ABSTRACT

Two main semantical approaches to possibilistic reasoning with classical propositions have been proposed in the literature. Namely, Dubois-Prade's approach known as possibilistic logic, whose semantics is based on a preference ordering in the set of possible worlds, and Ruspini's approach that we redefine and call similarity logic, which relies on the notion of similarity or resemblance between worlds. In this article we put into relation both approaches, and it is shown that the monotonic fragment of possibilistic logic can be semantically embedded into similarity logic. Furthermore, to extend possibilistic reasoning to deal with fuzzy propositions, a semantical reasoning framework, called fuzzy truth-valued logic, is also introduced and proved to capture the semantics of both possibilistic and similarity logics.

KEYWORDS: Possibilistic reasoning, possibilistic logic, similarity logic, fuzzy truth-valued logic

1. INTRODUCTION

Two main semantical approaches to possibilistic reasoning with classical propositions have been proposed in the literature. The first one, a prefer-
ence-based approach, has been proposed by Dubois, Prade, and colleagues in a wide series of works since the middle 1980s (see for instance [1, 2] for basic surveys), and the second one, a similarity-based approach, by Ruspini in [3] and [4]. Although with different motivation, both approaches share the goal of showing that the nature of incompleteness that possibilistic reasoning tries to capture departs from that captured by probabilistic reasoning or evidential reasoning. At the semantical level, the main difference between Dubois-Prade's and Ruspini's approaches is the following one:

- In Dubois-Prade's approach, known as possibilistic logic (PL), the incomplete description of a system state is done by means of a fuzzy set of possible worlds, or equivalently by a possibility distribution on the set of possible worlds, which induces a linear preordering on possible worlds, usually read as a preference ordering among possible complete descriptions.

- In Ruspini's approach, which we shall call similarity logic (SL), incompleteness in the description of a system state is modeled by means of a similarity relation on the set of possible worlds, together with the set of possible worlds that are compatible with the evidence, the so-called evidential set. In this way, the nature of possibilistic reasoning is based on metric notions rather than on preference notions.

Nevertheless, both approaches can be put into relation as soon as a correspondence between preference and similarity relations is established in both directions. A first correspondence will be based on a non-standard notion of fuzzy set (with non-functional connectives) as defined in [5] in terms of a primitive similarity relation. These fuzzy sets, interpreted as possibility distributions, will be taken as input models in the Dubois-Prade's approach. On the other direction, Valverde's representation theorem of T-similarity relations\(^1\) [6] provides a tool to build a T-similarity relation corresponding to each possibility distribution. The similarity relation obtained in this way provides the resemblance degree of each pair of worlds taking only into account the extent to which they are preferred according to a given possibility distribution.

On the other hand, the authors have proposed in [7] a semantical reasoning framework as an extension of possibilistic reasoning to deal with fuzzy propositions. Following this line, we introduce in this study what we call fuzzy truth-valued logic (FTL) where, as in PL, incompleteness is modeled as a fuzzy set of possible worlds, but it differs from PL in that FTL allows each possible world to be described by a many-valued interpretation of a set of logical formulas, instead of by a classical Boolean one.

\(^1\)Called T-indistinguishability relations in Valverde's paper.
Therefore, it seems interesting to study the above two approaches to possibilistic reasoning from the point of view of FTL.

The article is organized as follows. In the second section the basic notions of Dubois-Prade's and Ruspini's approaches are presented. Moreover, Ruspini's major constructs are reformulated from the measure theoretic and fuzzy sets points of view. These reformulations allow a further understanding of this approach. The third section is devoted to study in detail the above mentioned correspondences between the two approaches. We prove that Ruspini's approach subsumes Dubois-Prade's in the following sense: for each PL sentence and PL model, there exists an SL sentence and an SL model such that the PL model satisfies the PL sentence if, and only if, the corresponding SL model satisfies the corresponding SL sentence. In the last part of section 3, the relationship between the inference mechanisms provided by both approaches are studied. Section 4 presents FTL and establishes the links between PL and SL with FTL, and it is shown that FTL properly generalizes both. Finally, for interested readers, we provide a first annex summarizing the main results on T-similarity relations and its generators, and a second one giving a characterization for the subclass of SL models that correspond to the class of PL models.

Throughout this study we shall use the following notation conventions:

- $L$ will denote a propositional language.
- $W$ will denote a set of classical interpretations of $L$, and $W^*$ a set of many-valued interpretations of $L$ on the unit interval $[0, 1]$.\(^2\)
- For any $w \in W$ and for any proposition $p \in L$, $w \models p$ will denote that $p$ is true in the interpretation $w$, and $[p]$ will denote the set $\{w \in W \mid w \models p\}$ of models where the proposition $p$ is true.
- $T$ will denote a continuous t-norm, and $R_T$ the residuated implication generated by $T$, i.e. the implication function defined as $R_T(x, y) = \sup\{c \in [0, 1] \mid T(x, c) \leq y\}$.

## 2. TWO APPROACHES TO POSSIBILISTIC REASONING

### 2.1. Preference Relation on Possible Worlds: Dubois-Prade's Possibilistic Logic

The approach developed by Dubois, Prade, and colleagues since the 1980s [2, 8–10] has as its basic feature the representation of incomplete

\(^2\)In a general case, the unit interval $[0, 1]$ could be replaced by an arbitrary complete and bounded lattice.
information about the real state of affairs (evidence) by means of a possibility distribution \( \pi : W \rightarrow [0, 1] \), on the set \( W \) of possible interpretations (possible worlds) w.r.t. a propositional language \( L \). This distribution is usually read as a total preference ordering among worlds: \( w_1 \) is preferred to \( w_2 \) if \( \pi(w_1) \geq \pi(w_2) \). Each possibility distribution \( \pi \) induces a dual pair of possibility and necessity measures on the Boolean algebra of propositions \( L \), defined as

\[
Pos_{\pi}, \text{ Nec}_{\pi} : L \rightarrow [0, 1]
\]

\[
Pos_{\pi}(p) = \text{Sup}(\pi(w) \mid w \models p) \quad \text{Nec}_{\pi}(p) = 1 - Pos_{\pi}(\neg p)
\]

It is easy to check that these functions satisfy the axioms of possibility and necessity measures, i.e. the following conditions hold:

1) \( Pos_{\pi}(\top) = Nec_{\pi}(\top) = 1 \), and \( Pos_{\pi}(\bot) = Nec_{\pi}(\bot) = 0 \), being \( \top \) a tautology and \( \bot \) an anti-tautology

2) \( Pos_{\pi} \) and \( Nec_{\pi} \) are monotone w.r.t. the lattice ordering: \( p \leq q \) iff \( \neg p \lor q = \top \)

3) \( Pos_{\pi}(p \lor q) = \max(Pos_{\pi}(p), Pos_{\pi}(q)) \) and \( Nec_{\pi}(p \land q) = \min(Nec_{\pi}(p), Nec_{\pi}(q)) \)

Further well-known properties of such measures are that \( Pos_{\pi}(p) < 1 \) implies \( Nec_{\pi}(p) = 0 \), and \( Nec_{\pi}(p) > 0 \) implies \( Pos_{\pi}(p) = 1 \). Therefore, the graded uncertainty of a proposition \( p \) can range from \( Pos_{\pi}(p) = 0 \) (\( p \) is false) to \( Nec_{\pi}(p) = 1 \) (\( p \) is true), corresponding the case \( Pos_{\pi}(p) = 1 \) and \( Nec_{\pi}(p) = 0 \) to the total ignorance about the truth or falsity of \( p \). Such measures are the basis of the so-called possibilistic logic (PL), a weighted logic of uncertainty that handles possibility and necessity-valued formulas of classical logic. Possibilistic logic has been widely developed by its authors from many points of view and with many interesting features. Among them, the most remarkable ones are its deductive machinery [11] and its ability to cope with partial inconsistency [10] and non-monotonic inference patterns [9, 12]. However, because of the scope of this study, we will restrict ourselves to the monotonic propositional fragment of this logic, whose basic characteristics are given below.

**LANGUAGE**

Given the Boolean algebra of propositions \( L \), the extended possibilistic language \( L_{PL} \) is defined as

\[
L_{PL} = \{ (p, \Pi \alpha), (p, \text{Na}) \mid p \in L, \alpha \in [0, 1] \}
\]

\( PL \) formulas of type \( (p, \Pi \alpha) \) are called possibility-valued formulas, \( \Pi \alpha \) standing for a lower bound of the possibility degree of \( p \), whereas \( PL \) formulas of type \( (p, \text{Na}) \) are called necessity-valued formulas where \( \text{Na} \) stands for a lower bound of the necessity degree of \( p \).
SEMANTICS A model is a possibility distribution $\pi$ on $W$. The satisfaction relation for possibilistic formulas is defined as:

$$\pi \models_{PL} (p, \Pi \alpha) \text{ if, and only if, } Pos_{\pi} (p) \geq \alpha$$

$$\pi \models_{PL} (p, N \alpha) \text{ if, and only if, } Nec_{\pi} (p) \geq \alpha$$

INFERENGE Inference in possibilistic logic is mainly based on the generalization of the resolution principle to possibilistic clauses

$$\begin{align*}
(p \vee q, \nu_1) \\
(\neg p \vee r, \nu_2)
\end{align*}$$

$$\frac{q \vee r, \nu_1 \otimes \nu_2}{\cdot}$$

where the commutative operation $\otimes$ is defined by

$$\begin{align*}
N\alpha \otimes N\beta &= N\min(\alpha, \beta), \\
\Pi\alpha \otimes \Pi\beta &= \Pi0, \text{ and} \\
N\alpha \otimes \Pi\beta &= \begin{cases}
\Pi\beta, & \text{if } \alpha + \beta > 1 \\
\Pi0, & \text{if } \alpha + \beta \leq 1
\end{cases}
\end{align*}$$

It is worth pointing out that a proof procedure of refutation by resolution has been extended to $PL$, and soundness and completeness results for this procedure have been proved for necessity-valued formulas (see [10] for further related results).

2.2. Similarity Relation on Possible Worlds: Ruspini's Semantical Foundations of Fuzzy Logic

Ruspini's approach [3] to possibilistic reasoning is based on the introduction of a weighted binary relation $S$ in a set of possible worlds $W$ valued on $[0, 1]$. This relation assigns to every pair of possible worlds a similarity degree evaluated from the viewpoint of the particular problem being considered. In order to capture the intuitive notion of similarity, the function $S$ is required to fulfill the following three properties:

**Reflexivity:** $S(w, w) = 1$

**Symmetry:** $S(w, w') = S(w', w)$

**$T$-transitivity:** $S(w, w') \geq T(S(w, w''), S(w'', w'))$

being $T$ a continuous t-norm. To be closer to Ruspini's denotation, throughout this article such relations will be called $T$-similarity relations, although they are also known in the literature as $T$-indistinguishability.
relations (see Annex A for further details). For the sake of simplicity, $S$ is required to fulfil $S(w, w') = 1$ iff $w = w'$.

