modification is in that we may need to absorb a (single) parameter into the language and to shrink $\mathcal{C}$ if necessary.

Condition (1) here is quite strong, it is stronger than: $(\mathcal{C}, \leq)$ is a directed, well partial ordering of height $\omega$. We will see that its model-theoretic version, condition (D1) from Definition 2.1, describes it as a 'generic linearity', which seems to be a natural assumption needed for coding linear orders.

To describe our proof assume that $(\mathcal{M}, \leq, \ldots)$ is saturated and that $\mathcal{C} \subseteq \text{dcl}(\emptyset)$ satisfies the above two conditions. $\mathcal{C}$ determines a (possibly incomplete) 1-type $p_{\mathcal{C}}(x)$ which is, as a subset of $S_1(\emptyset)$, the set of all accumulation points of $\{\text{tp}(c) \mid c \in \mathcal{C}\}$. We will work exclusively inside $\mathcal{C} \cup p_{\mathcal{C}}(\mathcal{M})$, which turns out to be an initial segment of $(\mathcal{M}, \leq)$ with

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\( \mathcal{C} \subset p_{\mathcal{C}}(\mathcal{M}) \) (this is why we will call \( p_{\mathcal{C}} \) a \( \mathcal{C} \)-directed type). The proof is based on the redefined nearness relation and the related \( \mathcal{C} \)-independence. For \( a, b \in p_{\mathcal{C}}(\mathcal{M}) \), we will consider \( a \) to be near \( b \) if \( a \preceq b \) and \( tp(a/b) \) is not finitely satisfiable in \( \mathcal{C} \) (in our terminology: \( tp(a/b) \) is not a \( \mathcal{C} \)-type); \( (a_1, \ldots, a_n, \ldots) \) is \( \mathcal{C} \)-independent if each \( tp(a_{i+1}/a_1a_2\ldots a_i) \) is a \( \mathcal{C} \)-type (non-algebraic and finitely satisfiable in \( \mathcal{C} \)). Assuming smallness, we will prove that our \( T \) can be slightly modified (and \( \mathcal{C} \) replaced by a subset) so that the resulting \( p_{\mathcal{C}}(x) \) becomes complete and so called strongly \( \mathcal{C} \)-directed. Roughly speaking, if the type is not strongly directed (which is the case in Shelah’s \( T_1 \)), we will add a parameter to make it look like \( T_2 \), and then we will be able to get a strongly directed type. Thus we will reduce the general case to the \( T_2 \)-like case. In the case of a strongly directed type we will show that the equivalence relation generated by our nearness is particularly well behaved: it will turn out that being in distinct classes is the same as being \( \mathcal{C} \)-independent, and that the (complete) type of a \( \mathcal{C} \)-independent set is determined by its \( \prec \)-type; in other words, if we factor out the equivalence, the only structure on the quotient is the one induced by \( \prec \). Finally, we will show that any linear order can be coded in this way; thus, even in the case of Shelah’s \( T_2 \), our coding is finer than his original one.

**Theorem 1.** Suppose that \( T = \text{Th}(\mathcal{M}, \preceq, \ldots) \) is small, \( p_{\mathcal{C}}(x) \in S_1(\emptyset) \) is strongly \( (\mathcal{C}, \preceq) \)-directed, and \( M \models T \).

(a) If \( I \subset p_{\mathcal{C}}(M) \) is a maximal \( \mathcal{C} \)-independent set, then \( (I, \prec) \) is a linear order whose isomorphism type does not depend on the particular choice of \( I \).

(b) For every linear order there is \( N \models T \) such that the order type of a maximal \( \mathcal{C} \)-independent subset of \( p_{\mathcal{C}}(N) \) is isomorphic to it.

**Corollary 1.** If \( T \) is a complete, countable theory having a type over a finite domain which is directed by constants, then \( I(N_0, T) = 2^{\aleph_0} \).

The original motivation for my work comes from Anand Pillay’s work on elementary extensions of first-order structures in his Ph.D. thesis; see also [1–4]. There he proves that an arbitrary countable first-order structure has at least four countable elementary extensions which are non-isomorphic under isomorphisms fixing the ground structure point-by-point, and conjectures that the number must be infinite; in other words, he conjectures:

If \( T \) is the elementary diagram of a countable model then \( I(N_0, T) \geq \aleph_0 \).

This article contains one main ingredient of the proof, **Corollary 1** above. The other, a dichotomy theorem for minimal structures (every definable subset is either finite or co-finite), is contained in [9]. It asserts that either \( \text{Sem} \) (semi-isolation, defined below) is a pregeometry operator on the whole monster model, or there is a type (over a finite domain) directed by constants around. The two quickly produce the proof of the conjecture, as described in [9].

The paper is organized as follows: Section 1 contains a review of definitions and facts used later. In Section 2 we define \( \mathcal{C} \)-directed types and prove that the definition is equivalent to the one sketched above. In Section 3 we prove that every definable subset of \( \mathcal{C} \cup p_{\mathcal{C}}(\mathcal{M}) \) contains a minimal element, which will compensate the absence of Skolem functions. For the rest of the paper smallness of \( T \) will be assumed, and used essentially in the proofs. In Section 4 we consider intervals \([a, b]\) where \( tp(a/b) \) is a \( \mathcal{C} \)-type as ‘large’ ones, and prove that a ‘small’ interval cannot contain a large one; here, for \( a \preceq b \), \([a, b]\) is small means also that \( a \) is near \( b \). In Section 5 strongly directed types are defined, and it is described how they can be obtained from directed types. In the remaining sections we focus on the locus of a strongly directed type. In Section 6 we prove first, that the equivalence relation induced by the nearness is the semi-isolation \((x \in \text{Sem}(y))\), then that the quotient set is linearly ordered, and then the uniqueness of the type of a \( \mathcal{C} \)-sequence (of fixed length). The remaining needed fact for the proof of **Theorem 1** is the degeneracy of \( \mathcal{C} \)-independence: every pairwise \( \mathcal{C} \)-independent set is \( \mathcal{C} \)-independent; it also implies that the only structure on the quotient is the one induced by \( \preceq \). Degeneracy is proved in the last section, where **Theorem 1** is proved, too.

### 1. Preliminaries

Throughout the paper \((\mathcal{M}, \ldots)\) is a fixed uncountable, saturated first-order structure in a countable language, and \( T = \text{Th}(\mathcal{M}, \ldots) \) (\( T \) is assumed to be small from Section 4 on). \( a, b, \ldots, \bar{a}, \bar{b}, \ldots \) and \( A, B, \ldots \) will denote elements, tuples of elements and countable subsets of \( \mathcal{M} \); \( M, N, \ldots \) will denote its (countable) elementary submodels. By a type in variables \( \bar{x} \) we mean any consistent set of formulas all of whose free variables are among \( \bar{x} \); thus a type is not necessarily complete. For \( \phi(x) \) a formula, by \( \phi(A) \) we mean the set \( \{ a \in A \mid A \models \phi(a) \} \); similarly for \( \phi(M), p(A) \) and \( p(M) \) where \( p(x) \) is a type.

Important notions throughout the paper are those of a \( \mathcal{C} \)-type and of a \( \mathcal{C} \)-sequence. Since we will be interested only in 1-types the definition here is restricted to them: If \( A \subseteq B \) and \( \subseteq dcl(A) \) then by a \( \mathcal{C} \)-type (or a \( \mathcal{C} \)-coheir) we will mean a non-algebraic 1-type over \( B \) which is finitely satisfiable in \( \mathcal{C} \) (every formula from the type is satisfied by an element of \( \mathcal{C} \)). \((a_1, a_2, \ldots, a_n)\) is a \( \mathcal{C} \)-sequence over \( B \) if \( tp(a_{k+1}/a_1a_2\ldots a_kB) \) is a \( \mathcal{C} \)-type for all relevant \( k \).

**Definition 1.1.** Suppose \( \mathcal{C} \subseteq dcl(A) \) and \( I \subset \mathcal{M} \) is \( \mathcal{C} \)-independent over \( A \) if every finite subset of \( I \) can be arranged into a \( \mathcal{C} \)-sequence over \( A \).
Topological interpretation of \( C \)-types is that \( p \in S_1(A) \) is a \( C \)-type if and only if \( p \) is an accumulation point of \( \{tp(c/A) \mid c \in C \} \). The set of all \( C \)-types from \( S_1(A) \) forms a closed subset of \( S_1(A) \) which is defined by

\[
p_C(x) = \{ \phi(x) \mid \phi \text{ is over } A \text{ and } \phi(C) \text{ is co-finite in } C \};
\]

here \( \phi(C) \) is co-finite in \( C \) means that \( C \setminus \phi(C) \) is finite. Note that a 1-type \( p(x) \) over \( A \) is a \( C \)-type iff \( p(x) \vdash p_C(x) \).

**Definition 1.2.** Suppose \( C \subseteq dcl(A) \) and \( p \in S_1(A) \). We say that \( p \) is a \( C \)-isolated type if it is an isolated point in the space of all \( C \)-types in \( S_1(A) \).

**Fact 1.1.** (1) (Extension property of \( C \)-types) Let \( A \subseteq B \) and \( C \subseteq dcl(A) \). Any \( C \)-type over \( A \) can be extended to a complete \( C \)-type in \( S_1(B) \).

(2) If \( p \in S_1(A) \) is a \( C \)-type then there is \( C' \subseteq C \) such that \( p \) is the unique \( C' \)-type in \( S_1(A) \).

