Classification of generalized multiresolution analyses ☆

Lawrence W. Baggett [a], Veronika Furst [b], Kathy D. Merrill [c,*],
Judith A. Packer [a]

[a] *Department of Mathematics, University of Colorado, Boulder, CO 80309, USA*
[b] *Department of Mathematics, Fort Lewis College, Durango, CO 81301, USA*
[c] *Department of Mathematics, Colorado College, Colorado Springs, CO 80903, USA*

Abstract

We discuss how generalized multiresolution analyses (GMRAs), both classical and those defined on abstract Hilbert spaces, can be classified by their multiplicity functions m and matrix-valued filter functions H. Given a natural number valued function m and a system of functions encoded in a matrix H satisfying certain conditions, a construction procedure is described that produces an abstract GMRA with multiplicity function m and filter system H. An equivalence relation on GMRAs is defined and described in terms of their associated pairs (m, H). This classification system is applied to MRAs and other classical examples in $L^2(\mathbb{R}^d)$ as well as to previously studied abstract examples.
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1. Introduction

A generalized multiresolution analysis (GMRA) is a Hilbert space structure traditionally associated with classical wavelets, that is, functions whose dilates of translates provide an orthonormal basis for $L^2(\mathbb{R}^d)$. Given a wavelet, the nested sequence of subspaces V_j that result

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from taking only dilation powers less than j are dense and have trivial intersection, with V_{j+1} the dilate of V_j, and with V_0 invariant under translation. Such a structure is called a GMRA [6], and was developed to understand such wavelets as the famous example given by Journé, whose V_0 space does not have an orthonormal basis given by translates of a single function called a scaling function. When V_0 has this stronger property, the nested sequence $\{V_j\}$ is called a multiresolution analysis (MRA) [22,23]. Both MRAs and GMRAs have been extensively exploited to produce and understand wavelets, which in turn have proven useful for applications such as image and signal processing.

While wavelets and multiresolution structures were first studied in the Hilbert space $L^2(\mathbb{R}^d)$, analogous definitions make sense in other Hilbert spaces that have appropriate dilation and translation operators. Dutkay and Jorgensen [16] pioneered the study of wavelets in function spaces on fractals, with later work by D'Andrea et al. [14]. Larsen, Raeburn and coworkers then showed that these and other interesting examples can be constructed via direct limits [19,4,5]. Dutkay et al. [9,17] constructed MRAs and super-wavelets in Hilbert spaces formed by direct sums of $L^2(\mathbb{R}^d)$ to orthonormalize examples such as the Cohen wavelet. Tensor products of known examples lead to more exotic specimens (see Section 5). Our purpose in this paper is to construct a set of classifying parameters for GMRAs in order to unify and allow comparison of all these disparate examples. We also provide an explicit construction of a canonical GMRA equivalent to each of them.

Accordingly, we will consider GMRA structures in an abstract Hilbert space $\mathcal{H}$, equipped with "translations" given by a unitary representation π of a countable abelian group Γ acting in $\mathcal{H}$, and a "dilation" given by a unitary operator δ.

We assume that these operators are related by

$$\delta^{-1}\pi_\gamma\delta = \pi_{\alpha(\gamma)} \tag{1}$$

for all $\gamma \in \Gamma$, where α is an isomorphism of Γ into itself such that the index of $\alpha(\Gamma)$ in Γ equals $N > 1$, and such that $\bigcap \alpha^n(\Gamma) = \{0\}$. These definitions generalize the classical case of ordinary translation by the integer lattice in $L^2(\mathbb{R}^d)$, given by $\pi_n f(x) = f(x - n)$, and dilation by an expansive integer matrix A, given by $\delta f(x) = \sqrt{|\det A|}\, f(Ax)$.