The $\alpha$-cuts of the similarity function $S$ define a family of nested accessibility relations $R_\alpha$ on the set of possible worlds $W$:

$$R_\alpha(w, w') \text{ if, and only if, } S(w, w') \geq \alpha$$

For every $\alpha \in [0, 1]$, the accessibility relation $R_\alpha$ defines a Brouwerian Modal Logic with possibility and necessity modal operators $\Delta_\alpha$ and $\nabla_\alpha$ respectively:

$$w \models \Delta_\alpha p, \text{ if and only if, there exists a world } w' \text{ such that }$$

$$S(w, w') \geq \alpha \text{ and } w' \models p$$

$$w \models \nabla_\alpha p, \text{ if and only if, for every world } w' \text{ such that }$$

$$S(w, w') \geq \alpha \text{ it holds } w' \models p$$

Having defined a similarity relation $S$ on $W$, Ruspini generalizes the semantical entailment relationships between propositions in terms of a measure of neighborhood of some sets of possible worlds by defining the so-called degree of implication and the so-called degree of consistency as

$$I(p \mid q) = \inf_{w \models q} \sup_{w' \models p} S(w, w')$$

$$C(p \mid q) = \sup_{w' \models q} \sup_{w \models p} S(w, w')$$

respectively. These measures can also be expressed in terms of the above modal systems as next proposition shows.

**PROPOSITION 1**

$I(p \mid q) = \sup\{\alpha \in [0, 1] \mid q \models \Delta_\alpha p\}$ and $C(p \mid q) = \sup\{\alpha \in [0, 1] \mid q \not\models \nabla_\alpha \neg p\}$.

**Proof** We will prove only the first equality, the second is analogous. The proof is in two steps.

(i) Suppose $\alpha < I(p \mid q)$. By definition of $I(p \mid q)$, this implies that, for every $w'$ such that $w' \models q$, $\sup_{w \models p} S(w, w') > \alpha$, and therefore there exists $w''$ such that $w'' \models p$ and $S(w'', w') \geq \alpha$. Thus we have proved that $q \models \Delta_\alpha p$.

(ii) Suppose now $\alpha > I(p \mid q)$. This implies that there exists $w'$ such that $w' \models q$ and that for every $w''$ satisfying $w'' \models p$, $S(w', w'') < \alpha$. This simply means that $w' \not\models \Delta_\alpha p$, and thus $q \not\models \Delta_\alpha p$.

Therefore, from (i) and (ii), the first equality holds. 

Further interesting properties of $C$ and $I$ are that the inequality $C(p \mid q) \geq I(p \mid q)$ always holds, and that the degree of consistency $C$ is a
symmetric measure whereas the degree of implication $I$ is not. However, this latter measure keeps the $T$-transitivity property of the similarity $S$, i.e., $I(p \mid q) \geq T(I(p \mid r), I(r \mid q))$ for any propositions $p$, $q$, and $r$.

Information about a system state is modeled in this framework by an evidential set $E$ of possible worlds, i.e., the set of worlds that are compatible with the evidence. Given such an evidential set $E$, Ruspini introduces what he calls unconditioned and conditional possibility and necessity distributions, although these constructs have a meaning that does not totally agree with the standard one.

1) **Unconditioned distributions**

If $E$ is an evidential set, then functions $\text{Nec}(\cdot)$ and $\text{Pos}(\cdot)$ over propositions are called unconditioned necessity and possibility distributions respectively for $E$ if

$$\text{Nec}(p) \leq I(p \mid E)$$

$$\text{Pos}(p) \geq C(p \mid E)$$

2) **Conditional distributions**

If $E$ is an evidential set, then functions $\text{Nec}(\cdot \mid \cdot)$ and $\text{Pos}(\cdot \mid \cdot)$ over pairs of propositions are called conditional necessity and possibility distributions respectively for $E$ if

$$\text{Nec}(p \mid q) \leq \text{Inf}_{w \in E} \{R_T(I(q \mid w), I(p \mid w))\}$$

$$\text{Pos}(p \mid q) \geq \text{Sup}_{w \in E} \{R_T(I(q \mid w), I(p \mid w))\}$$

3) **Generalized Modus Ponens**

Ruspini defines a partition as a set of propositions $P = \{q_i \mid i \in I\}$ such that, for every $w \in E$ there exists $q_i \in P$ with $I(q_i \mid w) = 1$. For a given partition $P$, and based on the above definitions of possibility and necessity distributions, the following expressions of the Zadeh's generalized modus ponens are proposed by Ruspini.

i) for necessity functions: if every $q_i \in P$,

$$\text{Nec}(q_i) \leq I(q_i \mid E) \quad \text{and}$$

$$\text{Nec}(p \mid q_i) \leq \text{Inf}_{w \in E} \{R_T(I(q_i \mid w), I(p \mid w))\},$$

3Although $I(p \mid E)$ and $C(p \mid E)$ have not been formally defined ($E$ is not a proposition), they can be easily defined by extension as $I(p \mid E) = \text{Inf}_{w' \in E} \text{Sup}_{w = p} S(w', w)$, and $C(p \mid E) = \text{Sup}_{w' \in E} \text{Sup}_{w = p} S(w', w)$. 
then the following inequality holds
\[ \sup_{i \in I} \{ T(\text{Nec}(q_i), \text{Nec}(p \mid q_i)) \} \leq I(p \mid E) \]

ii) for possibility functions: if for every \( q_i \in P \),
\[ \text{Pos}(q_i) \geq C(q_i \mid E) \quad \text{and} \]
\[ \text{Pos}(p \mid q_i) \geq \sup_{w \in E} \{ R_T(I(q_i \mid w), I(p \mid w)) \}, \]

then the following inequality also holds
\[ \sup_{i \in I} \{ T(\text{Pos}(q_i), \text{Pos}(p \mid q_i)) \} \geq C(p \mid E) \]

In order to allow for a clearer comparison with the other approaches considered here, we reformulate in the rest of this section the above Ruspini's main constructs both from a measure theoretic (subsection 2.2.1.) and from a fuzzy sets theory (subsection 2.2.2.) points of view.

2.2.1. IMPLICATION AND CONSISTENCY MEASURES Given a similarity relation \( S \) and an evidential set \( E \), we introduce the pair of unconditioned measures \( I_{S,E} \) and \( C_{S,E} \) on \( L \) by defining:
\[ I_{S,E}(p) = \inf_{w' \in E} \sup_{w \models p} S(w, w') \]
\[ C_{S,E}(p) = \sup_{w' \in E} \inf_{w \models p} S(w, w') \]

These measures, as Ruspini points out, are lower and upper bounds of the resemblance degrees between worlds in \([p]\) and worlds in \( E \), from the point of view of \( E \). The value of \( I_{S,E}(p) \) provides the measure of what extent \( p \) can be considered certain w.r.t. the incomplete knowledge represented by \( S \) and \( E \). In particular, if \( I_{S,E}(p) = 1 \) then \( E \models p \). On the other hand, the value of \( C_{S,E}(p) \) provides the measure of what extent \( p \) can be considered compatible with the available knowledge. In particular, if \( C_{S,E}(p) = 1 \) then \( E \nvDash p \). Observe that when the evidential set is a singleton both measures coincide, i.e., \( I_{S,w}(p) = C_{S,w}(p) \) for any world \( w \) and any proposition \( p \). It is worth also noticing that \( C_{S,E} \) is in fact a possibility measure, whereas \( I_{S,E} \) is not a necessity measure, because in general only the inequality \( I_{S,E}(p \land q) \leq \min(I_{S,E}(p), I_{S,E}(q)) \) holds. Next proposition shows which inequalities relate both measures \( C_{S,E} \) and \( I_{S,E} \).

These measures correspond to those denoted by Ruspini as \( I(. \mid E) \) and \( C(. \mid E) \) respectively.
PROPOSITION 2 For any proposition \( p \), the following inequalities hold:

\[
C_{S,E}(p) \geq I_{S,E}(p) \geq 1 - C_{S,E}(\neg p) \\
C_{S,E}(p) \geq 1 - I_{S,E}(\neg p) \geq 1 - C_{S,E}(\neg p)
\]

Proof The inequality \( C_{S,E}(p) \geq I_{S,E}(p) \) follows by definition. To prove the other inequality, observe that \( S(w, w') = 1 \) for any world \( w \). Therefore, for any world \( w' \) it holds that \( \sup_{w \models p} S(w, w') + \sup_{w \models \neg p} S(w, w') \geq 1 \). Hence, \( I_{S,E}(p) = \inf_{w' \in E} \sup_{w \models p} S(w, w') = \inf_{w' \in E} (1 - \sup_{w \models \neg p} S(w, w')) = 1 - C_{S,E}(\neg p) \).

This proposition helps in the understanding of what implication and consistency degrees are. Namely, \( I_{S,E}(p) = 0 \) means that \( \neg p \) is fully compatible with the incomplete information \( (C_{S,E}(\neg p) = 1) \), and \( C_{S,E}(p) = 0 \) means that \( \neg p \) is certain \( (I_{S,E}(\neg p) = 1) \). Moreover, the case \( I_{S,E}(p) = 0 \) and \( C_{S,E}(p) = 1 \) means that both \( p \) and \( \neg p \) are totally unknown.

Next we define the pair of conditional measures \( I_{S,E}(\cdot | \cdot) \) and \( C_{S,E}(\cdot | \cdot) \) as follows:

\[
I_{S,E}(p | q) = \inf_{w \in E} \{R_T(I_{S,w}(q), I_{S,w}(p))\} \\
C_{S,E}(p | q) = \sup_{w \in E} \{R_T(C_{S,w}(q), C_{S,w}(p))\}^5
\]

Notice that in particular, when \( q = \top \) is a tautology, i.e. when \( [q] = W \), then we recover the previous unconditioned measures: \( I_{S,E}(p | \top) = I_{S,E}(p) \) and \( C_{S,E}(p | \top) = C_{S,E}(p) \). Notice also that the degree of implication and consistency between two propositions can be now expressed in terms of the new unconditioned measures as \( I(p | q) = I_{S,[q]}(p) \) and \( C(p | q) = C_{S,[q]}(p) \) respectively.

Finally, Ruspini's expressions of the generalized modus ponens can be now reformulated in terms of the implication and consistency measures as follows.