(3) If \( p \in S_1(A \overline{\text{-isolated}}) \), then there is a \( C(x) \) such that \( p = C(x) \) is the unique \( C \)-type in \( S_1(A) \).

Recall Pillay’s notion of semi-isolation for complete types. \( tp(\bar{a}/A) \) is semi-isolated over \( \bar{b} \) if there is a formula \( \phi(\bar{x}, \bar{b}) \in tp(\bar{a}/A) \) such that \( \phi(\bar{x}, \bar{b}) \vdash tp(\bar{a}/A) \). If \( A = \emptyset \) we simply say that \( \bar{a} \) is semi-isolated over \( \bar{b} \), or that \( \bar{b} \) semi-isolates \( \bar{a} \).

In this article we are interested in semi-isolation on the locus of a single 1-type \( p(x) \), which is not necessarily complete. The semi-isolation operator \( \text{Sem}_p \) is defined on subsets of \( p(M) \) if \( p(x) \) is over \( A \) and \( E \subseteq p(M) \) we define

\[
\text{Sem}_p(E) = \{ d \in p(M) \mid \text{there is } \phi(x) \in tp(d/EA) \text{ such that } \phi(x) \vdash p(x) \}.
\]

Whenever the meaning of \( p \) is clear from the context, we will write \( \text{Sem} \) instead of \( \text{Sem}_p \). As a binary relation \( (x \in \text{Sem}_p(y)) \), semi-isolation is easily seen to be reflexive and transitive but, in general, it is not symmetric.

The following fact follows from Proposition 1 in [8].

**Fact 1.2.** Suppose \( C \subseteq dcl(A) \), \( a \in M \) and \( \bar{b}_1, \bar{b}_2 \in M^n \) are such that

(i) \( tp(\bar{b}_1/A) = tp(\bar{b}_2/A) \),

(ii) \( tp(\bar{b}_2/A) \) is semi-isolated over \( \bar{b}_1 \), and

(iii) \( tp(a/A\bar{b}_1) \) is a \( C \)-type.

Then \( tp(a\bar{b}_1/A) = tp(a\bar{b}_2/A) \) and, in particular, \( tp(a/A\bar{b}_2) \) is a \( C \)-type.

We reserve symbols \( \leq \) (and also \( \leq_{\phi, \leq'_{\phi}} \)) for partial orders; \( < \) is reserved for quasi-orders (reflexive and transitive), they will be used only in Section 2. By ordered structures we mean those of the form \( (M, \leq, \ldots) \). In ordered structures by \( a < b \) we mean \( a \leq b \) and \( a \neq b \), while \( a \perp b \) stands for \( a \nleq b \) and \( b \nleq a \). \( a < B \) denotes \( a < b \) for all \( a \in A \) and \( b \in B \); similarly for \( A \leq B, a < A, a \leq A, A < a \) and \( A \leq a \). \( a \) is a minimal element of \( A \) if \( a \in A \) and there is no \( b \in A \) such that \( b \leq a \); similarly, maximal elements are defined.

In quasi-ordered structures \( a < b \) means \( a \leq b \) and \( b \neq a \). The associated partial order \( a \leq b \) is defined by \( a < b \lor a = b \). \((A, \leq)\) is directed if it is non-empty and for all \( a, b \in A \) there is \( d \in A \) such that \( a < d \) and \( b < d \).

Ordered structures \((M, \leq, \ldots)\), which we will be dealing with, satisfy certain chain conditions for definable sets:

(MIN) Every definable non-empty subset of \( M \) has a minimal element;

(MAX) Every definable non-empty subset of \( M \) has a maximal element;

Note that (MIN) implies that every non-maximal element (immediate) successor, while (MAX) implies the existence of predecessors of non-minimal elements. Further (MIN), and (MAX) as well, can be expressed by a set of first-order sentences, so that their presence is a property of \( \text{Th}(M, \leq, \ldots) \). Therefore, if (MIN) or (MAX) is satisfied in \((M, \leq, \ldots)\), then it is satisfied in any structure elementarily equivalent to it and vice versa.

The following simple, technical lemma and the proposition following it, when combined with the smallness, will replace the use of Skolem functions (note that (MIN) in linear orders implies existence of Skolem functions).

**Lemma 1.1.** Suppose \((M, \leq, \ldots)\) satisfies at least one of (MIN) or (MAX). Then \( p(M) \) is an antichain whenever \( p \in S_1(\emptyset) \) is isolated.

**Proof.** Suppose that \( p \in S_1(\emptyset) \) is isolated, by \( \phi(x) \) say, and that \( a, b \in \phi(M) \) satisfy \( a < b \). We will prove that \( \phi(M) \) has neither a maximal nor a minimal element.

\[ (\exists y)(\phi(y) \land a < y) \in tp(a) = p. \]

But \( p \) is isolated by \( \phi(x) \), so:

\[ \models \phi(x) \rightarrow (\exists y)(\phi(y) \land x < y). \]

Therefore \( \phi(M) \) does not have a maximal element. Similarly, \( \phi(M) \) does not contain a minimal element. \( \square \)

**Proposition 1.1.** Suppose that \( T = \text{Th}(M, \leq, \ldots) \) is small and that \( M \) is prime over a finite subset. If \( T \) satisfies (MIN) or (MAX) then \((M, <)\) does not contain a copy of the rationales.
Suppose that \( \{m_q \mid q \in Q \} \subseteq M \) is ordered as the rationales are and we prove that \( T \) is not small. Choose for each \( q \in Q \) a formula \( \phi_q(x) \) isolating \( tp(m_q) \). For each real \( r \) define \( p_r(x) \) to be the Dedekind cut for \( r \):

\[
(\exists y)(\phi_q(y) \land y < x) \mid q \in Q \land q < r) \cup (\exists y)(\phi_q(y) \land x < y) \mid q \in Q \land r < q).
\]

First we prove that each \( p_r(x) \) is consistent: Note that every finite subset of \( p_r(x) \) is contained in some \( p_q(x) \ (q \in Q) \), which is consistent being realized by \( m_q \); by compactness \( p_r(x) \) is consistent, too.

Further we prove that \( p_r \)'s are pairwise contradictory. For suppose \((r, r') \) is a non-empty interval and \( q \in Q \cap (r, r') \). Then:

\[
(\exists y)(\phi_q(y) \land y < x) \in p_r(x) \quad \text{and} \quad (\exists y)(\phi_q(y) \land x < y) \in p_r(x).
\]

We claim that the conjunction of these two formulas is inconsistent with \( T \): if it were satisfied by \( d \in M \) there would be \( a, b \in \phi_q(M) \) with \( a < d < b \). But this is not possible since, by Lemma 1.1, \( \phi_q(M) \) is an antichain.

Therefore \( p_r \)'s are distinct and pairwise contradictory and \( T \) is not small. \( \square \)

2. \( c \)-directed types

In this section we define types directed by constants. The definition is model-theoretical and in terms of quasi-orders. The reason for using quasi-orders is that in [9], assuming that \( \text{Sem} \) is not a pregeometry operator, a type directed by constant is found satisfying precisely conditions \( \text{D1} \) and \( \text{D2} \) below. In Proposition 2.1 below, we prove that quasi-orders can be replaced by directed partial orders, and then in the rest of the paper we will work exclusively with partial orders. Another slight inconvenience is a possible incompleteness of a type directed by constants (over a finite set, say). It will be fixed by Proposition 2.2 where, assuming in addition that \( T \) is small, we prove that such a type has a completion over the same domain which is directed by constants, too.

**Definition 2.1.** (a) An infinite set \( c \) directs a type over \( A \) if \( c \subseteq dcl(A) \) and there is an \( A \)-definable quasi-order \( \preceq \) on \( M \) such that the following two conditions are satisfied:

- (D1) If \((a, b) \) is a \( c \)-sequence over \( A \) then \( b \preceq a \);
- (D2) For all \( c \in c \): \( d \preceq c \) implies \( d \in c \).

(b) A type \( p(x) \) is \( c \)-directed over \( A \) if \( c \) directs a type over \( A \) and:

\[ p(x) = \{ \phi(x) \mid \phi(x) \text{ is over } A \text{ and } \phi(c) \text{ is co-finite in } c \} \]

We will also say that \( p(x) \) is \((c, \preceq)\)-directed over \( A \), where \( c \) witnesses that \( c \) directs a type over \( A \). If the parameter set of the type is clear from the context (e.g. if \( p(x) \in S_1(A) \)) then we simply say that the type is \( c \)-directed (or \((c, \preceq)\)-directed).

(c) A type is directed by constants if it is \( c \)-directed over \( A \) for some \( A \) and \( c \subseteq dcl(A) \).

Note that if \( c \) directs a type over \( A \) then there is a unique type which is \( c \)-directed over \( A \); it is \( p_c(x) \) (i.e. the set of all accumulation points of \( tp(c/A) \mid c \in c \) in \( S_1(A) \)). It may look a bit strange to the reader that \( p_c(x) \) does not depend on the particular choice of \( c \). The explanation is that we are interested in \( c \)-sequences (\( a, b \) as `independent' sequences (they denote that 'b is generic over a'), which gives rise to view \( b \preceq a \) in condition \( D1 \) as 'generic linearity', and then any two witnessing quasi-orders generically agree.