The structure of a GMRA, and thus the parameters that uniquely identify it, are revealed via Stone's Theorem on unitary representations of abelian groups. Using this theorem, we know that the representation π restricted to V_0 is completely determined by a measure μ on the dual group $\widehat{\Gamma}$ and a Borel multiplicity function $m : \widehat{\Gamma} \to \{0, 1, 2, \ldots, \infty\}$, which essentially describes how many times each character occurs in the decomposition of $\pi|_{V_0}$. There is a unitary equivalence J between the action of π on V_0 and multiplication by characters on $\bigoplus L^2(\sigma_i)$, where $\sigma_i = \{\omega : m(\omega) \geqslant i\}$. Because of this, we think of J as a partial alternative Fourier transform. For simplicity, in this paper we will restrict our attention to the commonly studied case where μ is Haar measure, and m is finite a.e.

The multiplicity function m is one of the parameters that determine a GMRA. As we will see in Section 4, the other parameter is a "filter" that shows how the operator J interacts with dilation. Classical filters were periodic functions h and g in $L^2(\mathbb{R}^d)$ that described inverse dilates of Fourier transforms of bases of V_1 in terms of those of V_0. Starting with an MRA in $L^2(\mathbb{R}^d)$, such functions could be shown to satisfy certain orthogonality relations. Mallat, Meyer and Daubechies [22,23,15] turned this process around by using functions h and g satisfying orthogonality together with additional low-pass and non-vanishing conditions to construct MRAs and wavelets. Lawton [20] and Bratteli and Jorgensen [12] were able to relax the non-vanishing

condition by allowing Parseval frames in place of orthonormal bases, and Baggett, Courter, Jorgensen, Merrill, Packer [1,3] generalized this work to the GMRA setting by replacing h and g by matrix-valued functions H and G. In [11], Bratteli and Jorgensen related filters h and g to Ruelle operators S_h and S_g, which satisfy relations similar to those of Cuntz operators, and can be used to represent inverse dilations. This work was extended to generalized filters in [3] and later [4].

In the next section, we recall the relationship between abstract GMRAs, multiplicity functions and generalized filters. In particular, we describe conditions on a multiplicity function m and a filter H that guarantee that they will produce a GMRA. It turns out that these conditions are considerably more relaxed in an abstract Hilbert space than in $L^2(\mathbb{R}^d)$. In Section 3 we describe a construction procedure that produces an abstract GMRA from any m and H meeting the required conditions. This construction gives an explicit realization of the abstract direct limit GMRAs built in [4]. While the procedure relies on first choosing a filter G complementary to H, we show in Section 4 that the equivalence between GMRAs does not depend on the choice of G. Thus, the classifying set described there depends only on the pair m and H. In this section, we also give a necessary and sufficient condition on the equivalence class of a filter so that S_H is a pure isometry and thus associated with a GMRA, in the case of a finite multiplicity function. We conclude in Section 5 with a variety of examples that illustrate our main theorems, including an example of a GMRA where the translation group is not isomorphic to $\mathbb{Z}^d$.

2. GMRAs, multiplicity functions and filters

Let $\mathcal{H}$ be an abstract, separable Hilbert space, equipped with operators π_γ and δ satisfying Eq. (1).

Definition 1. A collection $\{V_j\}_{-\infty}^\infty$ of closed subspaces of $\mathcal{H}$ is called a **generalized multiresolution analysis** (GMRA) relative to π and δ if

(1) $V_j \subseteq V_{j+1}$ for all j.
(2) $V_{j+1} = \delta(V_j)$ for all j.
(3) $\bigcap V_j = \{0\}$, and $\bigcup V_j$ is dense in $\mathcal{H}$.
(4) V_0 is invariant under the representation π.

The subspace V_0 is called the **core subspace** of the GMRA $\{V_j\}$.