PROPOSITION 3 Given a proposition \( p \) and an arbitrary family of propositions \( P = \{q_i \mid i \in I\} \), the following inference schemes hold:

\[
I_{S,E}(q_i) \geq \alpha_i, \forall i \in I \\
I_{S,E}(p | q_i) \geq \beta_i, \forall i \in I \\
I_{S,E}(p) \geq \sup_{i \in I} T(\alpha_i, \beta_i)
\]

\[
C_{S,E}(q_i) \leq \alpha_i, \forall i \in I \\
C_{S,E}(p | q_i) \leq \beta_i, \forall i \in I \\
C_{S,E}(p) \leq \sup_{i \in I} T(\alpha_i, \beta_i)
\]

\(^5\)These measures correspond to the upper and lower bounds of Ruspini's definition of conditional necessity and possibility distributions respectively.
provided that in the second rule the requirement $\text{Sup}_{i \in I} C_{S,E}(q_i) = 1$ is satisfied.

Proof For the first rule, a simple translation of Ruspini's proof applies, although $P$ need not be a partition. For the second rule, suppose that $\text{Sup}_{i \in I} C_{S,E}(q_i) = 1$. Then we have the following chain of inequalities:

$$ \text{Sup}_{i \in I} T(a_i, \beta_i) \geq \text{Sup}_{i \in I} T(C_{S,E}(q_i), C_{S,E}(p | q_i)) = \text{Sup}_{i \in I} T(C_{S,E}(q_i), \text{Sup}_{w \in E} (R_T(C_{S,w}(q_i), C_{S,w}(p)))) \geq \text{Sup}_{i \in I} T(C_{S,E}(q_i), \text{Sup}_{w \in E} C_{S,w}(p)) = \text{Sup}_{i \in I} T(C_{S,E}(q_i), C_{S,E}(p)) = T(\text{Sup}_{i \in I} C_{S,E}(q_i), C_{S,E}(p)) = T(1, C_{S,E}(p)) = C_{S,E}(p). \quad \blacksquare $$

These expressions generalize Ruspini's in two different points. First, the rule for necessity measures holds for arbitrary sets of propositions $P$, not only for partitions. This can be easily noted going through Ruspini's proof for that rule, so in this case we have only made this fact explicit. The second point is that the pre-condition required for the rule for possibility measures, i.e. $\text{Sup}_{i \in I} C_{S,E}(q_i) = 1$, is actually more general than the one implied by imposing $P$ to be a partition. Namely, our condition can be read in the finite case as "there exists $w \in E$ and $i \in I$ such that $C_{S,w}(q_i) = 1,"$ while the Ruspini's one reads "for each $w \in E$ there exists $i \in I$ such that $C_{S,w}(q_i) = 1." 

2.2.2. SIMILARITIES AND FUZZY SETS Both notions of fuzzy set and of similarity relation can be linked. On the one hand, for instance, in [5] a notion of fuzzy set is introduced as a generalization of the extension of ill-defined or vague concepts where the membership degrees are degrees of similarity. More specifically, suppose that a concept $A$ applies on a set of paradigmatic cases $A_p = \{a_1, a_2, \ldots, a_n\}$ from a universe $U$. The method proposed by Niiniluoto states that the concept $A$ will also apply on other cases if they are sufficiently similar to some of $A_p$. To this end, suppose also that a similarity measure $S$ between elements of $U$ is defined. Then, a natural generalization of the membership function for $A$ arises by defining for every object $x$ of $U$ its partial membership degrees to $A$ as:

$$ \mu_A(x) = \text{Sup}\{S(x, a_i) \mid a_i \in A_p\} $$

Niiniluoto does not go further in this proposal, and neither formal definitions nor conditions on the similarity function $S$ are given. Nevertheless, using this concept of fuzzy set, we are going to reinterpret the major constructs of Ruspini's approach next.

First, given a similarity relation $S$, and following Niiniluoto's definition, we associate to every proposition $p$ a fuzzy subset, denoted by $\tilde{p}$, of the set
of possible worlds with the following membership function:

\[ \mu_{\tilde{p}}(w) = \text{Sup}\{S(w, w') \mid w' \in [p]\} \]

This definition is related to the concept of fuzzy rough set [13]. In fact, the fuzzy set \( \tilde{p} \) is the fuzzy upper approximation of the classical set \([p]\) via the similarity relation \( S \). On the other hand, in terms of the generators of \( S \) (see Annex A), \( \tilde{p} \) is the smallest fuzzy set including the crisp set \([p]\) that is, at the same time, a generator of \( S \).

From the previous definition it holds that \( \mu_{\tilde{p}}(w) = I(p \mid w) = C(p \mid w) \), i.e. the membership degree of a world \( w \) to \( \tilde{p} \) is the maximum value \( \alpha \) such that \( w \models \Delta_\alpha p \), or equivalently such that \( w \not\models \bigwedge_\alpha \neg p \). It is also worth noticing that with this notion of fuzzy set, union of fuzzy sets coincides with the usual one, but it is not the case with intersection or complementation. In general we only have the following relations:

\[ \mu_{(p \lor q)}(w) = \text{Sup}\{S(w, w') \mid w' \in [p \lor q]\} = \max(\mu_{\tilde{p}}(w), \mu_q(w)) \]

\[ \mu_{(p \land q)}(w) = \text{Sup}\{S(w, w') \mid w' \in [p \land q]\} \leq \min(\mu_{\tilde{p}}(w), \mu_q(w)) \]

\[ \mu_{(-p)}(w) = \text{Sup}\{S(w, w') \mid w' \in [-p]\} \geq 1 - \mu_{\tilde{p}}(w) \]

It is interesting to observe that, if we denote by \( F_S \) the mapping from \( L \) to the set \([0, 1]^W\) of fuzzy sets on \( W \) assigning to each proposition \( p \) the fuzzy set \( \tilde{p} \), \((F_S(L), \lor, \land, \neg)\) is a Boolean algebra with operations defined as

\[ \tilde{p} \lor \tilde{q} = (p \lor q)\sim, \tilde{p} \land \tilde{q} = (p \land q)\sim, \sim \tilde{p} = (-p)\sim \]

However, these operations are not enough to describe Ruspini's generalized modus ponens. As it will be shown below, the formulation of the generalized modus ponens makes use of conditional statements "if \( p \) then \( q \)" whose associated fuzzy set, denoted by \( \tilde{p} \rightarrow \tilde{q} \) and defined as \( \mu_{\tilde{q} \rightarrow \tilde{p}}(w) = R_\Gamma(\mu_{\tilde{q}}(w), \mu_{\tilde{p}}(w)) \), does not belong necessarily to \( F_S(L) \). Notice that \( \tilde{p} \rightarrow \tilde{q} \) is different from \((p \rightarrow q)\sim = (-p \lor q)\sim \). This situation is very similar to that occurring in the probabilistic reasoning framework where conditionals statements are usually modeled by means of conditional probabilities and not as probabilities on material implications.

Second, the unconditioned measures \( I_{S,E} \) and \( C_{S,E} \) can be then expressed as

\[ I_{S,E}(p) = I(p \mid E) = \text{Inf}_{w \in E}\{\mu_{\tilde{p}}(w)\} \]

\[ C_{S,E}(p) = C(p \mid E) = \text{Sup}_{w \in E}\{\mu_{\tilde{p}}(w)\} \]
In other words, the interval \([I_{S,E}(p), C_{S,E}(p)]\) can be interpreted as the range of truth-values of \(\tilde{p}\), the "fuzzification" of \(p\), when \(E\) is taken as reference (see Figure 1).

This interpretation is therefore in complete accordance with the generalized Zadeh’s possibility and necessity measures of the fuzzy set \(\tilde{p}\) given \(E\), namely:

\[
\Pi_E(\tilde{p}) = \sup_{w \in \omega} \min(\mu_E(w), \mu_{\tilde{p}}(w)) = C_{S,E}(p)
\]
\[
N_E(\tilde{p}) = \inf_{w \in \omega} \max(1 - \mu_E(w), \mu_{\tilde{p}}(w)) = I_{S,E}(p)
\]

In the standard algebra of fuzzy sets, using max-min as union and intersection connectives, these two measures satisfy the characteristic properties of possibility and necessity measures, that is,

\[
\Pi_E(A \cup B) = \max(\Pi_E(A), \Pi_E(B)) \quad \text{and} \quad N_E(A \cap B) = \min(N_E(A), N_E(B)).
\]

However, in the non-standard fuzzy sets algebra \(\langle F_s(L), \tilde{\vee}, \tilde{\wedge}, \tilde{\neg} \rangle\), these properties are only partially satisfied. As we have already noticed, in Ruspini’s approach only union is defined as in standard fuzzy set theory but negation and intersection are not. This is expressed now by the following relations:

\[
\Pi_E(\tilde{p} \tilde{\vee} \tilde{q}) = \max(\Pi_E(\tilde{p}), \Pi_E(\tilde{q})) \quad \text{and} \quad N_E(\tilde{p} \tilde{\wedge} \tilde{q}) \leq \min(N_E(\tilde{p}), N_E(\tilde{q}))
\]

Third, given two fuzzy sets \(\tilde{p}\) and \(\tilde{q}\) we define the conditional fuzzy set \(\tilde{q} \rightarrow \tilde{p}\) by setting

\[
\mu_{\tilde{q} \rightarrow \tilde{p}}(w) = R_T(\mu_{\tilde{q}}(w), \mu_{\tilde{p}}(w))
\]

This is very usual in fuzzy logic, where conditions of type "if \(q\) then \(p\)" are modeled by means of implication functions. Accordingly, the conditional

![Figure 1. Graphical representation of implication and consistency measures.](image-url)
measures $I_{S,E}(\cdot | \cdot)$ and $C_{S,E}(\cdot | \cdot)$ can be expressed as:

$$I_{S,E}(p | q) = \inf_{w \in E} \left\{ \mu_{\bar{q} \rightarrow \bar{p}}(w) \right\}$$

$$C_{S,E}(p | q) = \sup_{w \in E} \left\{ \mu_{\bar{q} \rightarrow \bar{p}}(w) \right\}$$

Finally, given a set of propositions $P = \{q_i | i \in I\}$ just taking into consideration the above relations, the corresponding expressions of Ruspini's generalized modus ponens can now be formulated as follows:

$$\inf_{w \in E} \left\{ \mu_{\bar{q}_i}(w) \right\} \geq \alpha_i \quad \inf_{w \in E} \left\{ \mu_{\bar{q}_i \rightarrow \bar{p}}(w) \right\} \geq \beta_i \quad \inf_{w \in E} \left\{ \mu_{\bar{p}}(w) \right\} \geq \sup_{i \in I} T(\alpha_i, \beta_i)$$

$$\sup_{w \in E} \left\{ \mu_{\bar{q}_i}(w) \right\} \leq \alpha_i \quad \sup_{w \in E} \left\{ \mu_{\bar{q}_i \rightarrow \bar{p}}(w) \right\} \leq \beta_i \quad \sup_{w \in E} \left\{ \mu_{\bar{p}}(w) \right\} \leq \sup_{i \in I} T(\alpha_i, \beta_i)$$

provided that for the second rule the condition $\sup_{i \in I} \sup_{w \in E} \left\{ \mu_{\bar{q}_i}(w) \right\} = 1$. In particular, the simplest form these rules can take is:

$$\inf_{w \in E} \left\{ \mu_{\bar{q}}(w) \right\} \geq \alpha \quad \sup_{w \in E} \left\{ \mu_{\bar{q}}(w) \right\} = 1$$

$$\inf_{w \in E} \left\{ \mu_{\bar{q} \rightarrow \bar{p}}(w) \right\} \geq \beta \quad \sup_{w \in E} \left\{ \mu_{\bar{q} \rightarrow \bar{p}}(w) \right\} \leq \beta$$