**Example 2.1.** Here are some examples of types directed by constants:

1. (\( \omega, \preceq \)). Here the unique non-algebraic \( 1 \)-type \( p \in S_1(\emptyset) \) is \( \omega \)-directed (witnessed by \( \preceq \)).
2. (\( \omega + \omega^*, \preceq \)), where \( \omega^* \) is reversed ordered \( \omega \) put on top of \( \omega \). In this case the unique non-algebraic \( 1 \)-type \( p \in S_1(\emptyset) \) is both \( \omega \)-directed (witnessed by \( \preceq \)) and \( \omega^* \)-directed (witnessed by \( \succeq \)). However, \( p \) is not \( \omega \) or \( \omega^* \)-directed.
3. (\( 2^{<\omega}, \preceq, c_0 \), \( n \in \omega \)) where \( c_n : n \to \{0\} \) and \( \preceq \) is the inclusion. Here \( c_0 \subset c_1 \subset \cdots \) and our structure is a binary tree of height \( \omega \) with all the elements from the branch \( c = \{ c_n \mid n \in \omega \} \) named. The unique \( c \)-type \( p \in S_1(\emptyset) \) is \( c \)-directed.

**Lemma 2.1.** Suppose that \( p(x) \) is \((c, \preceq)\)-directed over \( A \).

(a) For \( a \models p \) let \( c_a = \{ c \in c \mid c \preceq a \} \). Then: \( c_a \) is co-finite in \( c \) and does not depend on the particular choice of \( a \models p \); we will call it \( c_p \).

(b) \( c_p \subset c(M) \) and \( b < a \) whenever \( (a, b) \) is a \( c \)-sequence.

(c) \( (c_p, \preceq) \) is directed.

(d) \( c_p = \{ x \in c \mid c < x \} \) is co-finite in \( c \) for all \( c \in c_p \).

**Proof.** Without loss of generality \( A = \emptyset \). Let \( a \models p \).

(a) By way of contradiction, suppose that \( c_a \) is not co-finite in \( c \). Then \( c \setminus c_a \) is infinite, and there is \( b \) such that \( tp(b/a) \) is a \((c \setminus c_a)\)-type. Since \( x \not\preceq a \in tp(c/a) \) for all \( c \in c \setminus c_a \), we derive \( x \not\preceq a \in tp(b/a) \). On the other hand, \( (a, b) \) is a \( c \)-sequence so, by \( D1 \), \( b \preceq a \). A contradiction.

Now we prove that \( c_a = c_d \) for all \( d \models p \). So suppose that \( tp(d) \) is a \( c \)-type. Then there is \( d' \) such that \( (a, d') \) is a \( c \)-sequence and \( tp(d') = tp(d) \); in particular, we have \( c_{d'} = c_d \). By \( D1 \) we have \( a \preceq d' \) and thus \( c_a \subseteq c_{d'} = c_d \). Similarly \( c_d \subseteq c_a \), thus \( c_a = c_d \).
(b) For any \( c \in C \), by (D2), \( x \not\le c \) implies \( x \in C \), so \( a \not\le c \). Thus for all \( c \in C_p \) we have \( \models c \not\le a \land a \not\le c \) i.e. \( c < a \). This proves \( C < p(M) \) and, since \( tp(b/a) \) is a \( C \)-type, \( b < a \).

(c) Suppose \( c, c' \in C_p \) and consider the formula \( c < x \land c' < x \). It is satisfied by \( a \) so, since \( tp(a) \) is a \( C \)-type, there is \( c'' \in C \) satisfying it as well.

(d) If \( C \setminus C^c \) were infinite for some \( c \in C_p \), there would be \( b \) realizing a \( C_p \setminus C^c \)-type, and thus \( c \not\le b \). On the other hand \( tp(b) \) is a \( C \)-type so, by (b) \( c < b \). A contradiction. \( \square \)

**Proposition 2.1.** An infinite set \( C \subset dcl(A) \) directs a type over \( A \) if and only if there is an \( A \)-definable partial ordering \( \le \) on \( M \) such that:

1. \( \{ x \in C \mid c \le x \} \) is a co-finite subset of \( C \) (for all \( c \in C \)), and
2. \( (C, \le) \) is an initial part of \( (M, \le) \).

**Proof.** Without loss of generality let \( A = \emptyset \). First suppose that \( \le \) witnesses that \( C \) directs a type. Since, by Lemma 2.1(a), \( C_p \) is co-finite in \( C \), after slightly redefining \( \le \), we may assume that it is chosen so that \( C_p = C \). Let \( \le \) be the associated partial order. (2) follows from (D2) and (1) follows from Lemma 2.1(d).

For the other direction, suppose \( (C, \le) \) satisfies (1) and (2). (D2) is clearly satisfied, so suppose that \( (a, b) \) is a \( C \)-sequence and we prove \( b \le a \). Since for all \( c \in C \) \( C^c = \{ x \in C \mid c \le x \} \) is co-finite in \( C \), \( tp(a) \) is a \( C^c \)-type and thus \( c \le a \). Further, \( tp(b/a) \) is a \( C \)-type and \( c \le a \) (for all \( c \in C \)), imply \( b \le a \), thus (D1) is satisfied. \( \square \)

Having proved the proposition, we can deal only with partial orders; moreover, by Lemma 2.1(c), we can assume that \( (C, \le) \) is also directed.

**Convention.** From now on we will assume that \( (C, \le) \), witnessing that a type is directed by constants, is always chosen to be a directed partial order satisfying (1) and (2).

Another consequence of Proposition 2.1 is that if \( C \) directs a type over \( A \), and if we expand \( M \) (by adding new definable subsets), then \( C \) will direct a type over \( A \) in the new structure, too. Also, if \( (C, \le) \) directs a type and we take a reduct in which \( \le \) is definable then \( (C, \le) \) still directs a type in the new structure. For example, consider \( T = Th(\omega, \le, U_k)_{k \ge 2} \) where \( U_k \)'s are unary relations defined by: \( U_k(n) \) holds iff \( k \) divides \( n \). If we add to the language binary relations \( S_k(x, y) \) for \( x + k = y \) then we have elimination of quantifiers. The \( \omega \)-directed type here is incomplete (it does not decide whether \( U_k(x) \) or \( \neg U_k(x) \) holds). Moreover, no complete type from \( S_1(0) \) is directed by constants, since any \( C \) and \( \le \) can decide whether \( U_k(x) \) is in the \( C \)-type or not for only finitely many \( k \)'s (details are left to the reader).

Suppose that an incomplete type \( p(x) \) is \( (C, \le) \)-directed over \( A \). If we want to find a completion of \( p_C(x) \) which is directed by constants, we should consider subsets of \( C \) which direct a type over \( A \). Since \( p_C(x) \) is incomplete there is \( \phi(x) \) (over \( A \)) such that \( \phi(C) \) is infinite and co-infinite in \( C \). Thus \( C \) is split into two infinite pieces \( C_1 = \phi(C) \) and \( C_2 = \neg \phi(C) \). Note that condition (1) implies that there is an increasing \( \omega \)-sequence of members of \( C \) whose members with even indices are from \( C_1 \) and the others are from \( C_2 \); it witnesses that \( (C_1, \le) \) does not satisfy condition (2) (although it satisfies (1)). However, we can easily modify \( \le \) to get (2) satisfied, by simply making all the elements from \( C_2 \) incomparable to \( C_1 \), as explained in the following proposition.

**Proposition 2.2.** Suppose that \( p(x) \) is \( (C, \le) \)-directed over \( A \).

(a) If \( \phi(x) \) is over \( B \supseteq A \) and \( C \phi = \phi(C) \) is infinite, then \( (C_\phi, \le_\phi) \) directs a type over \( B \) where \( x \le_\phi y \) is defined by: \( \phi(x) \land \phi(y) \land x \le y \).

(b) ( \( T \) small) If \( A \) is finite then there is \( q \in S_1(A) \) which is directed by a subset of \( C \). \( q \) can be chosen to be \( (C_\phi, \le_\phi) \)-directed, where \( \phi(x) \) is any formula isolating a \( (\le_\phi) \)-type among the completions of \( p \).

**Proof.** (a) Suppose that \( c \in C_\phi \). Then, by (1) for \( (C, \le) \), \( c \le c' \) holds for all but finitely many \( c' \in C \). Thus \( c \le_\phi c' \) holds for all but finitely many \( c' \in C_\phi \) and condition (1) is satisfied. To verify (2), suppose that \( d \le_\phi c \). Then, by (2) for \( (C, \le) \), we have \( d \in C \) and \( d \in C_\phi \) follows.

(b) Suppose that \( \phi(x) \) isolates \( q(x) \in S_1(A) \) among the \( C_\phi \)-types from \( S_1(A) \). Then for any formula \( \psi(x) \) over \( A \) exactly one of \( \psi(C_\phi) \) and \( \neg \psi(C_\phi) \) is infinite, otherwise there would be at least two distinct complete \( C_\phi \)-types in \( S_1(A) \), one of which is a \( \psi(C_\phi) \)-type and the other a \( \neg \psi(C_\phi) \)-type. Thus \( p_{C_\phi}(x) \) is complete and, since \( q \) is a \( C_\phi \)-type, we have \( q(x) = p_{C_\phi}(x) \). By part (a) \( p_{C_\phi}(x) \) is \( (C_\phi, \le_\phi) \)-directed. \( \square \)

Minimal, ordered structures are a source of complete types which are directed by constants: if \( (M_0, \le, \ldots) \) is a minimal structure with an infinite \( \le \)-increasing chain, then it is not hard to realize that the unique non-algebraic \( 1 \)-type \( p \in S_1(M_0) \) is \( C \)-directed for all initial parts \( C \subset M_0 \) (see [7]). For example: \( (\omega, \le) \) and \( (\omega + \omega^*, \le) \) are minimal structures and the unique non-algebraic \( 1 \)-type is \( \omega \)-directed. It is interesting whether any complete type directed by constants originate from a minimal structure.