Let $\{V_j\}$ be a GMRA in a Hilbert space $\mathcal{H}$. For each j, write W_j for the orthogonal complement to V_j in V_{j+1}. It follows that $\mathcal{H} = \bigoplus_{j=-\infty}^\infty W_j$. Also, for each $j \geqslant 0$, W_j is an invariant subspace for the representation π. We apply Stone's Theorem on unitary representations of abelian groups to the subrepresentations of π acting in V_0 and W_0. Accordingly, there exists a finite, Borel measure μ (unique up to equivalence of measures) on $\widehat{\Gamma}$, Borel subsets $\sigma_1 \supseteq \sigma_2 \supseteq \cdots \widehat{\Gamma}$ (unique up to sets of μ measure 0), and a (not necessarily unique) unitary operator $J : V_0 \to \bigoplus_i L^2(\sigma_i, \mu)$ satisfying

$$\big[J\big(\pi_\gamma(f)\big)\big](\omega) = \omega(\gamma)\big[J(f)\big](\omega)$$

for all $\gamma \in \Gamma$, all $f \in V_0$, and μ almost all $\omega \in \widehat{\Gamma}$. We write m for the function on $\widehat{\Gamma}$ given by $m(\omega) = \sum_i \chi_{\sigma_i}(\omega)$, and call it the **multiplicity function** associated to the representation $\pi|_{V_0}$. The GMRA $\{V_j\}$ is an MRA if and only if $m \equiv 1$.

Analogously, there exists a finite, Borel measure $\widetilde{\mu}$, Borel subsets $\widetilde{\sigma}_k$, and an operator $\widetilde{J} : W_0 \to \bigoplus_k L^2(\widetilde{\sigma}_k, \widetilde{\mu})$ satisfying

$$\left[\widetilde{J}(\pi_\gamma(f))\right](\omega) = \omega(\gamma)\left[\widetilde{J}(f)\right](\omega)$$

for all $\gamma \in \Gamma$, $f \in W_0$, and $\widetilde{\mu}$ almost all ω. We write $\widetilde{m}$ for the function on $\widehat{\Gamma}$ given by $\widetilde{m}(\omega) = \sum_k \chi_{\widetilde{\sigma}_k}(\omega)$, and call it the multiplicity function associated to the representation $\pi|_{W_0}$.

In this paper, we will assume that the measures μ and $\widetilde{\mu}$ are absolutely continuous with respect to Haar measure, and thus take μ and $\widetilde{\mu}$ to be the restrictions of Haar measure to the subsets σ_1 and $\widetilde{\sigma}_1$, respectively. We also assume that the multiplicity function m associated to the representation $\pi|_{V_0}$ is finite almost everywhere.

Let α^* be the dual endomorphism of $\widehat{\Gamma}$ onto itself defined by $[\alpha^*(\omega)](\gamma) = \omega(\alpha(\gamma))$, and note that the kernel of α^* contains exactly N elements and that α^* is ergodic with respect to the Haar measure μ on $\widehat{\Gamma}$. Indeed, suppose that for some $\gamma \in \Gamma$ and some $n > 0$, we have $\gamma \circ (\alpha^*)^n = \gamma$. Since the characters of Γ separate the points of Γ, it follows that $\alpha^n \gamma = \gamma$, and hence $\gamma \in \bigcap \alpha^n \Gamma = \{0\}$. That is, the trivial character is the only $\gamma \in \Gamma$ that satisfies $\gamma \circ (\alpha^*)^n = \gamma$ for some $n > 0$, and it follows that any function $f \in L^2(\widehat{\Gamma})$ that satisfies $f \circ \alpha^* = f$ must be a constant function. The last statement is equivalent to the definition of the ergodicity of α^*.

Using α^* to relate the representations $\pi|_{V_1}$ and $\pi|_{V_0}$, it is shown in [6] and more generally in [4] that multiplicity functions for a GMRA must satisfy the following consistency equation:

$$m(\omega) + \widetilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta). \tag{2}$$

It follows, since the function m is finite a.e., that the sets σ_i and $\widetilde{\sigma}_k$ are completely determined by the multiplicity function m. It also follows that a multiplicity function m associated with a GMRA must satisfy the consistency inequality:

$$m(\omega) \leqslant \sum_{\alpha^*(\zeta)=\omega} m(\zeta). \tag{3}$$

We will see in the next section that the consistency inequality is a sufficient as well as necessary condition for a function $m : \widehat{\Gamma} \to \{0, 1, 2, \ldots\}$ to be a multiplicity function associated to an abstract GMRA. Accordingly, we make the following definition.