3. CORRESPONDENCE BETWEEN DUBOIS-PRADE'S AND RUSPINI'S APPROACHES

It is clear from last section that at first sight there is a main difference between both approaches at the inference level. In $SL$, inference is formulated through conditional measures, whereas in $PL$ conditional statements are modeled by means of material implication. For this reason, we will mainly focus our study on relating both approaches at the semantical level (subsections 3.1, 3.2, and 3.3), devoting the last part of this section (subsection 3.4) to point out some comments on the inference machinery. To this end, and to better explicate the relationship between both approaches, it seems more advantageous to define model theoretically Ruspini's $SL$ logic in a more formal way as follows:

LANGUAGE Given a Boolean algebra of propositions $L$, the extended language $L_{SL}$ is defined as

$$L_{SL} = \{(p, [\alpha, \beta]) | p \in L, \alpha \in [0, 1], \beta \in [0, 1], \alpha \leq \beta\}$$
In a \( SL \) formula \( (p, [\alpha, \beta]) \), \( \alpha \) stands for a lower bound for the implication degree of \( p \), and \( \beta \) stands for an upper bound of the consistency degree of \( p \). The formula \( (p, [\alpha, \beta]) \) expresses that \( p \) is considered to be true at least to the degree \( \alpha \) and compatible at most to the degree \( \beta \) w.r.t. an state of incomplete knowledge.

SEMANTICS A model is a pair \((S, E)\) modeling a state of incomplete knowledge, where \( S \) is a \( T \)-similarity relation on the set \( W \) of possible worlds or interpretations of \( L \), and \( E \) is a subset of \( W \). The satisfaction relation is defined as:

\[
(S, E) \models_{SL} (p, [\alpha, \beta]) \text{ if, and only if, } \alpha \leq I_{S,E}(p) \text{ and } C_{S,E}(p) \leq \beta
\]

In what follows we are going to show how some constructs of possibilistic logic can be described within the Ruspini's similarity approach, and vice versa.

3.1. Mapping \( SL \) into \( PL \)

The idea of mapping \( SL \) models into \( PL \) models is to take as possibility distributions those induced by the "fuzzification" of the evidential sets via a similarity relation, using the approach of Niiniluoto. More formally, given a \( SL \) model \((S, E)\), we consider the fuzzy subset \( \tilde{E} \) of \( W \) defined as:

\[
\tilde{E}(w) = \sup \{ S(w, w') \mid w' \in E \}
\]

The possibility distribution \( \pi_{S,E} \) induced on \( W \) by \( \tilde{E} \), that is, \( \pi_{S,E}(w) = \mu_{\tilde{E}}(w) \) for every \( w \) of \( W \), will be then used as the basis for defining a possibilistic logic semantics on top of the same propositional language \( L \). Namely, the possibility distribution \( \pi_{S,E} \) generates, in the usual way, a pair of dual possibility/necessity measures on propositions of \( L \):

\[
\text{Pos}_{\pi_{S,E}}(p) = \sup \{ \pi_{S,E}(w) \mid w \models p \}
\]

\[
\text{Nec}_{\pi_{S,E}}(p) = 1 - \text{Pos}_{\pi_{S,E}}(\neg p)
\]

It is worth noticing the coincidence between degrees of possibility in \( \pi_{S,E} \) and degrees of unconditioned consistency in \((S, E)\).

PROPOSITION 4 For every proposition \( p \), it holds \( \text{Pos}_{\pi_{S,E}}(p) = C_{S,E}(p) \).

Proof \( \text{Pos}_{\pi_{S,E}}(p) = \sup \{ \pi_{S,E}(w) \mid w \models p \} = \sup_{w' \in E} \sup_{w' \models p} S(w, w') = C_{S,E}(p) \). \[\blacksquare\]

However no relation can be established between degrees of necessity and degrees of implication, mainly because Ruspini's measures on a
proposition \( p \) do not relate at all with the measures on the negated proposition \( \neg p \). Therefore, the links we can establish between satisfaction relations for corresponding models of both approaches are those given in the following theorem.

**Theorem 1** For any model \((S, E)\) the following properties hold:

i) If \((S, E) \models_{SL} (p, [\alpha, \beta])\) then \( \pi_{S,E} \models_{PL} (\neg p, N_{1-\beta}) \) and \( \pi_{S,E} \models_{PL} (p, \Pi_{\alpha}) \)

ii) If \( \pi_{S,E} \models_{PL} (p, N_{\alpha}) \) then \((S, E) \models_{SL} (\neg p, [0, 1-\alpha]) \) and \((S, E) \models_{SL} (p, [\alpha, 1]) \)

**Proof**

i) If \((S, E) \models (p, [\alpha, \beta])\), then by definition \( I_{S,E}(p) \geq \alpha \) and \( C_{S,E}(p) \leq \beta \), but from the above remark we have \( Pos_{\pi_{S,E}}(p) = C_{S,E}(p) \geq I_{S,E}(p) \geq \alpha \) and \( Nec_{\pi_{S,E}}(\neg p) = 1 - C_{S,E}(p) \geq 1 - \beta \). Thus it follows that \( \pi_{S,E} \models (p, \Pi_{\alpha}) \) and \( \pi_{S,E} \models (\neg p, N_{1-\beta}) \).

ii) If \( \pi_{S,E} \models (p, N_{\alpha}) \) then \( 1 - C_{S,E}(\neg p) = Nec_{\pi_{S,E}}(p) \geq \alpha \), and thus \((S, E) \models (\neg p, [0, 1-\alpha]) \). On the other hand it is \( I_{S,E}(p) \geq 1 - C_{S,E}(\neg p) = Nec_{\pi_{S,E}}(p) \geq \alpha \), and thus we also have \((S, E) \models (p, [\alpha, 1]) \).

### 3.2. Mapping \( PL \) and \( SL \)

To express \( PL \) constructs in terms of \( SL \), we need first of all a procedure to obtain a similarity relation from a possibility distribution. This problem can be considered as a particular case of the following fuzzy clustering problem (see for instance [14] or [15]). Let \( U \) be a set of objects and \( A \) a set of attributes, and let \( h_a(x) \) the extent to which the proposition “the object \( x \) has attribute \( a \)” is considered to be true. The problem then is how to get a fuzzy classification (induced by a \( T \)-similarity relation) on the set \( U \) with respect to the set of attributes \( A \). More concretely, the matter is how to define a \( T \)-similarity relation on \( U \) given a family \( \{h_a\}_{a \in A} \) of fuzzy subsets, each one modeling how a particular attribute applies on the objects of \( U \). The solution is given by the following Valverde’s representation theorem of \( T \)-similarity relations in [6].

**Theorem 2** A binary function \( S \), mapping pairs of objects from a universe \( U \) into \([0, 1]\), is a \( T \)-similarity relation if, and only if, there exists a family \( H \) of fuzzy subsets of \( U \) such that

\[
S(w, w') = \inf_{h \in H} \{ \min(R_T(h(w), h(w')), R_T(h(w'), h(w))) \}
\]

for all \( w \) and \( w' \) in \( U \). Each fuzzy subset \( h \in H \) is said to be a generator of \( S \).
Notice that, if we consider the T-similarities generated by each \( h \in H \), i.e., \( S_h(w, w') = \min(R_T(h(w), h(w')), R_T(h(w'), h(w))) \), the above theorem says that \( S = \inf_{h \in H} S_h \), and obviously it is \( S_h \geq S \), for every generator \( h \) of \( S \) (see Annex A for further details).

Thus, given a possibilistic model \( \pi \), i.e. a possibility distribution \( \pi : W \to [0, 1] \) on the set of possible worlds, the T-similarity relation that measures resemblance of worlds only from the point of view of \( \pi \) is, according to the representation theorem, the T-similarity \( S_\pi \) generated by \( \pi \) and defined as

\[
S_\pi(w, w') = \min(R_T(\pi(w), \pi(w')), R_T(\pi(w'), \pi(w)))
\]

On the other hand, it seems natural to consider the evidential set associated to \( \pi \) to be the set of worlds fully compatible or maximally preferred w.r.t. \( \pi \), i.e., \( E_\pi = \text{Core}(\pi) = \{w \in W \mid \pi(w) = 1\} \). Choosing in this way the evidential set, the following proposition holds.

**PROPOSITION 5** Let \( \pi \) be a possibility distribution on \( W \). Take \( S_\pi \) to be the T-similarity relation defined above and \( E_\pi = \text{Core}(\pi) \). Then the following properties hold:

(i) \( \pi_{S_\pi, E_\pi} = \pi \).

(ii) \( I_{S_\pi, E_\pi}(p) = C_{S_\pi, E_\pi}(p) = \text{Pos}_\pi(p) \), for every proposition \( p \).

**Proof**

(i) \( \pi_{S_\pi, E_\pi}(w) = \sup\{S_\pi(w, w') \mid w' \in E_\pi\} \)

\[
= \sup\{\min(R_T(\pi(w), \pi(w')), R_T(\pi(w'), \pi(w))) \mid w' \in E_\pi\}
\]

\[
= \min(R_T(\pi(w), 1), R_T(1, \pi(w))) = \pi(w).
\]

(ii) \( C_{S_\pi, E_\pi}(p) = C(p \mid E_\pi) = \sup_{w' \models E_\pi} \sup_{w \models p} S_\pi(w, w') \)

\[
= \sup_{w' \models E_\pi} \sup_{w \models p} \min(R_T(\pi(w), \pi(w')), R_T(\pi(w'), \pi(w)))
\]

\[
= \sup_{w \models p} \min(R_T(\pi(w), 1), R_T(1, \pi(w))) = \text{Pos}_\pi(p).
\]

\( I_{S_\pi, E_\pi}(p) = I(p \mid E_\pi) = \inf_{w' \models E_\pi} \sup_{w \models p} S_\pi(w, w') \)

\[
= \inf_{w' \models E_\pi} \sup_{w \models p} \min(R_T(\pi(w), \pi(w')), R_T(\pi(w'), \pi(w)))
\]

\[
= \sup_{w \models p} \min(R_T(\pi(w), 1), R_T(1, \pi(w))) = \text{Pos}_\pi(p).
\]
The equality (i) states that we recover the original possibility distribution \( \pi \) from the similarity \( S_\pi \) and any evidential set \( E_\pi \) associated to \( \pi \) itself. Equalities (ii) show that the implication and consistency degrees induced by \( (S_\pi, E_\pi) \) coincide, and are equal to the possibility degree induced by \( \pi \). This is a direct consequence of making indistinguishable, through \( S_\pi \), all the elements of \( \text{Core}(\pi) \). Another consequence of the latter is that the membership function of \( \beta, \mu_\beta(w) = \text{Sup}\{S_\pi(w, w') \mid w' \in [p]\} \), has constant value \( \text{Pos}_\pi(p) \) for any \( w \in \text{Core}(\pi) \) (see Figure 2). Finally, notice that the proposition would still be valid if we take as evidential set not the whole \( \text{Core}(\pi) \) but a subset of it.