**Question.** If \( p \in S_1(\emptyset) \) is \( (C, \le) \)-directed, must \( (C, \le) \) be a minimal structure?

It is not hard to violate the linearity in \( (\omega, \le) \) keeping the minimality unharmed: simply replace each \( n \in \omega \) by an \( n \)-element antichain, and make its elements greater than the elements from \( m \)-element antichains for all \( m < n \). This
modification is inessential because it interprets \((\omega, \leq)\): \(x \perp y\) defines an equivalence relation and, if we factor it out, we end up with \((\omega, \leq)\). I could not find an example of a minimal, ordered structure which is ‘essentially’ non-linear, but I believe that such structures exist.

**Question.** Is there a minimal structure \((M, \leq)\) which is directed and does not interpret \((\omega, \leq)\)?

### 3. Min–max conditions

In this section \(p\) is a fixed \((\mathcal{E}, \leq)\)-directed type (over \(\emptyset\), for simplicity). Note that we do not assume completeness of \(p\), although the facts proved here will be applied to complete types. From now on, the object of our study is \((\mathcal{E} \cup p(\mathcal{M}), \leq)\) and the impact of external parameters will not be interesting for us. In the next lemma we collect facts that will be used further in the text without specific mentioning.

**Lemma 3.1.** \((a)\) \(\{x \in \mathcal{E} \mid x \leq c\}\) is finite for all \(c \in \mathcal{E}\).
\((b)\) \(\mathcal{E} < p(\mathcal{M})\).
\((c)\) \(\mathcal{E} \cup p(\mathcal{M})\) is an initial part of \(\mathcal{M}\) (\(p(\mathcal{M})\) is convex).
\((d)\) Every \(\mathcal{E}\)-sequence is decreasing.
\((e)\) Every \(\mathcal{E}\)-independent set is linearly ordered by \(<\).

**Proof.** Here only part \((c)\) requires proof. So let \(e \in p(\mathcal{M})\), \(a \notin \mathcal{E}\) and \(a \leq e\). It suffices to prove that \(a \in p(\mathcal{M})\). Suppose \(\psi(x) \in \text{tp}(a)\). Then:

\[\models (\exists y) (\psi(y) \land y \leq e).\]

Since \(\text{tp}(e)\) is a \(\mathcal{E}\)-type there is \(c \in \mathcal{E}\) satisfying:

\[\models (\exists y) (\psi(y) \land y \leq c).\]

Therefore \(\models \psi(d)\) holds for some \(d \leq c\) and, by \((D2)\), \(d \in \mathcal{E}\). We have just shown that any \(\psi(x) \in \text{tp}(a)\) is satisfied by an element of \(\mathcal{E}\) and, since \(a \notin \mathcal{E}\), it follows that \(\text{tp}(a)\) is a \(\mathcal{E}\)-type. Thus \(a \models p.\)

In the next lemma we establish a connection between \(\mathcal{E}\)-coheirs and semi-isolation. Recall:

\[\text{Sem}_p(E) = \{d \in p(\mathcal{M}) \mid \text{there is } \phi(x) \in \text{tp}(d/EA) \text{ such that } \phi(x) \vdash p(x)\}.

**Lemma 3.2.** Suppose \(\emptyset \neq E \subset p(\mathcal{M})\) and \(a \in p(\mathcal{M})\). Then:

\(\text{tp}(a/E)\) is a \(\mathcal{E}\)-type if and only if \(a < \text{Sem}_p(E)\).

In particular: if \(\text{tp}(a/E)\) is a \(\mathcal{E}\)-type then \(a < E\).

**Proof.** First assume that \(a < \text{Sem}_p(E)\) and let \(\phi(x, E) \in \text{tp}(a/E)\). It suffices to show that \(\phi(x, E)\) is satisfied by an element of \(\mathcal{E}\). If \(e \in E\) then \(\phi(x, E) \land x < e\) \(\in \text{tp}(a/E)\) and, since \(a \notin \text{Sem}_p(E)\), there is \(d \notin p(\mathcal{M})\) such that \(\models \phi(d, E) \land d < e\). But \(d < e\) implies \(d \in \mathcal{E} \cup p(\mathcal{M})\). Thus \(d \in \mathcal{E}\).

For the other direction, suppose that \(\text{tp}(a/E)\) is a \(\mathcal{E}\)-type and \(d \in \text{Sem}_p(E)\). We will prove \(a < d\). Choose \(\theta(E, y) \in \text{tp}(d/E)\) witnessing \(d \in \text{Sem}_p(E)\): \(\theta(E, y) \vdash p(y)\). Since \((c < y) \in p(y)\) for all \(c \in \mathcal{E}\) we conclude:

\[\models (\forall y) (\theta(E, y) \rightarrow c < y).\]

Since \(\text{tp}(a/E)\) is a \(\mathcal{E}\)-type we derive

\[\models (\forall y) (\theta(E, y) \rightarrow a < y).\]

Finally \(\models \theta(E, d) \implies a < d.\)

In general, in ordered structures with \(\mathcal{E}\)-directed types we cannot expect neither (MIN) nor (MAX) to be necessarily satisfied: simply add to the structure a disjoint unary predicate with the ordered rationals in there. However, we will see in the next proposition that the two are locally satisfied: for definable subsets of \(\mathcal{E} \cup p(\mathcal{M})\). (b) says that we have even a strong (MIN) condition.

**Proposition 3.1.** Suppose \(e \in p(\mathcal{M})\) and \(D \subseteq \mathcal{M}\) is definable (with parameters) and non-empty.

(a) If \(D \subseteq \{x \in \mathcal{M} \mid x < e\}\) then \(D\) has a maximal and a minimal element.

(b) If \(D \subseteq p(\mathcal{M})\) then \(b < D\) for some \(b \in p(\mathcal{M})\). Moreover, if \(\mathcal{T}\) is small then \(b\) can be found realizing an isolated type over the parameters needed to define \(D\).

**Proof.** (a) Let \(D\) be defined via \(\psi(x, e, \bar{b})\) and \(\models \psi(x, e, \bar{b}) \rightarrow x < e\). There is a formula \(\phi(z)\) (without parameters) expressing:

‘for all \(\bar{y}\) if \(\emptyset \neq \psi(\mathcal{M}, z, \bar{y}) < z\) then \(\psi(\mathcal{M}, z, \bar{y})\) contains a maximal and a minimal element.’

For all \(c \in \mathcal{E}\) the set \(\{z \in \mathcal{M} \mid z < c\}\) is finite, so we have:
for all $\bar{y}$ if $\emptyset \neq \psi(\mathcal{M}, c, \bar{y}) < c$ then $\psi(\mathcal{M}, c, \bar{y})$ is finite and contains a maximal and a minimal element.

Therefore $\models \phi(c)$ holds for all $c \in \mathcal{C}$. Since $\text{tp}(e)$ is a $\mathcal{C}$-type we conclude $\models \phi(e)$.

(b) Suppose $D \subseteq p(\mathcal{M})$ and let $D' = \{x \in \mathcal{M} \mid x < D\}$ be the set of lower bounds of $D$. Clearly $\mathcal{C} \subseteq D' \subseteq p(\mathcal{M}) \cup \mathcal{C}$, $D'$ is definable so, by (a), it contains a maximal element $b$. But $\mathcal{C}$ has no maximal elements, since it is directed, so $b \notin \mathcal{C}$. Thus $b \in p(\mathcal{M})$.

To prove the ‘moreover’ part just notice that $D'$ is definable over the parameters needed to define $D$. □

Having established local versions of min–max conditions we can suitably reformulate Proposition 1.1, leaving the proof to the reader:

**Proposition 3.2.** If $M$ is prime over a finite subset then the rationales cannot be embedded into $(p(\mathcal{M}), \leq)$.

**Assumption.** From now on we assume that $T$ is small.

4. $\mathcal{C}$-intervals

In this section $p \in S_1(\emptyset)$ is a fixed ($\mathcal{C} \subseteq \leq$)-directed type. In ordered structures any pair of elements $a < b$ can be identified with the corresponding interval $[a, b] = \{x \in \mathcal{M} \mid a \leq x \leq b\}$. In our situation, for $a, b \in p(\mathcal{M})$, the pairs where $\text{tp}(a/b)$ is a $\mathcal{C}$-type are intuitively considered as ‘large’ intervals (a is ‘generic’ over b), and all the others are ‘small’; ‘$\{a, b\}$ is small’ we interpret also as ‘$a$ is near $b$’. Thus $[a, b]$ is a large interval iff $(a, b)$ is a $\mathcal{C}$-sequence (there is no danger of confusing sequences and intervals since we will use only closed intervals).

The main technical result in this section is Proposition 4.1, in which we justify the intuition by proving that a small interval cannot contain a large one.

**Definition 4.1.** Let $\mathcal{C} \subseteq \mathcal{C}$ be infinite and $a, b \in p(\mathcal{M})$, $b \leq a$.

(a) $[a, b]$ is a $\mathcal{C}$-interval iff $(a, b)$ a $\mathcal{C}$-sequence;

(b) $\mathcal{C}$-isolated interval is a $\mathcal{C}$-interval $[a, b]$ where $\text{tp}(b/a)$ is a $\mathcal{C}$-isolated type.