Definition 2. A **multiplicity function** is a Borel function $m : \widehat{\Gamma} \to \{0, 1, 2, \ldots\}$ that satisfies the consistency inequality (3).

In contrast, Bownik, Rzeszotnik and Speegle [10] and Baggett and Merrill [7] showed that an additional technical condition related to dilates of the translates of the support of m is required for m to be a multiplicity function for a GMRA in $L^2(\mathbb{R}^d)$. We will need the following observation about multiplicity functions.

Proposition 3. *Suppose* $m : \widehat{\Gamma} \to \{0, 1, 2, \ldots\}$ *satisfies the consistency inequality. If* m *is not identically* 0*, then there exists a set* F *of positive measure in* $\widehat{\Gamma}$ *such that*

$$m(\omega) < \sum_{\alpha^*(\zeta)=\omega} m(\zeta)$$

for all $\omega \in F$*. That is, the consistency inequality is a strict inequality on a set of positive measure.*

Proof. Suppose

$$m(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta)$$

for almost all $\omega \in \widehat{\Gamma}$. It follows directly by induction that

$$m(\omega) = \sum_{\alpha^{*n}(\zeta)=\omega} m(\zeta)$$

for almost all ω. Let k be a positive integer for which there exists a set $E \subseteq \widehat{\Gamma}$ of positive measure such that $m(\omega) \leqslant k$ for all $\omega \in E$. Choose n such that $N^n > k$. Then, for almost every $\omega \in \alpha^{*-n}(E)$, we have

$$\sum_{\zeta \in \ker(\alpha^{*n})} m(\omega\zeta) = m\big(\alpha^{*n}(\omega)\big) \leqslant k,$$

implying that there exists some $\zeta \in \ker(\alpha^{*n})$, and a subset $E' \subseteq \alpha^{*-n}(E)$ of positive measure, such that $m(\omega\zeta) = 0$ for all $\omega \in E'$. Hence, $m(\omega) = 0$ on a set F of positive measure. But, from the equation

$$m\big(\alpha^*(\omega)\big) = \sum_{\alpha^*(\zeta)=1} m(\omega\zeta),$$

it follows that the sequence $\{m(\alpha^{*n}(\omega))\}$ is nondecreasing. Because α^* is ergodic, we must have that the sequence $\{\alpha^{*n}(\omega)\}$ intersects the set F infinitely often for almost all ω. Hence $m(\omega) = 0$ a.e. $\quad\square$

The other ingredients we will need for our GMRA construction are filters, which are defined in terms of a multiplicity function m as follows:

Definition 4. Let m be a multiplicity function, and write $\sigma_i = \{\omega\colon m(\omega) \geqslant i\}$. Set

$$\widetilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta) - m(\omega),$$

and set $\widetilde{\sigma}_k = \{\omega\colon \widetilde{m}(\omega) \geqslant k\}$. Let $H = [h_{i,j}]$ and $G = [g_{k,j}]$ be (possibly infinite) matrices of Borel, complex-valued functions on $\widehat{\Gamma}$ such that for every j, $h_{i,j}$ and $g_{k,j}$ are supported in σ_j. Suppose further that H and G satisfy the following "filter equations":

$$\sum_{\alpha^*(\zeta)=\omega} \sum_{j} h_{i,j}(\zeta)\overline{h_{i',j}(\zeta)} = N\delta_{i,i'}\chi_{\sigma_i}(\omega), \tag{4}$$

$$\sum_{\alpha^*(\zeta)=\omega} \sum_{j} g_{k,j}(\zeta)\overline{g_{k',j}(\zeta)} = N\delta_{k,k'}\chi_{\widetilde{\sigma}_k}(\omega), \tag{5}$$

and

$$\sum_{\alpha^*(\zeta)=\omega} \sum_{j} g_{k,j}(\zeta)\overline{h_{i,j}(\zeta)} = 0. \tag{6}$$

Then H is called a **filter** relative to m and α^*, and G is called a **complementary filter** to H.