As a direct consequence of the last proposition, next theorem shows how \( PL \) can be embedded in \( SL \).

**THEOREM 3** For any \( PL \) model \( \pi \), the following equivalencies hold:

i) \( \pi \models_{PL} (p, \Pi \alpha) \) if, and only if, \( (S_\pi, E_\pi) \models_{SL} (p, \lfloor \alpha, 1 \rfloor) \)

ii) \( \pi \models_{PL} (p, \operatorname{Na}) \) if, and only if, \( (S_\pi, E_\pi) \models_{SL} (\neg p, \lfloor 0, 1 - \alpha \rfloor) \)

iii) \( \pi \models_{PL} (p, \Pi \alpha) \) and \( \pi \models_{PL} (\neg p, \Pi 1 - \beta) \) if, and only if, \( (S_\pi, E_\pi) \models_{SL} (p, \lfloor \alpha, \beta \rfloor) \).

Figure 3 summarizes results of the previous and present subsections on relating semantics of \( PL \) and \( SL \). In this picture, the plain arrow from \( PL \) to \( SL \) stands for the faithful semantical embedding of \( PL \) into \( SL \) given by the equivalencies on top of the arrow. The dashed arrow from \( SL \) to \( PL \) stands for the partial embedding from the \( SL \) fragment of \( SL \) formulas of type \( (p, \lfloor 0, \alpha \rfloor) \) into the \( PL \) fragment of necessity-valued formulas given by the equivalence below the arrow. Besides, a weaker additional relation involving arbitrary \( SL \) formulas and possibility-valued \( PL \) formulas is also provided.

### 3.3. Closing Relations Between \( PL \) and \( SL \) Models

In the previous subsections we have been concerned with correspondences from \( SL \) to \( PL \) models and vice versa. Now we are interested in
how they relate one each other. To this end, denote by $F$ the mapping from $SL$ to $PL$ models defined in subsection 3.2., i.e.

$$F(S, E) = \pi_{S,E}$$

and denote by $G$ the mapping from $PL$ to $SL$ models defined in subsection 3.3., i.e.

$$G(\pi) = (S_{\pi}, \text{Core}(\pi))$$

In Annex A, it is shown that $\pi_{S,E}$ is itself a generator of the similarity relation $S$, and therefore the similarity $S_{\pi_{S,E}}$ generated by $\pi_{S,E}$ is always greater or equal than $S$ for any $E$. This property, together with property (i) of Proposition 5, enables us to give the following theorem describing the properties of the compositions of mappings $F$ and $G$, where the relation $\geq$ in the set of $SL$ models is defined as

$$(S, E) \geq (S', E') \iff \pi_{S,E} \models (p, [\alpha, \beta])$$

**THEOREM 4** If $\text{Id}_{SL}$ and $\text{Id}_{PL}$ denote the identity mapping on the sets of $SL$ and $PL$ models respectively, then the mappings $F$ and $G$ satisfy the following properties.6

(i) $G \circ F \geq \text{Id}_{SL}$ and $F \circ G = \text{Id}_{PL}$,

(ii) $G \circ F$ is a closure operator on $SL$ models, and a model $(S, E)$ is closed w.r.t. this operator if, and only if, $S$ is the unidimensional similarity relation generated by $\pi_{S,E}$ and $E = \text{Core}(\pi_{S,E})$.

In Annex B, a characterization of closed similarity models is fully described. Moreover, it is shown that, if $T$ is an archimedian $t$-norm and $(S, E)$ is a closed similarity model, the preference preorder of its corresponding possibilistic model and the preorder defined by any generator of

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6A pair $(F, G)$ satisfying (i) is known as a Galois Connection.
the $T$-similarity relation $S$ are either the same or one the inverse of the other.

Finally it is easy to check that, although $\text{GoF} \geq \text{Id}_{SL}$, the consistency degree is always preserved by this transformation, that is, $C_{S,E}(p) = C_{\text{GoF}(S,E)}(p)$ holds for every $p$.

3.4. Relating Inference Mechanisms

To study the relationship between the inference mechanisms proposed in $PL$ and $SL$ it is necessary to start out from two considerations:

- $PL$ uses an extended resolution principle as a general inference rule.
- The modus ponens rule proposed for $SL$ cannot be considered as a proper inference rule because it relies on the use of measures of conditional statements that do not belong to the language defined above.

So, to relate the inference mechanisms provided by both logics we need

1) either to define a resolution scheme in $SL$, or
2) to consider how conditioning is performed in $PL$.

With respect to the first point, because of $C_{S,E}$ is a possibility measure but $I_{S,E}$ is not a necessity measure, the only resolution-like inference rule that can be formulated in $SL$ is that corresponding to the dual form of the resolution principle in $PL$ for necessity-valued clauses. Taking into account that a necessity-valued formula $(p, N\alpha)$ corresponds to the $SL$ formula $(\neg p, [0, \alpha])$, the following inference scheme is valid in $SL$:

\[
(p \land q, [0, \alpha]) \\
(\neg p \land r, [0, \beta]) \\
(q \land r, [0, \max(\alpha, \beta)])
\]

As far as conditioning is concerned, two further facts have to be taken into consideration:

- The notion of conditional possibility measure $\Pi(q \mid p)$ has been defined by Dubois and Prade [9] as the least specific solution of the equation $\Pi(p \land q) = T(\Pi(q \mid p), \Pi(p))$, being $T$ a t-norm. Such measure is thus defined as $\Pi(q \mid p) = R_T(\Pi(p), \Pi(p \land q))$. For instance

\[
\Pi_{\text{min}}(q \mid p) = \begin{cases} 
1, & \text{if } \Pi(p \land q) = \Pi(p) \\
\Pi(p \land q), & \text{otherwise}
\end{cases}
\]

\[
\Pi_{\text{prod}}(q \mid p) = \begin{cases} 
1, & \text{if } \Pi(p \land q) = \Pi(p) \\
\frac{\Pi(p \land q)}{\Pi(p)}, & \text{otherwise}
\end{cases}
\]

where $T = \text{minimum}$ and $T = \text{product}$ respectively.
• The consistency measure $C_{S,E}(\cdot | \cdot)$ of conditioned statements used in the modus ponens-like inference mechanism in $SL$ is not a conditional possibility measure in the above sense because, in general, only the inequality

$$C_{S,E}(p \land q) \leq T(C_{S,E}(q | p), C_{S,E}(p)).$$

is satisfied. In spite of that, for every $q$, $C_{S,E}(\cdot | q)$ is indeed a possibility measure in the sense of $PL$. Therefore it seems difficult to define in $PL$ an inference mechanism corresponding to the Ruspini's modus ponens-like inference mechanism in $SL$, mainly because the conditional measures used by Ruspini are not conditional measures in $PL$.

4. DEALING WITH FUZZY PROPOSITIONS

The main difference in dealing with fuzzy propositions with respect to non-fuzzy propositions is to consider a possible world as a many-valued interpretation instead of a classical interpretation. That is, fuzzy propositions are allowed to be valued with intermediate degrees of truth. The set of possible worlds or many-valued interpretations (not necessarily truth-functional) will be denoted by $W^*$. For the sake of simplicity we will consider the unit interval $[0, 1]$ as the set of truth-values, identifying 0 and 1 with the classical truth-values false and true respectively. As a matter of fact, the set of “classical” possible worlds $W$, i.e. the set of interpretations on the binary set $\{0, 1\}$, is a proper subset of $W^*$. In this setting, given a proposition $p$, we generalize the concept of set of $p$-worlds $[p]$ to the concept of sets of $\alpha$-p-worlds, for every $\alpha \in [0, 1]$, defining $[p]_\alpha = \{w \in W^* | w(p) = \alpha\}$. So the key idea to extend possibilistic reasoning to cope with fuzzy propositions is to consider $W^*$ instead of $W$, and to consider classical measures on sets of $\alpha$-p-worlds.

Dubois, Lang, and Prade have proposed two other approaches [1, 8, 10, 11] to cope with fuzzy propositions in the framework of possibilistic logic. In [8, 10], a fuzzy proposition $p$ with membership function $\mu_p$ is modeled by means of a family of necessity-valued formulas $((p_a, \alpha_a) | a \in U)$ where $U$ is the universe of discourse for $p$ (it would correspond to $W$ in our case) and $\alpha_a$ is the necessity degree of $p_a$ generated from the possibility distribution $\pi = \mu_p$. In [1, 11], the resolution principle with necessity-valued clauses of $PL$ has been extended to the case when fuzzy propositional variables are involved, using the concept of generalized necessity and possibility measures for fuzzy propositions (see section 2.2.2.). In this approach, a formula $(p, N\alpha)$ corresponds to a possibility distribution $\pi_{p,\alpha}(w) = \max(1 - \alpha, \mu_p(w))$. 
4.1. Fuzzy Truth-valued Logic

Let $L$ denote a logical propositional language, consisting of classical well-formed formulas, and $W^*$ the set of many-valued interpretations on $L$. Many-valued interpretations will be considered as functions $w: L \rightarrow [0,1]$ that assign a truth-degree to every proposition such that at least preserves the classical properties on $\{0,1\}$.

Every $[0,1]$-valued interpretation will correspond to a complete description of the real state of a system to be reasoned about. Then, every proposition $p$ can be identified with a fuzzy subset $[p]^* \subseteq W^*$ of possible worlds having as membership function $\mu_{[p]^*}: W^* \rightarrow [0,1]$ defined by $\mu_{[p]^*}(w) = w(p)$, for all $w \in W^*$.