**Lemma 4.1.** (a) If $a, b \in p(\mathcal{M})$ and $b < a$ then the following three are all equivalent:

$\quad [a, b]$ is a $\mathcal{C}$-interval, $\text{tp}(b/a)$ is a $\mathcal{C}$-type, $b \notin \text{Sem}(a)$.

(b) If $[a, d]$ is a $\mathcal{C}$-interval and $b \in \text{Sem}(d)$ then $\text{tp}(da) = \text{tp}(ba)$; in particular $[a, b]$ is a $\mathcal{C}$-interval.

**Proof.** (a) follows immediately from the above definition and Lemma 3.2. (b) is a reformulation of Fact 1.2: note that $\text{tp}(a/d)$ is a $\mathcal{C}$-type and $b \in \text{Sem}(d)$ imply, by Fact 1.2, $\text{tp}(da) = \text{tp}(ba)$. □

**Proposition 4.1.** If $a \leq b < d \leq e$ are from $p(\mathcal{M})$ and $[b, d]$ is a $\mathcal{C}$-interval then $[a, e]$ is a $\mathcal{C}$-interval, too.

**Proof.** By way of contradiction, suppose that $\text{tp}(a/e)$ is not a $\mathcal{C}$-type. Then, by Proposition 3.1(b), we may assume that $\text{tp}(a/e)$ is an isolated type. Then consider the following definable set:

$$D = \{a_1, e_1 \mid a < a_1 < e_1 \leq e \text{ and } \text{tp}(ae) = \text{tp}(ae_1)\}.$$  

Choose $d'$ with $\text{tp}(d'd) = \text{tp}(ae)$. Since $\text{tp}(b/d)$ is a $\mathcal{C}$-type and $\text{tp}(d'/d)$ is isolated, by Lemma 4.1(b), we have $\text{tp}(bd) = \text{tp}(bd')$. In particular, we have $a \leq b < d'$, and hence $(d', d) \in D$ (so $D$ is non-empty). By (MAX), there is $a_1 \in D$ such that $a_1$ is maximum among such $(a_1, e_1)$. Since $\text{tp}(ae) = \text{tp}(a_1e_1)$, there is an automorphism $\sigma$ with $\sigma(ae) = a_1e_1$. Then the pair $(a_1, e_1) \in D$ contradicts the maximality.

We conclude the section with a technical lemma which will be used in Section 6.

**Lemma 4.2.** Suppose that $[a, d]$ is a $\mathcal{C}$-isolated interval, $a < b < d$ and $\text{tp}(b/ad)$ is isolated. Then if $[b, d]$ is a $\mathcal{C}$-interval, it must be a $\mathcal{C}$-isolated interval.

**Proof.** Suppose $\phi(d, y, a) \vdash \text{tp}(b/ad)$ and $a(b/a)$ is isolated by $R(d, x)$ among the $\mathcal{C}$-types from $S_1(\mathcal{M})$. Also, suppose that $[b, d]$ is a $\mathcal{C}$-interval. Then:

$$\models a < b < d \land \phi(d, b, a) \land R(d, a).$$

By Lemma 1.1 $\phi(d, \mathcal{M}, a)$ is an antichain, so there is a formula in $\text{tp}(b/d)$ expressing:

$$(\exists y)\left(\phi(d, \mathcal{M}, y) \text{ is an antichain} \land y < d < d \land R(d, y)\right).$$

We will prove that this formula witnesses $\mathcal{C}$-isolation of $[b, d]$: assuming that $[b', d]$ is a $\mathcal{C}$-interval with $\text{tp}(b'/d)$ containing the formula, we will show $\text{tp}(bd) = \text{tp}(bd')$. So suppose $[b', d]$ is a $\mathcal{C}$-interval and let $a'$ witnesses that $b'$ satisfies the formula:

$$\phi(d, \mathcal{M}, a') \text{ is an antichain} \land a' < b' < d \land \phi(d, b', a') \land R(d, a').$$

Then $a' \notin \mathcal{C}$: otherwise, $\phi(d, \mathcal{M}, a')$ would contain the set of all realizations of $\text{tp}(b'/d)$, which is not an antichain since $\text{tp}(b'/d)$ is a $\mathcal{C}$-type. Finally, $a' < b' < d$ implies $a' \in p(\mathcal{M}) \cup \mathcal{C}$ and thus $a' \in p(\mathcal{M})$.

$$a' < b' < d \text{ and } [b', d] \text{ is large, by Proposition 4.1, imply that } a' = R(d, a') \text{, by the isolation property of } R(d, y) \text{, implies } \text{tp}(a'd) = \text{tp}(ad'). \text{Then the isolation property of } \phi(d, y, a) \text{ and } \models \phi(d, b', a') \text{ implies } \text{tp}(a'bd) = \text{tp}(abd). \text{Thus } \text{tp}(bd) = \text{tp}(bd').$$

1 The following proof, much shorter than the original one, was made by the referee. Thanks to him/her.
5. Strongly directed types

In this section we introduce strongly directed types and show that any type directed by constants (in a small theory) can be extended to a strongly directed one: for this, it suffices to absorb into the language a single parameter and shrink \( C \) if necessary.

**Definition 5.1.** (a) \( p(x) \in S_1(\emptyset) \) is strongly \( C \)-directed via \( \phi(x) \in p(x) \) if there is an \( A \)-definable partial order \( \leq \) such that \( p(x) \) is \((C, \leq)\)-directed, \((\phi(M), \leq, \ldots)\) satisfies both (MIN) and (MAX), and:

(D3) For all \( r(x) \in S_1(\emptyset) \) with \( \phi(x) \in r(x) \) exactly one of the following holds:

\[
    r(M) < p(M); \quad r = p; \quad p(M) < r(M).
\]

In this case we will also say that \( p(x) \) is strongly \((C, \leq)\)-directed via \( \phi(x) \).

(b) \( p(x) \in S_1(\emptyset) \) is strongly \( C \)-directed if there is \( \phi(x) \in p(x) \) such that \( p(x) \) is strongly \( C \)-directed via \( \phi(x) \).

(c) \( p(x) \in S_1(\emptyset) \) is strongly directed if there are \( A = dcl(A) \) and \( \phi(x) \in p(x) \) such that \( p(x) \) is strongly \( C \)-directed via \( \phi(x) \).

**Remark 5.1.** (1) If \( p(x) \in (\omega, \leq) \)-directed, then \( r(M) < p(M) \) in (D3) is equivalent to: \( r = tp(c/A) \) for some \( c \in C \).

(2) \( (\omega + \omega, <) \) the unique \( \omega \cup \omega^* \)-type \( p \in S_1(\emptyset) \) is \( \omega \)-directed via \( \omega = x \); it is strongly \( \omega^* \)-directed via \( \omega = x \), as well. On the other hand, due to the failure of (MAX), in \((\omega, \leq)\) the \( \omega \)-directed type is not strongly directed; the same is with the binary tree with a branch named.

(3) Had we required in the definition that \((M, \leq, \ldots)\) satisfies both (MAX) and (MIN) nothing would have been substantially changed: simply replace \( \leq \) by \( \leq^\phi \) (where \( x \leq^\phi y \) is \( \phi(x) \land \phi(y) \land x < y \)). This is not so important since for all our purposes we will be able to assume in addition that \( \phi(x) \) is \( x = x \). The main advantage provided by \( \phi(x) \) is that from now on the object of our study is no longer \( \mathcal{E} \cup p(M) \), which is not definable, but a definable set \( \phi(M) \), where we have both (MAX) and (MIN) satisfied and our \( p(M) \) nested between \( \mathcal{E} \) and the realizations of all other types containing \( \phi(x) \).

**Question.** If \((M, \leq, \ldots)\) satisfies both (MAX) and (MIN) and \( p \) is a \((\mathcal{E}, \leq)\)-directed complete type, must \( p \) be strongly \( \mathcal{E} \)-directed?

**Proposition 5.1.** Suppose that \( p(x) \in S_1(\emptyset) \) is \((\mathcal{E}, \leq)\)-directed and \( e \models p \).

(a) Suppose that \( q(x) \in S_1(e) \) is \( \mathcal{E} \)-isolated type extending \( p(x) \) and that \( \phi(x) \in q(x) \) isolates it among the \( \mathcal{E} \)-extensions of \( p(x) \); also, suppose (wlog) that \( \models \phi(x) \rightarrow x < e \). Then \( q(x) \) is strongly \((\mathcal{E}_\phi, \leq^\phi)\)-directed via \( \phi(x) \), where \( \mathcal{E}_\phi = \phi(e, \mathcal{E}) \) and \( x \leq^\phi y \) is defined by: \( \phi(x) \land \phi(y) \land x < y \).

(b) There exists an infinite \( \mathcal{E} \subseteq \mathcal{E} \) and a strongly \( \mathcal{E} \)-directed \( q(x) \in S_1(e) \) which extends \( p(x) \).

(c) If \( p(x) \) is strongly \((\mathcal{E}, \leq)\)-directed via \( \phi(x) \) and \( q(x) \in S_1(e) \) is the unique \( \mathcal{E} \)-type extending \( p(x) \), then \( q(x) \) is strongly \((\mathcal{E}, \leq)\)-directed via \( \phi(x) \).