We note that it will sometimes be useful to consider filters and complementary filters to be matrix valued functions on $\widehat{\Gamma}$ rather than a matrix of complex valued functions. It is then a consequence of the definition above that the nonzero portion of the matrix $H(\omega)$ is contained in the upper left block of dimensions $m(\alpha^*(\omega)) \times m(\omega)$, while the nonzero portion of $G(\omega)$ is contained in the upper left block of dimensions $\widetilde{m}(\alpha^*(\omega)) \times m(\omega)$.

Given a filter H relative to m and α^*, we may define a "Ruelle" operator S_H on $\bigoplus_i L^2(\sigma_i)$ by

$$\big[S_H(f)\big](\omega) = H^t(\omega)f\big(\alpha^*(\omega)\big).$$

Similarly, a complementary filter G defines a Ruelle operator S_G from $\bigoplus_i L^2(\widetilde{\sigma}_i)$ to $\bigoplus_i L^2(\sigma_i)$ by

$$\big[S_G(f)\big](\omega) = G^t(\omega)f\big(\alpha^*(\omega)\big).$$

The filter equations satisfied by H and G translate to the following Cuntz-like conditions for the Ruelle operators (see [3,4]):

Lemma 5. *If H is a filter relative to m and α^*, and G is a complementary filter to H, then the Ruelle operators they define satisfy*

(1) $S_H^* S_H = I$, $S_G^* S_G = \widetilde{I}$,
(2) $S_H^* S_G = 0$, *and*
(3) $S_H S_H^* + S_G S_G^* = I$,

where I is the identity operator on $\bigoplus_i L^2(\sigma_i, \mu)$ and $\widetilde{I}$ is the identity operator on $\bigoplus_k L^2(\widetilde{\sigma}_k, \widetilde{\mu})$.

Filters, like multiplicity functions, arise naturally out of GMRAs. Let $\{V_j\}$ be a GMRA (with finite multiplicity function and associated measure absolutely continuous with respect to Haar), and let μ, $\{\sigma_i\}$, J, $\widetilde{\mu}$, $\{\widetilde{\sigma}_k\}$, and $\widetilde{J}$ be as in the Stone's Theorem discussion above. Write C_i for the element of the direct sum space $\bigoplus_j L^2(\sigma_j, \mu)$ whose ith coordinate is χ_{σ_i} and whose other coordinates are 0, and $\widetilde{C}_k$ for the element in $\bigoplus_l L^2(\widetilde{\sigma}_l, \widetilde{\mu})$ whose kth coordinate is $\chi_{\widetilde{\sigma}_k}$ and whose other coordinates are 0. Let $\bigoplus_j h_{i,j}$ be the element $J(\delta^{-1}(J^{-1}(C_i)))$ and $\bigoplus_j g_{k,j}$ be the element $J(\delta^{-1}(\widetilde{J}^{-1}(\widetilde{C}_k)))$, both in $\bigoplus_j L^2(\sigma_j, \mu)$. It was shown in [2] that the matrix $H = [h_{i,j}]$

is then a filter relative to m and α^*, and the matrix $G = [g_{k,j}]$ is a complementary filter to H. We call these **filters constructed from a GMRA**, and note that they are not unique, but rather depend on the choice of the maps J and $\widetilde{J}$. The operators $J \circ \delta^{-1} \circ J^{-1}$ and $J \circ \delta^{-1} \circ \widetilde{J}^{-1}$ are the corresponding Ruelle operators S_H and S_G respectively. It follows directly from their definitions that S_H and S_G are isometries, and the GMRA requirement that $\bigcap V_j = \{0\}$ implies that $S_H = J \circ \delta^{-1} \circ J^{-1}$ is a pure isometry.