As in possibilistic logic, the incomplete available information about a system state is modeled by means of a possibility distribution $\pi$ on the set $W^*$ of possible worlds, that is, $\pi: W^* \rightarrow [0,1]$. Under such information, the (fuzzy) set of possible truth-values (together with their possibility degrees) for any proposition $p$ is estimated by the so-called inverse truth-functional modification $\tau_p/\pi$ (see [16]), i.e., given a proposition $p$ and a possibility distribution $\pi$, the function $\tau_p/\pi : [0,1] \rightarrow [0,1]$ is defined as:

$$\tau_p/\pi(z) = \sup_{w \in \pi^{-1}(\{z\})} \mu_{[p]^*}(w),$$

which turns to be equivalent to

$$\tau_p/\pi(z) = \sup_{w \in \pi^{-1}(\{z\})} \pi(w) = \text{Pos}_{\pi}(\{z\}),$$

that is, for each $z$, $\tau_p/\pi(z)$ can be understood as the possibility measure of the set of $z$-p-worlds. In other words $\tau_p/\pi$, known as the fuzzy truth-value of $p$ given $\pi$, provides a (possibility) measure for each horizontal cut of the membership function $\mu_p$. This differs from [8] in that measures are provided there for a family of non-fuzzy propositions associated to $p$, corresponding in some sense to vertical cuts of $\mu_p$. On the other hand, $FTL$ is related with the approach of [1] in the sense that a certainty qualified formula $(p, Na)$ corresponds to the $FTL$ formula $(p, r_{\alpha})$, where $r_{\alpha}(z) = \max(1 - \alpha, z)$.

An earlier semantical formalization of fuzzy logic as a fuzzy truth-valued logic was given in [7]. We now define the language and the semantics of the fuzzy truth-valued logic $FTL$ as follows:

**LANGUAGE** Given a propositional language $L$, the extended language $L_{FTL}$ of $FTL$ is defined by the set of formulas

$$L_{FTL} = \{(p, \tau) \mid p \in L, \tau \text{ is a fuzzy truth value on } [0,1], \text{i.e. } \tau \in [0,1]^{[0,1]}\}$$
That is, \textit{FTL} formulas are pairs composed of propositions and fuzzy truth-values.

**SEMANTICS** A model is a possibility distribution \( \pi \) on \( W^* \). The satisfaction relation between models and \textit{FTL}-formulas is defined for every model \( \pi \) and every \textit{FTL}-formula \( (p, \tau) \) as:

\[
\pi \models_{\text{FTL}} (p, \tau) \text{ if, and only if, } \tau_{p/\pi} \leq \tau
\]

**4.2. Relating \textit{PL} and \textit{SL} to \textit{FTL}**

Possibilistic logic can be interpreted in terms of the fuzzy truth-valued logic (\textit{FTL}) in different ways. The most direct way to do it is to consider only classical propositions or, in other words, to consider as the set of possible worlds not the whole \( W^* \) but only the subset of classical interpretations \( W \). This can be easily achieved by extending any possibility distributions \( \pi \) on \( W \) to a possibility distribution on \( W^* \), written also \( \pi \), in such a way that it makes impossible all the worlds of \( W^* - W \), i.e., \( \pi(w) = 0 \) if \( w \in W^* - W \). Therefore, from now on, we will use the same symbol to denote, without distinction, both a possibility distribution on \( W \) and its extension to \( W^* \). In this context, given a possibility distribution \( \pi \) on \( W \), and considering the parametric family \( \tau_{\alpha, \beta} \) of fuzzy truth-values

\[
\tau_{\alpha, \beta}(x) = \begin{cases} 
\alpha, & \text{if } x = 0 \\
\beta, & \text{if } x = 1 \\
0, & \text{otherwise}
\end{cases}
\]

for all \( \alpha, \beta \in [0, 1] \), possibility and necessity values are related to these fuzzy truth-values as follows.

**PROPOSITION 6** Let \( \pi : W \to [0, 1] \) be a possibility distribution. Then the following conditions hold:

i) \( \text{Pos}_\pi(p) = 1 \) and \( \text{Nec}_\pi(p) = \alpha \) if, and only if, \( \tau_{p/\pi} = \tau_{1-a,1} \).

ii) \( \text{Pos}_\pi(p) = \alpha \) and \( \text{Nec}_\pi(p) = 0 \) if, and only if, \( \tau_{p/\pi} = \tau_{1-a} \).

Proof. By definition we have \( \tau_{p/\pi}(z) = \text{Pos}_\pi([p]_z) \). Since \( W \) is the set of classical interpretations, \( [p]_1 = \{ w \in W \mid w \models p \} \), \( [p]_0 = \{ w \in W \mid w \models \neg p \} \cup (W^* - W) \), and \( [p]_z = \emptyset \) for \( 0 < z < 1 \). Therefore \( \tau_{p/\pi}(1) = \text{Pos}_\pi(p), \tau_{p/\pi}(0) = \text{Pos}_\pi(\neg p) = 1 - \text{Nec}_\pi(p), \) and \( \tau_{p/\pi}(z) = \text{Pos}_\pi(\emptyset) = 0 \) for \( 0 < z < 1 \). As a consequence, (i) and (ii) follow directly.
Notice that, from the above properties, it holds that

\[ \text{Nec}_{\pi}(p) \geq \alpha \text{ if, and only if, } \tau_{p/\pi} \leq \tau_{1-\alpha,1} \]

\[ \text{Pos}_{\pi}(p) \geq \alpha \text{ if, and only if, } \tau_{p/\pi} \geq \tau_{1,\alpha} \]

Therefore, the semantical entailment in \( PL \) of necessity-valued formulas has a direct translation into \( FTL \) as it is expressed in the following theorem.

**Theorem 5** Let \( \pi : W \to [0, 1] \) be a possibility distribution. Then the following equivalence holds:

\[ \pi \models_{PL} (p, N_\alpha) \text{ if, and only if, } \pi \models_{FTL} (p, \tau_{1-\alpha,1}). \]

Notice that, using the interpretation of possibility and necessity values as fuzzy truth-values given in proposition 6, the semantical entailment of \( PL \) possibility-valued formulas does not have a direct translation into \( FTL \). However, taking into account that \( PL \) semantics can be embedded in \( SL \) semantics (see section 3.2.), we will see how that translation will be possible after first having faithfully translated any \( SL \) semantical entailment to a corresponding one in \( FTL \).

Given a \( T \)-similarity relation \( S \) on \( W \), we define a function \( f_S : W \to W^* \) mapping each classical world \( w \) to its corresponding many-valued world \( f_S(w) \), defined by \( f_S(w)(p) = \text{Sup}_{w', p} S(w, w') \), for all \( p \in L \).\(^8\) Now, given a \( SL \) model \((S, E)\), we build its corresponding \( FTL \) model by considering the possibility distribution \( \pi_{S,E} : W^* \to [0, 1] \) defined by:

\[ \forall w^* \in W^*, \pi_{S,E}(w^*) = \begin{cases} 1, & \text{if } \exists w \in E \text{ such that } w^* = f_S(w) \\ 0, & \text{otherwise} \end{cases} \]

Considering these particular kind of \( FTL \) models it is easy to prove that, from the semantical point of view, \( FTL \) logic embeds \( SL \) logic.

**Theorem 6** For any \( SL \) model \((S, E)\) and for any \( SL \) sentence \((p, [\alpha, \beta])\) the following equivalence holds:

\[ (S, E) \models_{SL} (p, [\alpha, \beta]) \text{ if, and only if, } \pi_{S,E} \models_{FTL} (p, \kappa_{\alpha,\beta}), \]

where \( \kappa_{\alpha,\beta} \) is the fuzzy truth-value defined as

\[ \kappa_{\alpha,\beta}(x) = \begin{cases} 1, & \text{if } \alpha \leq x \leq \beta \\ 0, & \text{otherwise} \end{cases} \]

\(^8\) Notice that \( f_S(w) \) is not necessarily a functional many-valued interpretation.
Proof Let us compute $\tau_{p/\tilde{S},E}$. By definition $\tau_{p/\tilde{S},E}(z) = \text{Sup}(\hat{\pi}_{S,E}(w^*) \mid w^* \in W^*, w^*(p) = z)$, and therefore either $\tau_{p/\tilde{S},E}(z) = 1$, if there exists $w \in E$ such that $w^* = f_S(w)$ and $f_S(w)(p) = \text{Sup}_{w' \in p} S(w, w') = z$, or $\tau_{p/\tilde{S},E}(z) = 0$ otherwise. Thus, if $\tau_{p/\tilde{S},E}(z) = 1$ then necessarily $z \in [I_{S,E}(p), C_{S,E}(p)]$. Moreover, the interval $[I_{S,E}(p), C_{S,E}(p)]$ is the smallest closed interval including the set $\{z \in [0, 1] \mid \tau_{p/\tilde{S},E}(z) = 1\}$. Now suppose that $(S, E) \models_{SL} (p, [\alpha, \beta])$ holds. Then $\{z \in [0, 1] \mid \tau_{p/\tilde{S},E}(z) = 1\} \subset [I_{S,E}(p), C_{S,E}(p)] \subset [\alpha, \beta]$. Therefore $\tau_{p/\tilde{S},E} \leq \kappa_{\alpha,\beta}$. Reciprocally, if $\tau_{p/\tilde{S},E} \leq \kappa_{\alpha,\beta}$ then $\{z \in [0, 1] \mid \tau_{p/\tilde{S},E}(z) = 1\} \subset [\alpha, \beta]$, and therefore we also have $[I_{S,E}(p), C_{S,E}(p)] \subset [\alpha, \beta]$. ■

Finally, this result, together with Theorem 3, where $SL$ is proved to capture $PL$, leads directly to the following corollary showing how $FTL$ captures $PL$ as well.

**COROLLARY** For any $PL$ model $\pi$ and any $PL$ sentences $(p, \Pi \alpha)$ and $(p, \Pi \alpha)$ the following equivalence holds:

i) $\pi \models_{PL} (p, \Pi \alpha) \iff (S_n, E_n) \models_{SL} (p, [\alpha, 1]) \iff \hat{\pi}_{S_n,E_n} \models_{FTL} (p, \kappa_{\alpha,1})$

ii) $\pi \models_{PL} (p, \Pi \alpha) \iff (S_n, E_n) \models_{SL} (\neg p, [0, 1 - \alpha]) \iff \hat{\pi}_{S_n,E_n} \models_{FTL} (\neg p, \kappa_{0,1-\alpha})$.