**Proof.** (a) \( q(x) \) is the unique \( \mathcal{E}_\phi \)-type in \( S_1(e) \) by our choice of \( \phi \). **Proposition 2.2(a)** applies and \( q(x) \) is \((\mathcal{E}_\phi, \leq^\phi)\)-directed. To show that \((\mathcal{M}, \leq, \ldots)\) satisfies both (MAX) and (MIN) suppose that \( D \subseteq \mathcal{M} \) is definable and non-empty. If \( D \not= \emptyset \) then any element from the intersection is both \( \leq^\phi \)-minimal and \( \leq^\phi \)-maximal (being not \( \leq^\phi \)-comparable to any other element of \( \mathcal{M} \)). Now assume that \( D \subseteq \phi(e, \mathcal{M}) \) is non-empty. Then \( \leq \) and \( \leq^\phi \) agree on \( D \) and, in particular, any \( \leq \)-maximal (minimal) element of \( D \) is also \( \leq^\phi \)-maximal (minimal). By our assumptions on \( \phi(x) \) we have \( D \subseteq \{ x \in \mathcal{M} \mid x < e \} \), so, by **Proposition 3.1(a), D** contains a \( \leq^\phi \)-minimally and \( \leq^\phi \)-maximally element.

To prove (D3) suppose that \( \phi(x) \in r \in S_1(e) \), \( r(x) \not= q(x) \) and \( r(x) \not= tp(c/e) \) for all \( c \in \mathcal{E}_\phi \). It suffices to prove \( q(M) \not< r(M) \Rightarrow \models \phi(x) \rightarrow x < e \) implies \( r(M) \subseteq p(M) \); since \( q(x) \) is the unique \( \mathcal{E} \)-type containing \( \phi(x) \) in \( S_1(e) \) there is \( \theta(e, x) \in r(x) \) which is satisfied by no \( c \in \mathcal{E}_\phi \). Thus \( \theta(e, x) \land \phi(e, x) \models p(x) \) and we have:

\[
    \models (\forall x) \left( (\theta(e, x) \land \phi(e, x)) \rightarrow c <^\phi x \right) \quad \text{for all } c \in \mathcal{E}_\phi.
\]

Since \( q(x) \in S_1(e) \) is a \( \mathcal{E}_\phi \)-type:

\[
    (\forall x) \left( \theta(e, x) \land \phi(e, x) \right) \rightarrow y <^\phi x \in q(y),
\]

which combined with \( \theta(e, x) \land \phi(e, x) \in r(x) \) implies \( q(M) \not< r(M) \).

(b) follows from (a), by smallness.

(c) Only (D3) requires verification. Suppose \( r(x) \in S_1(e) \) contains \( \phi(x) \) and \( r(x) \models x < e \). If \( r(x) \models \emptyset \not=p(x) \) then \( q(M) \not< r(M) \) follows from the fact that \( p(x) \) is strongly directed. Otherwise, \( r(M) \subseteq p(M) \) and, using the uniqueness, we get \( r(M) \subseteq \mathcal{E}_\phi \). But, by **Lemma 3.2**, \( q(M) \not< \mathcal{E}_\phi \) and thus \( q(M) \not< r(M) \). □

The next lemma contains properties of strongly \( \mathcal{E} \)-directed types which are not shared by all \( \mathcal{E} \)-directed types. For example, none of (a)-(c) is valid in a binary tree with elements of a branch named. However, they are valid in \((\omega, <)\), although the type in question is not strongly \( \mathcal{E} \)-directed.

**Lemma 5.1.** Suppose \( p \in S_1(\emptyset) \) is strongly \((\mathcal{E}, \leq)\)-directed via \( \phi(x) \in p, a, b \in p(M) \) and \( c \in \mathcal{E} \).

(a) \( (\phi(x) \land a \preceq x) \models p(x); (\phi(y) \land c \preceq y) \models y \in \mathcal{E} \).

(b) If \( b \in p(M) \) and \( b \not< b' \) then \( b' \in p(M) \cup \mathcal{E} \).

(c) If \( \{a, d\} \) is a \( \mathcal{E} \)-interval, \( (\forall x) \models \phi(a') \land a' \preceq x \) then \([a', d] \) is a \( \mathcal{E} \)-interval.
Proof. Without loss of generality assume that $\phi(x) = x = x$. (a) and (b) follow immediately from (D3).

(c) Let $\psi(d, y) \in \text{tp}(a'/d)$. Then:

$$\models (3y)(a \perp y \wedge \psi(d, y)).$$

Since $\text{tp}(a/d)$ is a $C$-type there is $c \in C$ such that:

$$\models (3y)(c \perp y \wedge \psi(d, y)).$$

By (a) $c \perp y$ implies $y \in C$, so $\psi(d, y)$ is satisfied by an element of $C$. It follows that $\text{tp}(a'/d)$ is a $C$-type. \hfill \Box

Lemma 5.2. Suppose $p \in S_1(\emptyset)$ is strongly $(C, \leq)$-directed via $\phi(x) \in p$ and

$$U = \{x \in \phi(M) \mid p(M) \prec x\}.$$ 

Then $U$ does not have a minimal element.

Proof. Without loss of generality assume that $\phi(x) = x = x$. Suppose, on the contrary, that $u \in U$ is minimal. Then by (D3): $\models [x \in \mathcal{M} \mid x < u] = p(M) \cup C$. It follows that $p(M) \cup C$ is definable, which is not possible by (MAX). \hfill \Box

Proposition 5.2. Suppose $p \in S_1(\emptyset)$ is strongly $C$-directed via $\phi(x) \in p$ and $D \subseteq p(M)$ is definable and non-empty. Then there are $a, b \in p(M)$ such that $b < D < a$. Moreover, $a, b$ can be found each realizing an isolated type over the set of parameters used to define $D$.

Proof. Without loss of generality assume that $\phi(x) = x = x$. $b \in p(M)$ satisfying $b < D$ exists by Proposition 3.1(b); $a$ satisfying $D \prec a$ will be found in a similar way: Let $D^a = \{x \in \mathcal{M} \mid D < x\}$ be the set of upper bounds of $D$. By (D3) we get:

$$U = \{x \in \mathcal{M} \mid p(M) < x\} \subseteq D^a \subseteq p(M) \cup U.$$ 

$D^a$ is definable so, by (MIN), it contains a minimal element $a \in D^a$. By Lemma 5.2 $U$ has no minimal elements, so $a \notin U$ and thus $a \in p(M)$. \hfill \Box

6. Uniqueness of extensions

Throughout the section fix $p \in S_1(\emptyset)$ which is strongly $(C, \leq)$-directed via $\phi(x)$. Since we will be working exclusively within $\phi(M)$, for simplicity we will assume that $\phi(x) = x = x$.

Recall [from [2]] that a complete type over $A$ is called good if whenever $a, b$ are its realizations and $\text{tp}(a/bA)$ is isolated then $\text{tp}(b/aA)$ is isolated, too. In other words a type is good iff isolation is a symmetric (binary) relation on its locus. In the following lemma we will prove that a strongly $C$-directed type is good, and even more: semi-isolation is symmetric on the set of realizations of a strongly $C$-directed type.

Lemma 6.1. (a) $p$ is good.

(b) For all $a, b \in p(M)$: $a \in \text{Sem}(b)$ iff $b \in \text{Sem}(a)$.

Proof. (a) Suppose $a, b \in p(M)$ and $\text{tp}(a/b)$ is isolated, by $\psi(b, x)$ say. We will show that $\text{tp}(b/a)$ is isolated, too. By Lemma 1.1 $\psi(b, M)$ is an antichain, so there is a formula $\psi(y, a) \in \text{tp}(b/a)$ expressing:

$$\psi(y, a) \wedge \forall x (\psi(y, x, M) \text{ is an antichain}).$$

If $b'$ is a minimal element of $\phi(M, a)$ then $b \nless b'$ so, by Lemma 5.1(b), $b' \in p(M) \cup C$. $b' \in C$ is impossible: $b' \in C$ and

$$\models \psi(b', a) \iff \psi(b', a') \text{ for all } a' \in p(M),$$

in which case $\psi(b', M)$ is not an antichain. We conclude that the set of minimal elements of $\phi(M, a)$, call it $D$, is definable over $a$ and $D \subseteq p(M)$.

Let $b'' \in D$ be such that $\text{tp}(b''/a)$ is isolated. Then $b'' \in p(M)$ and

$$\models \psi(b'', a) \text{ imply } \text{tp}(b''/a) = \text{tp}(ba) \iff \text{tp}(b''/a) \text{ is isolated.}$$

(b) Suppose $a \nless b \in p(M)$ and $b \in \text{Sem}(a)$, and we will prove $a \in \text{Sem}(b)$. Let $\theta(a, y) \in \text{tp}(b/a)$ and $\theta(a, y) \models p(y)$. We have three possibilities: $a \perp b$, $b < a$ and $b < a$. Note that the first, by Lemma 5.1(a), implies $a \in \text{Sem}(b)$.

Case 1. $b < a$.

By Proposition 5.2 there is $b' \in p(M)$ such that:

$$b' < \theta(a, M) \text{ and } \text{tp}(b'/a) \text{ is isolated.}$$

$\models \theta(a, b)$ implies $b' < b$. By (a), $\text{tp}(b'/a)$ is isolated implies that $\text{tp}(a/b')$ is isolated, too; in particular $a \in \text{Sem}(b')$ and $b' \in \text{Sem}(a)$. By Lemma 4.1(a) $[b', a]$ is a small interval so, by Proposition 4.1, $b' < b < a$ implies that $[b', b]$ is small, too. Thus $b' \in \text{Sem}(b)$ and, by transitivity, $a \in \text{Sem}(b)$.