Just as with multiplicity functions, this necessary condition on a filter to be associated with a GMRA turns out to be sufficient as well. In Theorem 5 of [4], it is shown that if S_H is a pure isometry on a Hilbert space $\bigoplus L^2(\sigma_i)$, then it is possible to construct a generalized multiresolution analysis via a direct limit process. Our construction in the next section will give a concrete realization under the same hypotheses. Again, as with multiplicity functions, we see that this necessary and sufficient condition on the filter H is much weaker than what is required for a filter to be associated with a GMRA in $L^2(\mathbb{R}^d)$. For example, in that context, the "refinement equation",

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{|\det A|}} H(A^{t^{-1}}\omega)\widehat{\phi}(A^{t^{-1}}\omega), \tag{7}$$

suggests some sort of convergence of the infinite product $\prod_{j=1}^{\infty} \frac{1}{\sqrt{|\det A|}} H((A^t)^{-j}\omega)$, which in turn requires that the filter H satisfies some low-pass condition of being close to $\sqrt{|\det A|}$ times a partial identity near the origin [3,4]. Theorems from [2,5] indicate that in the abstract setting, a much weaker condition is sufficient to guarantee that S_H is a pure isometry. In the case where the matrix H is 1×1, the simple condition that $|H(\omega)| \neq 1$ on a set of positive measure is sufficient to show that S_H is a pure isometry [11,5]. In particular, filters traditionally labeled "high-pass" can be used as H. Proposition 19 in Section 4 of this paper gives a new, more general result of this type.

3. Explicit construction of GMRAs on abstract Hilbert spaces

Let m be a multiplicity function on $\widehat{\Gamma}$, as in Definition 2 and let H be a filter relative to m and α^*. Using Proposition 3, define

$$\widetilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta) - m(\omega), \tag{8}$$

and define the sets $\{\sigma_i\}$ and $\{\widetilde{\sigma}_k\}$ as in the preceding section. As is shown in [1], given a filter H relative to m and α^*, there always exists a complementary filter G. For the purposes of this construction, let G be any filter complementary to H.

Because a J map guaranteed by Stone's Theorem takes the core subspace V_0 of any GMRA to $\bigoplus L^2(\sigma_i)$, this direct sum of L^2 spaces is a natural candidate for the core subspace of an abstract GMRA built out of a multiplicity function and a filter. The group Γ acts on this space in a natural way via multiplication by characters. Similarly, the space $\bigoplus L^2(\widetilde{\sigma}_k)$ is an obvious candidate for the abstract $W_0 = V_1 \ominus V_0$, and the relationships $J \circ \delta^{-1} \circ J^{-1} = S_H$ and $J \circ \delta^{-1} \circ \widetilde{J}^{-1} = S_G$ suggest that the Ruelle operators S_H and S_G provide natural abstract inverse dilations on these spaces. Thus the main task remaining in building a GMRA given m and H is to describe positive dilates of W_0; such subspaces could then be used to fill out the rest of the Hilbert space.

If the group $\Gamma = \mathbb{Z}^d$, then embedding $\widehat{\Gamma} = \mathbb{T}^d$ as $[-\frac{1}{2}, \frac{1}{2}]^d$ in $\mathbb{R}^d$ provides us with a simple candidate for the dilate of our constructed $W_0 = \bigoplus L^2(\widetilde{\sigma}_k)$. Since in this case, α is an isomorphism of $\mathbb{Z}^d$, we must have $\alpha(n) = An$, for a matrix A. We define positive dilations $\mathcal{D}^j : \bigoplus L^2(\widetilde{\sigma}_k) \to \bigoplus L^2(A^t \widetilde{\sigma}_k)$ by

$$\mathcal{D}^j \left(\bigoplus_k f_k(\omega) \right) = \bigoplus_k \frac{1}{\sqrt{|\det A|}^j} f_k\big((A^t)^{-j}\omega\big).$$

For more general Γ, we will use an abstract construction to define the positive dilation $\mathcal{D}$ in terms of a cross section for the map α^*. Just as in the case of $\Gamma = \mathbb{Z}^d$, our dilated space will be a direct sum of L^2 spaces such that the map $f \to \sqrt{N} f \circ \alpha^*$ determines an isometry of the dilated space onto the original one.