5. CONCLUSIONS

We have established semantical relationships between the two main approaches to possibilistic reasoning, Dubois-Prade's approach known as possibilistic logic and Ruspini's approach whose formalization we have called similarity logic. We have also proposed an extension of $PL$ to deal with vague propositions called fuzzy truth-valued logic that has been related to the previous two approaches. The main semantical links among these logics can be summarized in Figure 4. In the scheme of this figure, plain arrows stand for faithful semantical embeddings of logics, whereas the dashed arrow stands for the partial embedding of $PL$ into $FTL$, more concretely, of the $PL$ fragment of necessity-valued formulas into $FTL$. However, notice again that a faithful semantical embedding of $PL$ into $FTL$ can be obtained by composing the two plain arrows, according to the previous corollary. Due to the two possible paths from $PL$ to $FTL$ (only for necessity-valued formulas), it is interesting to remark that $\pi \models_{PL} (p, \Pi \alpha), \hat{\pi}_{S_n,E_n} \models_{FTL} (\neg p, \kappa_{0,1-\alpha})$ and $\pi \models_{PL} (p, \tau_{1-\alpha,1})$ are all of them equivalent expressions. In addition, the study contains some other relations from $SL$ to $PL$ and from $FTL$ to $PL$ or $SL$, but all of them are partial—they are not embeddings.
As concluding remarks and for future work it is interesting to point out the following:

First, in subsection 2.2.2., the main constructs of Ruspini's approach have been redefined from the point of view of fuzzy sets. Given a similarity model \((S, E)\) the consistency and implication degrees of a classical proposition \(p\) have been obtained as the supremum and infimum over the evidential set \(E\) of the membership function of \(\tilde{p}\), the "fuzzification" of the set of \(p\)-worlds \([p]\). In this approach consistency is a possibility measure in the sense of Dubois and Prade but implication is not a necessity measure. On the other hand, in [13] the notion of fuzzy rough set has been introduced as pair of lower and upper approximations of a original fuzzy set with respect to \(T\)-similarity relation. In this context, the fuzzification \(\tilde{p}\) of a classical proposition \(p\) is the upper approximation of the rough fuzzy set defined from \([p]\) with respect to the \(T\)-similarity relation \(S\). Therefore, there exists another possible "fuzzification" of \(p\) with respect to \(S\) given by the lower approximation \(p\) defined as \(\mu_p(w) = \text{Inf}_{w' \in E}(1 - S(w, w'))\). Then, analogous corresponding definitions of implication and consistency measures can be defined from \(p\), and it is easy to see that in this case implication is a necessity measure in the sense of Dubois-Prade but consistency is not a possibility measure. In this way, we could define a "dual" similarity logic where \(PL\) could also be embedded.
The second remark concerns to the inference mechanisms of \( PL, SL, \) and \( FTL. \) As we pointed out in subsection 3.4., resolution in \( PL \) can be translated in a dual form into \( SL. \) But it seems most interesting to study the notion of conditional proposition in both approaches. Ruspini gives as inference mechanism the generalized modus ponens, using conditional consistency and implication measures but his definitions do not coincide with the notion of conditional measures in \( PL. \) On the other hand, inference in \( FTL \) has not been addressed too. Therefore it remains for future work to further study relationships at inference level, as well as to identify the possible different kinds of conditional objects behind all these approaches, in the line studied in [2].

ANNEX A: ON T-SIMILARITY RELATIONS AND ITS GENERATORS

From a general point of view, the problem of distinguishability has been always considered as a basic notion to build any theory. Roughly speaking, the problem is to find a binary relation representing a general notion of indistinguishability between elements of a set of objects. It seems reasonable that this kind of relations should be reflexive and symmetric but not transitive as it was cleverly pointed out by Poincare. In the literature, different notions of nontransitive indistinguishability relations have been proposed. For example, Menger in [17] defined the notion of probabilistic relation that assigns to every pair of elements of a probabilistic space the probability of these elements to be indistinguishable. But it is in the frame of fuzzy relations where the notion of indistinguishability has been mainly developed. Zadeh initiated the studies of such domain defining the so-called similarity relations [18]. He proved that similarity relations are associated in a natural way to partition trees and, by duality, to ultrametrics as well. Ruspini, in the frame of clustering problems [14], defined a notion of likeness relation. But it was Trillas [19, 20] who gave a general definition of a \( T \)-indistinguishability relation related to a \( t \)-norm \( T, \) which contains as particular cases all previous definitions.

Given a \( t \)-norm \( T \) and a set \( X, \) a fuzzy relation \( S : X \times X \to [0, 1] \) is said to be a \( T \)-indistinguishability relation if the following conditions hold:

i) For every \( x \) of \( X, \) \( S(x, x) = 1 \) (Reflexivity)

ii) For every \( x, y \) of \( X, \) \( S(x, y) = S(y, x) \) (Symmetry)

iii) For every \( x, y, z \) of \( X, \) \( T(S(x, y), S(y, z)) \leq S(x, z) \) (\( T \)-Transitivity)

This definition contains all previous definitions as particular cases. The probabilistic relation of Menger is obtained taking the \( t \)-norm \( T \) as the
product operator, i.e. \( T(x, y) = xy \), the Zadeh's similarity relation is obtained taking \( T \) as the minimum operator, i.e. \( T(x, y) = \text{min}(x, y) \), and the Ruspini's likeness relation is obtained taking \( T \) as the t-norm of Lukasiewicz, i.e. \( T(x, y) = \text{max}(0, x + y - 1) \). For simplicity reasons, and following the denotation used by Ruspini in [3], from now on, all these kinds of relations will be referred to as \( T \)-similarity relations.

As usual, every \( T \)-similarity relation on a finite set is given by a matrix whose elements are the values \( S(x, y) \). As usual also, the t-norm \( T \) used in a specific application depends on the context of the application. In his study about the semantics of fuzzy logic, Ruspini uses the general definition of \( T \)-similarity relation and particularizes to the t-norm of Lukasiewicz in some examples, but all the results in Ruspini's and in this study are valid for any t-norm, unless explicit indication is made.

A procedure to obtain a \( T \)-similarity relation from a family of fuzzy sets is given by the Valverde's representation theorem [6], already introduced in section 3.2., but we repeat it here in order for this annex to be self-contained.

**Theorem** Let \( T \) be a continuous t-norm. A fuzzy relation \( S \) on a set \( X \) is a \( T \)-similarity relation if, and only if, there exists a family \( H = \{h_j\}_{j \in J} \) of fuzzy subsets of \( X \) such that

\[
S(x, y) = \text{Inf}_{j \in J} \left\{ \text{min} \left( R_T(h_j(x), h_j(y)), R_T(h_j(y), h_j(x)) \right) \right\}
\]

for all \( x, y \) of \( X \) and being \( R_T \) the residuated implication defined by \( T \).

In other words, this theorem says that the \( T \)-similarity relation generated by a family \( H \) of fuzzy sets is \( S = \text{Inf}_{j \in J} S_j(x, y) \), where \( S_j(x, y) = \text{Min}(R_T(h_j(x), h_j(y)), R_T(h_j(y), h_j(x))) \), that is, \( S \) is the infimum of the \( T \)-similarity relations \( S_j : X \times X \to [0, 1] \) generated by each fuzzy set \( h_j \) of \( H \). In the frame of clustering problems, this has been interpreted as follows. Let \( X \) be a set of objects and let \( A \) be a set of attributes such that for every attribute \( a \) of \( A \) there exists a fuzzy set \( h_a : X \to [0, 1] \) assigning to every element \( x \) of \( X \) the degree of applicability of the attribute \( a \) to the element \( x \). Then, for every attribute \( a \), the fuzzy set \( h_a \) generates a \( T \)-similarity relation \( S_a \), being \( S_a(x, y) \) the similarity degree between objects \( x \) and \( y \) with respect only to the attribute \( a \). Finally, the \( T \)-similarity relation \( S \) defined by \( S(x, y) = \text{Inf}_{a \in A} S_a(x, y) \) gives the similarity degrees between elements of \( X \) with respect to the whole set of attributes \( A \).

From the above representation theorem, a notion of what a generator of a \( T \)-similarity relation is has been introduced and studied in [15] and [21].
A function $h : X \rightarrow [0, 1]$ is said to be a generator of a T-similarity relation $S$ if the T-similarity relation $S_h$ generated by $h$ is greater or equal to $S$. Moreover, a family of functions $\{h_i \mid i \in I\}$ is said to be a generating family of $S$ if for every $x, y$ of $X$, $S(x, y) = \inf_{i \in I} S_i(x, y)$.

Obviously, if $F$ is a generating family of $S$, and $G$ is a family of generators of $S$, then $F \cup G$ is also a generating family of $S$. The following proposition characterizes the set of generators of a given T-similarity relation.

**Proposition** For every T-similarity relation $S$ on $X$, let $\Phi_S : [0, 1]^X \rightarrow [0, 1]^X$ defined by $\Phi_S(h)(x) = \sup_{y \in X} T(S(x, y), h(y))$. If $H_S$ denotes the set of generators of $S$, then $H_S = \Phi_S([0, 1]^X)$, that is, the mapping $h : X \rightarrow [0, 1]$ is a generator of $S$ if, and only if, there exists a function $f : X \rightarrow [0, 1]$ such that $\Phi_S(f) = h$.

The following examples are useful:

1) Any constant function is a generator of any T-similarity relation $S$.

**Proof** If $K$ is the constant mapping $K(x) = k$ for every $x$ of $X$, being $k$ a fixed number of $[0, 1]$, a simple computation shows that $\Phi_S(K) = K$, for every T-similarity relation $S$. The result is obvious as the T-similarity relation generated by $K$ is $S_K(x, y) = 1$ for every $x, y$ of $X$.

2) For every element $a$ of $X$, the function $h_a : X \rightarrow [0, 1]$ defined by $h_a(x) = S(a, x)$—the corresponding column of the matrix of $S$—is a generator of $S$.

**Proof** For every $a$ of $X$, let $f_a$ be the characteristic function of the singleton $\{a\}$, that is,$$
f_a(x) = \begin{cases} 1, \text{ if } x = a \\ 0, \text{ if } x \neq a \end{cases}$$A simple computation shows that $\Phi_S(f_a) = h_a$, and so $h_a$ is a generator of $S$. Hence, if $X$ is finite, every column of the matrix $S(x, y)$ is a generator of $S$.

3) For every subset $E$ of $X$ and every T-similarity relation $S$ on $X$, the possibility distribution $\pi_{S,E}$, defined as $\pi_{S,E}(x) = \sup_{y \in E} S(x, y)$, is a generator of $S$.

**Proof** Let $f_E$ be the characteristic function of the set $E$, that is,$$
f_E(x) = \begin{cases} 1, \text{ if } x \in E \\ 0, \text{ otherwise} \end{cases}$$As in the previous cases, it is easy to check that $\Phi_S(f_E) = \pi_{S,E}$ and hence $\pi_{S,E}$ is a generator of $S$.

T-similarity relations generated by one function, i.e., those that have a generating family with only one element, are said to be unidimensional and
play an important role in this work. A first interesting property of a unidimensional $T$-similarity $S$ is that if $f$ and $g$ are generators such that \{a | f(a) = 1\} = \{a | g(a) = 1\}$, then necessarily $f = g$. A characterization of unidimensional $T$-similarity relations such that $S(x, y) \neq 0$ for every $x$ and $y$, is given in [21] for archimedean $t$-norms. A $t$-norm $T$ is archimedean if $T(x, x) < x$ for any $x$ of the open unit interval $(0, 1)$.

**Theorem** If $T$ is an archimedean $t$-norm, a $T$-similarity relation $S$ on $X$, satisfying $S(x, y) \neq 0$ for every $x$ and $y$ of $X$, is unidimensional if, and only if, there exists a total order in $X$ ($\leq$) with first and last elements, denoted by $a$ and $b$ respectively, such that for every $x$, $y$, and $z$ of $X$ with $a \leq x < y < z < b$, then $T(S(x, y), S(y, z)) = S(x, z) > 0$.