Case 2. $a < b$.

The proof is dual to the previous. By Proposition 5.2 there is $b' \in p(M)$ with:

$$\theta(a, M) < b' \text{ and } \text{tp}(b'/a) \text{ is isolated.}$$

As before we derive: $a < b < b'$ and $a \in \text{Sem}(b')$. Then $[a, b']$ is a small interval, so $[a, b]$ is small, too. Thus $a \in \text{Sem}(b)$.
**Proposition 6.1.** $x \in \text{Sem}(y)$ is an equivalence relation on $p(M)$ (which is induced by $x$ is near $y$). Its classes are convex, and the quotient set is linearly ordered by $\prec$.

**Proof.** $x \in \text{Sem}(y)$ is, always, reflexive and transitive. By previous lemma it is symmetric, so is an equivalence relation. To show that it is induced by the nearness (recall that $x$ is near $y$ if $x < y$ and $[x,y]$ is a small interval) it suffices, assuming $a \in \text{Sem}(b)$, to find $d \in p(M)$ such that both $[d,a]$ and $[d,b]$ are small intervals. Note that any $d$ which is maximal satisfying $x < a \land x < b$ works. Classes are convex by Proposition 4.1. It remains to show that the quotient is linearly ordered. Let $a', b' \in p(M)$ be from distinct classes. By Lemma 5.1(a) $a' \nprec b'$ is impossible, so $a' < b'$ or $b' < a'$. □

**Lemma 6.2.** If $e \in p(M)$ then there is a unique $\mathcal{C}$-type in $S_1(e)$.

**Proof.** Suppose, on the contrary, that there are at least two $\mathcal{C}$-extensions of $p$. Then, by smallness, there are two distinct $\mathcal{C}$-isolated intervals. Eventually (in Claim 4 below) we will prove that every $\mathcal{C}$-isolated interval $[a, d]$ contains two distinct $\mathcal{C}$-isolated subintervals $[a', b]$ and $[b, d]$ such that $\text{tp}(ba/ad)$ is isolated; by iterating this process we produce an embedding of the rationales into a model prime over $ad$, which is in contradiction with Proposition 1.1.

Suppose that there are two distinct $\mathcal{C}$-isolated types $p_1$, $p_2 \in S_1(e)$ isolated among the $\mathcal{C}$-types by $R(x, e)$ and $B(x, e)$ respectively. Such $R(x, y)$ and $B(x, y)$ can be chosen each implying $x < y$ and such that $R(x, e) \land B(x, e)$ is inconsistent, so from now on assume that it has been done.

Let $C_\mathcal{C} = R(C, e)$, $C_\mathcal{B} = B(C, e)$. Note that $p_1$ is the unique $C_\mathcal{B}$-type and $p_2$ is the unique $C_\mathcal{B}$-type in $S_1(e)$. Also, $C_\mathcal{B}$-intervals and $C_\mathcal{B}$-intervals are $\mathcal{C}$-isolated intervals.

**Claim 1.** If $[a, d]$ is a $C_\mathcal{B}$-interval and $[a, b]$ is a $C_\mathcal{B}$-interval then either $[b, d]$ is a $\mathcal{C}$-interval or $[d, b]$ is a $\mathcal{C}$-interval.

**Proof.** Firstly, we rule out the possibility $b \perp d$: $b \perp d$ implies by Lemma 5.1(a) $b \in \text{Sem}(d)$, then $b \in \text{Sem}(d)$ and $[a, b]$ is a $\mathcal{C}$-interval imply, by Lemma 4.1(b), $\text{tp}(da) = \text{tp}(ba)$ which is not the case.

Assume $b < d$ and we show that $[b, d]$ is a $\mathcal{C}$-interval (similarly $d < b$ implies that $[d, b]$ is a $\mathcal{C}$-interval): by Lemma 4.1(b) $b < d$ and $\text{tp}(da) \neq \text{tp}(ba)$ imply $b \notin \text{Sem}(d)$ so, by Lemma 4.1(a), $\text{tp}(b/d)$ is a $\mathcal{C}$-type and $[b, d]$ is a $\mathcal{C}$-interval.

**Claim 2.** There are $a < b < d$ with $\text{tp}(bd/ad)$ isolated and $[b, d]$ a $\mathcal{C}$-interval such that either:

1. $[a, d]$ is a $\mathcal{C}$-interval and $[a, b]$ is a $\mathcal{C}$-interval; or
2. $[a, d]$ is a $\mathcal{C}$-interval and $[a, b]$ is a $\mathcal{C}$-interval.

**Proof.** Choose $a', b', d \in p(M)$ such that $[a, d]$ is a $\mathcal{C}$-interval and $[a, b']$ is a $\mathcal{C}$-interval. By Claim 1 we have two cases: either $[b', d]$ is a $\mathcal{C}$-interval or $[d, b']$ is a $\mathcal{C}$-interval. In the first case we shall show that condition (1) is satisfied; a similar argument shows that in the second case (2) is satisfied.

So suppose that $[b', d]$ is a $\mathcal{C}$-interval. Then:

\[ \models a < b' < d \land R(a, b') \land B(a, d). \]

By (MAX) the set $D = \{ y \in M \mid \models a < y < d \land R(a, y) \}$ contains a maximal element $b$; moreover, assume that it is chosen such that $\text{tp}(b/d)$ is isolated. Thus:

\[ \models a < b < d \land R(a, b) \land B(a, d). \]

We will prove that $[b, d]$ is a $\mathcal{C}$-interval and that $[a, b]$ is a $\mathcal{C}$-interval.

If $[b, d]$ were not a $\mathcal{C}$-interval we would have $b \in \text{Sem}(d)$. Then, since $[a, d]$ is a $\mathcal{C}$-interval, Lemma 4.1(b) implies $\text{tp}(da) = \text{tp}(ba)$ and thus $[a, b]$ is a $\mathcal{C}$-interval, which is in contradiction with $\models R(b, a)$. Thus $[b, d]$ is a $\mathcal{C}$-interval.

It remains to prove that $[a, b]$ is a $\mathcal{C}$-interval. Since $b$ is a maximal element of $D$ and $b' \in D$ either $b \perp b'$ or $b' \leq b$ holds:

$\models b \perp b'$ implies $b \in \text{Sem}(b')$, by Lemma 5.1(a). By Lemma 4.1(b) $[a, b']$ is a $\mathcal{C}$-interval and $b \in \text{Sem}(b')$ imply $\text{tp}(ba) = \text{tp}(b/a)$ and $[a, b]$ is a $\mathcal{C}$-interval.

$b \leq b'$ implies $a < b' \leq b$. Since $[a, b']$ is a large interval $[a, b]$ is a 'large' interval, too. In both cases $[a, b]$ is a $\mathcal{C}$-interval; $\models R(a, b)$ implies that it is a $\mathcal{C}$-interval.

**Claim 3.** Suppose $[a, d]$ is a $\mathcal{C}$-interval and

\[ D = \{ x \in p(M) \mid \models a < x < d \land R(x, d) \}. \]

Then $D \neq \emptyset$ and if $d'$ is minimal in $D$ then $[a', d]$ is a $\mathcal{C}$-interval.

**Proof.** First we show that there is $a'' \in D$ such that $[a'', d]$ is a $\mathcal{C}$-interval. For let $[0, d_0]$ be a $\mathcal{C}$-interval and let $\text{tp}(a_0/da_0)$ be a $\mathcal{C}$-type. Then $[a_0, d_0]$ is a $\mathcal{C}$-interval and in particular $\text{tp}(da_0/a_0) = \text{tp}(da)$. $M$ is saturated, so there is $a'' \in M$ such that $\text{tp}(da_0/a_0) = \text{tp}(d'a''a')$; then $[a', d]$ is a $\mathcal{C}$-interval since $[0, d_0]$ is so. Clearly, $a'' \in D$.

Continuing the proof of the claim note that it suffices to prove that $[a', d]$ is a $\mathcal{C}$-interval; then $\models R(a', d)$ would imply that it is a $\mathcal{C}$-interval. Having found $a''$ as above, by the minimality condition we have two possibilities: $a' \perp a''$ and $a' \leq a''$. In the first $[a'', d]$ is a $\mathcal{C}$-interval and $a' \perp a''$, by Lemma 5.1(c), imply that $[a', d]$ is a $\mathcal{C}$-interval. In the second $[a'', d]$ is a $\mathcal{C}$-interval and $a' \leq a''$, by Proposition 4.1, imply that $[a', d]$ is a $\mathcal{C}$-interval.
Claim 4. If [a, d] is a $C^*$-interval then there are $b, a'$ such that $tp(ba'/da)$ is isolated, $a \leq a' < b < d$ and both $[b, d]$ and $[a, b]$ are $C^*$-isolated intervals.

Proof. Without loss of generality suppose $[a, d]$ is a $C^*$-interval. According to Claim 2 we have the following two cases:

Case 1: Possibility (1) from Claim 2 occurs: there are $a < b < d$ such that $tp(b/da)$ is isolated, $[b, d]$ is a $C^*$-interval and $[a, b]$ is a $C^*$-interval. In this case let $a' = a$; then, by Lemma 4.2, $[b, d]$ is a $C^*$-isolated interval and we are done.