Let c be a regular Borel section for the map α^*; i.e., c is a Borel map from $\widehat{\Gamma} \equiv \alpha^*(\widehat{\Gamma}) \equiv \widehat{\Gamma}/\ker(\alpha^*)$ into $\widehat{\Gamma}$, for which $\alpha^*(c(\omega)) = \omega$ for all $\omega \in \widehat{\Gamma}$. (See, e.g. [21, Lemma 1.1].) Define $\tau : \widehat{\Gamma} \to \ker(\alpha^*)$ by

$$\tau(\omega) = c\big(\alpha^*(\omega)\big)\omega^{-1}.$$

For example, in the simple case associated with dilation by 2 in $L^2(\mathbb{R})$, where $\widehat{\Gamma} = \mathbb{T} \equiv [-\frac{1}{2}, \frac{1}{2})$, with $\alpha^*(\omega) = 2\omega$, we can take $c(\omega) = \frac{\omega}{2}$. Thus, here

$$\tau(\omega) = \begin{cases} 0 & \text{if } \omega \in [-\frac{1}{4}, \frac{1}{4}), \\ \frac{1}{2} & \text{otherwise.} \end{cases} \qquad (9)$$

Now, let ν be a finite Borel measure on $\widehat{\Gamma}$. Let E be a Borel subset of $\widehat{\Gamma}$, let ζ be an element of the kernel of α^*, and set

$$E_\zeta = \big\{ \omega \in E \colon \tau(\omega) = \zeta \big\}.$$

Proposition 6. *The set E is the disjoint union $\bigcup_\zeta E_\zeta$, and α^* is 1–1 on each E_ζ into $\widehat{\Gamma}$.*

Proof. The first statement is clear.

For $\omega \in E_\zeta$ we have

$$c\big(\alpha^*(\omega)\big) = \tau(\omega)\omega = \zeta\omega,$$

which shows that α^* must be 1–1 on E_ζ. $\quad\square$

Now let $E_1, E_2, \dots$ be a (countable) collection of Borel subsets of $\widehat{\Gamma}$. For each i let ν_i be the restriction to E_i of the measure ν. Write $E_{i,\zeta}$ for $[E_i]_\zeta$, and let $\nu_{i,\zeta}$ be the restriction of ν_i to the subset $E_{i,\zeta}$ of E_i. Write $\mathcal{K} = \bigoplus_i L^2(E_i, \nu_i)$. For each $\zeta \in \ker(\alpha^*)$, define $E'_{i,\zeta} = \alpha^*(E_{i,\zeta})$, and set $\nu'_{i,\zeta}$ equal to the measure $N\alpha^*_*(\nu_{i,\zeta})$ that is defined on $E'_{i,\zeta}$ by

$$\nu'_{i,\zeta}(F) = N\nu_{i,\zeta}\big(\alpha^{*-1}(F)\big).$$

For example, in the simple setting described by Eq. (9), if we take $E_1 = [-\frac{3}{8}, \frac{3}{8})$ and $E_2 = [\frac{1}{8}, \frac{1}{2})$, then

$$E_{1,0} = \left[-\frac{1}{4}, \frac{1}{4}\right), \qquad E_{1,\frac{1}{2}} = \left[-\frac{3}{8}, -\frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right),$$

$$E_{2,0} = \left[\frac{1}{8}, \frac{1}{4}\right), \qquad E_{2,\frac{1}{2}} = \left[\frac{1}{4}, \frac{1}{2}\right), \qquad (10)$$

so that

$$E'_{1,0} = \left[-\frac{1}{2}, \frac{1}{2}\right), \qquad E'_{1,\frac{1}{2}} = \left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right),$$

$$E'_{2,0} = \left[\frac{1}{4}, \frac{1}{2}\right), \qquad E'_{2,\frac{1}{2}} = \left[-\frac{1}{2}, 0\right).$$

The measure $v'_{i,\zeta}$ here is just Haar measure restricted to $E'_{i,\zeta}$.