For a better understanding of this proposition we need the notion of what a betweenness relation is. A ternary relation $B \subseteq X \times X \times X$ is a betweenness relation [22] if $(x, y, z) \in B$ implies that:

1. $x \neq y \neq z \neq x$
2. $(z, y, x) \in B$
3. $(y, z, x) \in B$ and $(z, x, y) \in B$
4. if $(x, z, t) \in B$ then $(x, y, t) \in B$ and $(y, z, t) \in B$

In such a case we say that $y$ is between $x$ and $z$. Two interesting examples of betweenness relations are the following ones. In these examples it is assumed $x \neq y \neq z \neq x$.

**A)** Given a mapping $h: X \to [0, 1]$, we can define the following betweenness relation:

$$y \text{ is between } x \text{ and } z \text{ if } x \leq_h y \leq_z z \text{ or } z \leq_h x \leq_y y,$$  \hspace{1cm} (a)

where $\leq_h$ is the total preorder in $X$ generated by $h$, i.e. $x \leq_h y$ iff $h(x) \leq h(y)$.

**B)** Given a $T$-similarity relation $S$ on $X$, where $T$ is archimedean, the following betweenness relation can be properly defined:

$$y \text{ is between } x \text{ and } z \text{ if } T(S(x, y), S(y, z)) = S(x, z) > 0$$  \hspace{1cm} (b)

Having in mind these examples, the above theorem says that, if $T$ is an archimedean $t$-norm, a $T$-similarity relation $S$ is unidimensional if, and only if, the betweenness relation (b) generated by $S$ and the betweenness relation (a) generated by any generator $h$ of $S$ are the same. Moreover, noticing that any pre-order and its dual generates the same betweenness relation, we can say that if $S$ is unidimensional, the total preorderings defined by any two different generators of $S$ must be either the same or one is the dual of the other.
It is worth noticing that this result is not true in general for non-archimedian \( T \)-norms, as the following example shows. Take \( T = \text{min} \), \( X = \{x, y, z\} \) and let \( h \) and \( g \) be the fuzzy sets on \( X \) defined by
\[
h(x) = g(x) = 1/3, \quad h(y) = g(z) = 1, \quad h(z) = g(y) = 1/2.
\]
A simple computation shows that the \( T \)-similarities generated by \( h \) and \( g \) are the same but the betweenness relations defined by the preorders associated to \( h \) and \( g \) are different. Moreover, the relation (b) defined above is not, in general, a betweenness relation when \( T \) is not archimedian. For instance, take the \( \text{min} \)-similarity relation \( S \) generated by a function \( h : X \to [0, 1] \) such that there exist \( x, y, \) and \( z \), three different elements of \( X \), with \( h(x) < h(y) < h(z) \). Then, it is easy to check that
\[
\text{Min}(S(x, y), S(y, z)) = S(x, z) \quad \text{and} \quad \text{Min}(S(x, z), S(z, y)) = S(x, y)
\]
This would mean that, on the one hand, \( y \) should be between \( x \) and \( z \), and the other hand \( z \) should be between \( x \) and \( y \), being \( y \neq z \). This situation prevents the relation (b) to be a betweenness relation in this particular case.

ANNEX B: CHARACTERIZATION OF CLOSED SIMILARITY MODELS

In section 3, the relation between similarity and possibilistic models has been studied. In subsection 3.3., the mapping \( F \) from \( SL \) to \( PL \) models and the mapping \( G \) from \( PL \) to \( SL \) models have been defined and proved that \( F \circ G = \text{Id}_S \) and \( G \circ F \geq \text{Id}_P \). The question is now to characterize the \( SL \) models \( (S, E) \) such that \( G \circ F(S, E) = (S, E) \), that is, to describe the set of the closed \( SL \) models by the closure operator \( G \circ F \). The following lemma is needed to prove a characterization theorem for closed similarity models which is the main result of this section.

**Lemma** For any \( T \)-norm \( T \), if \( x < y < z \), then \( R_T(R_T(z, y), R_T(z, x)) = R_T(y, x) \).

Proof Paalman de Miranda's representation theorem of \( T \)-norms [23] postulates that, given a \( T \)-norm \( T \), there exists a family of disjoint intervals \( \{(a_i, b_i) \mid i \in I\} \) and a corresponding family \( \{f_i \mid i \in I\} \) of continuous and strictly decreasing functions \( f_i : [a_i, b_i] \to [0, +\infty] \) with \( f_i(b_i) = 0 \) such that
\[
T(x, y) = \begin{cases} 
    f_i^{-1}(f_i(x) + f_i(y)), & \text{if } x, y \in [a_i, b_i] \text{ for some } i \in I \\
    \text{Min}(x, y), & \text{otherwise}
\end{cases}
\]
Here, $f_i^{-1}$ is the so-called pseudo-inverse of $f_i$ and defined as

$$f_i^{-1}(x) = \begin{cases} a_i, & \text{if } x < 0 \\ f_i^{-1}(x) = f_i^{-1}(x), & \text{if } x \in [0, f(a_i)] \\ f_i^{-1}(x) = b_i, & \text{if } x > f(a_i). \end{cases}$$

As a consequence, the corresponding residuated implication $R_T$ generated by $T$ can be expressed as

$$R_T(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ f_i^{-1}(f_i(y) - f_i(x)), & \text{if } x > y \text{ and } x, y \in [a_i, b_i] \\ y, & \text{otherwise}. \end{cases}$$

Taking this representation into account, four different cases have to be considered to prove the lemma:

1. $x, y,$ and $z$ belong to some interval $[a_i, b_i]$ of the family defined by $T$. In such a case, $R_T(R_T(z, y), R_T(z, x)) = f_i^{-1}(f_i(f_i^{-1}(f_i(x) - f_i(z)))) - f_i^{-1}(f_i(y) - f_i(z))) = f_i^{-1}(f_i(x) - f_i(y)) = R_T(y, x)$.

2. $z$ and $y$ belong to a same interval $[a_i, b_i]$ but $x$ does not. In this case, since $R_T(z, y) \geq y$, $R_T(z, y)$ and $x$ cannot belong to the same interval, and thus, $R_T(R_T(z, y), R_T(z, x)) = R_T(R_T(z, y), x) = x = R_T(y, x)$.

3. $x$ and $y$ belong to a same interval $[a_i, b_i]$ but $z$ does not. In this case, $R_T(z, y) = y$ and $R_T(z, x) = x$, and thus $R_T(R_T(z, y), R_T(z, x)) = R_T(y, x)$.

4. $x, y,$ and $z$ belong to three different intervals. In this last case, we have that $R_T(z, y) = y$, $R_T(z, x) = x$ and $R_T(y, x) = x$, and therefore $x = R_T(y, x) = R_T(R_T(z, y), R_T(z, x))$.

Therefore the lemma is proved.

**Theorem** Let $T$ be a t-norm and let $S$ be a $T$-similarity relation on a set of worlds $W$. A similarity model $(S, E)$ is closed if, and only if, $S$ is unidimensional and there exists a generator $h$ of $S$ such that $E = \{w \in W | h(w) = M\}$ being $M$ the maximum value of the image set $h(W)$.

**Proof** Suppose that $S$ is generated by a function $h : W \rightarrow [0, 1]$ and $E = \{w \in W | h(w) = M\}$, being $M = \text{Max}\{h(w) | w \in W\}$. In this case, the $T$-similarity relation $S$ is defined as

$$S(x, y) = \begin{cases} 1, & \text{if } h(x) = h(y) \\ R_T(h(x), h(y)), & \text{if } h(x) > h(y) \\ R_T(h(y), h(x)), & \text{if } h(x) < h(y). \end{cases}$$
On the other hand, given the similarity model \((S, E)\), the possibility distribution \(\pi_{S,E} = F(S, E)\) reduces to

\[
\pi_{S,E}(w) = \begin{cases} 
1, & \text{if } w \in E \\
R_T(M, h(w)), & \text{if } w \notin E.
\end{cases}
\]

Observe here that if \(M = 1\), it must be necessarily \(h = \pi_{S,E}\). Now, it is easy to prove that the \(T\)-similarity relation \(S_{\pi_{S,E}}\) generated by \(\pi_{S,E}\) is equal to \(S\):

- if \(h(w) = h(w')\) then \(\pi_{S,E}(w) = \pi_{S,E}(w')\) and so, \(S_{\pi_{S,E}}(w, w') = S(w, w') = 1\).
- if \(M > h(w) > h(w')\) then, applying the result of the previous lemma, it is evident that \(S_{\pi_{S,E}}(w, w') = R_T(\pi_{S,E}(w), \pi_{S,E}(w')) = R_T(R_T(M, h(w)), h(w')) = S(w, w')\).

Moreover, since \(R_T(x, y) < 1\) iff \(x > y\), it is easy to see that \(\pi_{S,E}(w) = 1\) iff \(w \in E\), that is, \(\text{Core}(\pi_{S,E}) = E\) and thus \(G(F(S, E)) = (S, E)\). Therefore \((S, E)\) is a closed similarity model.

The reciprocal of the theorem is evident because if \((S, E)\) is a closed similarity model, then \(G(\pi_{S,E}) = (S, E)\), and thus \(\pi_{S,E}\) generates \(S\), and \(E = \text{Core}(\pi_{S,E}) = \{w \mid \pi_{S,E}(w) = 1\}\). Therefore the theorem is proved.

**Corollary**  If \((S, E)\) is a closed similarity model, then \(\pi_{S,E}\) is the only normalized possibility distribution on \(W\) that generates \(S\).

For archimedian t-norms, taking into account the characterization theorem of unidimensional \(T\)-similarity relations (see Annex A), we can give a stronger characterization of closed similarity models than the above theorem.

**Theorem**  Let \(T\) be an archimedian t-norm and let \(S\) be a \(T\)-similarity relation on a set of worlds \(W\) satisfying \(S(w, w') > 0\), for every \(w, w' \in W\). A similarity model \((S, E)\) is closed if, and only if, \(S\) is unidimensional and for every generator \(h\) of \(S\) either \(E = \{w \in W \mid h(w) = M\}\) or \(E = \{w \in W \mid h(w) = m\}\), being \(M\) and \(m\) the maximum and minimum value of the image set \(h(W)\) respectively.

Finally, it is interesting to notice that, as a consequence of this theorem, if \(T\) is an archimedian t-norm, the betweenness relation generated by the total preference preorder given by a possibilistic model \(\pi\), is the same betweenness relation defined by the unidimensional \(T\)-similarity relation \(S_\pi\). In the other way round, closed similarity models are those pairs \((S, E)\), where \(S\) defines the same betweenness relation than the one defined by the preference preorder associated to \(\pi_{S,E}\) and \(E\) is the core of \(\pi_{S,E}\).
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References