Case 2: Whenever $[a', d]$ is a $C^*$-interval then there is $b$ such that $tp(b/da')$ is isolated, $a' < b < d$, $[b, d]$ is a $C^*$-interval and $[a', b]$ is a $C^*$-interval. In this case let $[a, d]$ be a $C^*$-interval. Find $a'$ satisfying:

$$tp(a'/da)$$

is isolated and $a'$ is minimal such that $a < a' < d \land R(a', d)$.

By Claim 3 $[a', d]$ is a $C^*$-interval. Further, find $b$ such that $a' < b < d$ satisfy the assumptions of this case:

$$tp(b/da')$$

is isolated, $[b, d]$ is a $C^*$-interval and $[a', b]$ is a $C^*$-interval; moreover, we can choose $b$ so that $tp(b/da')$ is isolated. By transitivity of isolation $tp(ba'/da)$ is isolated, and by Lemma 4.2 $[b, d]$ is a $C^*$-isolated interval. Therefore, $a < a' < b < d$ satisfy all the required conditions. □

**Proposition 6.2.** Let $(e_1, e_2, \ldots, e_n) \in M^\ell$ be a $C$-sequence.

(a) There is a unique $C$-type in $S_1(e_1 e_2 \ldots e_n)$; it is good and strongly $C$-directed (via $x = x$).

(b) There is a unique type of a $C$-sequence of fixed length.

**Proof.** (a) By previous lemma, there is a unique $C$-type in $S_1(e_1)$. It is strongly $C$-directed via $x = x$ by Proposition 5.1(c) and good by Lemma 6.1. Absorb $e_1$ into the language and continue.

(b) follows immediately from (a). □

By Proposition 6.2, for all $C$-sequences $E = (e_1, e_2, \ldots, e_n)$ there is a unique $C$-type in $S_1(E)$, which we will denote by $p_E$. So $p_E$ is good and strongly directed via $x = x$.

**Question.** Is there a unique $C$-type in $S_1(E)$ for all finite $E \subseteq p(M)$?

7. **Proof of Theorem 1**

Throughout this section let $p \in S_1(\emptyset)$ be strongly ($C, \leq$)-directed via $\phi(x)$; again, since we will be working within $\phi(M)$, we may assume that $\phi(x)$ is $x = x$. Recall that $A \subseteq p(M)$ is $C$-independent if every finite subset of $A$ can be arranged into a $C$-sequence over $B$. The following lemma states that this dependence is degenerated.

**Lemma 7.1.** $A \subseteq p(M)$ is $C$-independent iff every pair of elements of $A$ is $C$-independent.

**Proof.** First we prove the case when $A$ has three elements. So, suppose $A = \{a, b, d\}$, $b < a < d$ and that both $[b, a]$ and $[a, d]$ are $C$-intervals. Assuming that $tp(b/da)$ is not a $C$-type we will find a copy of the rationales embedded into a model which is prime over a finite subset; this is in contradiction with Proposition 3.2. Thus $tp(b/da)$ has to be a $C$-type, and it follows that $(d, a, b)$ is a $C$-sequence.

Suppose $tp(b/da)$ is not a $C$-type and find $\psi(d, a, x) \models tp(b/da)$ as a witness:

$$\psi(d, a, x) \models tp(x) \land \models \psi(d, a, x) \rightarrow x < a.$$  

By a first-order formula with parameters $da$ we can express:

`x is a minimal element of $\psi(d, a, M)'`.

Choose $b'$ satisfying the formula with $tp(b'/da)$ isolated. $b' \in p(M)$ and $b' < a$ follow from our assumptions on $\psi(d, a, x)$. The two combined with $a \models p_d$ imply $b' \models p_d$. We have two realizations $a, b'$ of $p_d$ with $tp(b'/da)$ isolated; since $p_d$ is good (by Proposition 6.2) we conclude that $tp(a/b'd)$ is isolated, too. Finally, from the minimality of $b' \in p(M)$ we get $b \neq b'$ which, together with $b \models p_d$, implies $b' \models p_d$ by Lemma 5.1(b). We have just found a pairwise $C$-independent triple $\{b', a, d\}$ with $b' < a < d$ and $tp(a/da')$ isolated. It follows that $(p(M_0), \cdot)$, where $M_0$ is prime over $db'$, contains a densely ordered subset.

The general case follows by induction: suppose $b < a < d < e_1 < \cdots < e_n$ is pairwise $C$-independent. By the induction hypothesis $b < a < d$ are pairwise $C$-independent (over $\emptyset = e_1 e_2 \ldots e_n$) realizations of $p_d$; by the above $(d, a, b)$ is $C$-independent over $\emptyset$ and $\{e_1, e_2, \ldots, e_n, d, a, b\}$ is $C$-independent. □

**Proof of Theorem 1(a).** Suppose $M \models T$ and $A_I = \{a_i | i \in I\} \subseteq p(M)$ is a maximal (under inclusion) $C$-independent set. We will prove that $(A_I, \cdot)$ is a linear order whose isomorphism type does not depend on the particular choice of $A_I$. Let $A_I = \{a_i | i \in I\} \subseteq p(M)$ be another maximal $C$-independent subset of $p(M)$. By degeneracy and the maximality of $A_I$, for each $i \in I$ there is a unique $j \in I$ such that $a_i \in \text{Sem}(a_j)$. Conversely, for each $j \in J$ there is a unique $i \in I$ such that $a_j \in \text{Sem}(a_i)$. In this way we get a bijection between $A_I$ and $A_J$, which is easily seen to be order-preserving. □
Lemma 7.2. If \([a_1, \ldots, a_n]\) is \(C\)-independent, \(b \in p(M)\) and \(tp(b/a_1a_2 \ldots a_n)\) is isolated then either \([b, a_1, a_2, \ldots, a_n]\) is \(C\)-dependent or \(b \in [a_1, \ldots, a_n]\).

**Proof.** Without loss of generality let \(a_1, a_2, \ldots, a_n\) be a \(C\)-sequence. Assuming that \([b, a_1, a_2, \ldots, a_n]\) is \(C\)-independent we shall show that \(tp(b/a_1a_2 \ldots a_n)\) is non-isolated. Assume that \([b, a_1, a_2, \ldots, a_n]\) is \(C\)-independent and let

\[
a_n < \cdots < a_{k+1} < b < a_k < \cdots < a_1
\]

be its arrangement into a \(C\)-sequence. Find \(b' \in p(M)\) such that

\[
a_n < \cdots < a_{k+1} < b' < b < a_k < \cdots < a_1
\]

is a \(C\)-sequence. Note that by uniqueness of \(C\)-extensions

\[tp(a_1 \ldots a_kb_{k+1} \ldots a_n) = tp(a_1 \ldots a_kb_{k+1} \ldots a_n)\]

and thus \(tp(b/a_1a_2 \ldots a_n) = tp(b'/a_1a_2 \ldots a_n)\). Therefore the set of realizations of \(tp(b/a_1a_2 \ldots a_n)\) is not an antichain since it contains \(b' < b\). Lemma 1.1 applies and \(tp(b/a_1a_2 \ldots a_n)\) is non-isolated. \(\square\)

**Proof of Theorem 1(b).** Suppose that \((L, <_L)\) is a linear order and we will find \(N \models T\) such that the order type of a maximal \(C\)-independent subset of \(p(N)\) is isomorphic to \((L, <_L)\). Let \(A = (a_i)_{i \in I} \subseteq p(M)\) be a \(C\)-independent set ordered in the order type of \((L, <_L)\): \(a_i < a_j\) iff \(i < j\). By smallness there is a countable model \(N \supseteq A\) satisfying:

for all \(d \in N\) there is a finite \(A_0 \subseteq A\) such that \(tp(d/A_0)\) is isolated.

It suffices to show that \(A\) is a maximal \(C\)-independent subset of \(p(N)\). So, suppose that \(b \in p(N) \setminus A\). Then \(tp(b/A_0)\) is isolated (for some finite \(A_0 \subseteq A\)) and, by Lemma 7.2, \(b \cup A_0\) is \(C\)-dependent. Thus \(A\) is maximal, and the conclusion follows by part (a). \(\square\)

Now we can summarize the facts, leaving the proof to the reader:

**Theorem 2.** Suppose that \(T\) is small, \(M \models T\), and \(p \in S_1(\emptyset)\) is strongly \((C, \leq)\)-directed via \(\phi(x)\).

(a) \(x \in \text{Sem}(y)\) is an equivalence relation (call it \(\sim_M\)) on \(p(M)\), whose classes are linearly ordered by \(<\).

(b) \(a_1, a_2, \ldots, a_n\) are from distinct \(\sim_M\)-classes iff \([a_1, a_2, \ldots, a_n]\) is \(C\)-independent (of size \(n\)).

(c) If \(a_1 < a_2 < \cdots < a_n\) are from distinct \(\sim_M\)-classes, and \(b_1 < b_2 < \cdots < b_n\) are, too, then \(tp(a_1a_2 \ldots a_n) = tp(b_1b_2 \ldots b_n)\).

In our situation \(\text{Sem}_p\) is not a pregeometry operator. So we first proved that \(\text{Sem}_p\), as a binary relation, is an equivalence relation \(\sim\) which agrees with \(<\), and then that the whole induced structure on \(p(M)/\sim\) is induced by \(<\) alone. This suggests that the \(C\)-independence, as defined here, induces a kind of ‘linear order dimension’ and that strongly directed types may be viewed as ‘minimal-regular’ among the ordered types (in a small theory).

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**References**