Using our newly defined sets and measures, we let

$$\mathcal{K}' = \bigoplus_{i,\zeta} L^2\left(E'_{i,\zeta}, v'_{i,\zeta}\right).$$

Proposition 7. *For each $f \in \mathcal{K}$, set $\mathcal{D}(f)$ equal to the element of $\mathcal{K}'$ given by*

$$\left[\mathcal{D}(f)\right]_{i,\zeta}(\omega) = \frac{1}{\sqrt{N}} f_i\left(\zeta^{-1}c(\omega)\right).$$

Then the operator $\mathcal{D}$ is an isometry of $\mathcal{K}$ onto $\mathcal{K}'$.

Proof.

$$\left\|\mathcal{D}(f)\right\|^2 = \sum_i \sum_\zeta \int_{E'_{i,\zeta}} \left|\left[\mathcal{D}(f)\right]_{i,\zeta}(\omega)\right|^2 dv'_{i,\zeta}(\omega)$$

$$= \frac{1}{N} \sum_i \sum_\zeta \int_{E'_{i,\zeta}} \left|f_i\left(\zeta^{-1}c(\omega)\right)\right|^2 dv'_{i,\zeta}(\omega)$$

$$= \sum_i \sum_\zeta \int_{E_{i,\zeta}} \left|f_i\left(\zeta^{-1}c\left(\alpha^*(\eta)\right)\right)\right|^2 dv_{i,\zeta}(\eta)$$

$$= \sum_i \sum_\zeta \int_{E_{i,\zeta}} \left|f_i\left(\zeta^{-1}\tau(\eta)\eta\right)\right|^2 dv_{i,\zeta}(\eta)$$

$$= \sum_i \sum_\zeta \int_{E_{i,\zeta}} \left|f_i(\eta)\right|^2 dv_{i,\zeta}(\eta)$$

$$= \sum_i \int_{E_i} \bigl|f_i(\eta)\bigr|^2 \, d\nu_i(\eta)$$

$$= \|f\|^2,$$

where the second to last step is justified because, for $\eta \in E_{i,\zeta}$, we have $\tau(\eta) = \zeta$. Thus, $\mathcal{D}$ is an isometry.

To see that $\mathcal{D}$ is onto $\mathcal{K}'$, it suffices to note that the inverse of $\mathcal{D}$ is given by

$$\bigl[\mathcal{D}^{-1}(f)\bigr]_i(\omega) = \sqrt{N}\, f_{i,\tau(\omega)}\bigl(\alpha^*(\omega)\bigr). \qquad \square$$

We will refer to the space $\mathcal{K}' = \mathcal{D}(\mathcal{K})$ as a *dilation by α^** of $\mathcal{K}$. Note that this general definition of $\mathcal{D}$ is consistent with the definition given at the beginning of this section for the special case of $\Gamma = \mathbb{Z}^d$. For example, in the situation defined by Eq. (10), we can see this equivalence by using integer translation in $\mathbb{R}$ to identify $L^2([-\frac{1}{2}, \frac{1}{2})) \oplus L^2([-\frac{1}{2}, -\frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2}))$ with $L^2([-\frac{3}{4}, \frac{3}{4}))$ and $L^2([\frac{1}{4}, \frac{1}{2})) \oplus L^2([-\frac{1}{2}, 0))$ with $L^2([\frac{1}{4}, 1))$.

We are now ready to construct explicitly a GMRA from the parameters m, H, and G.

Theorem 8. *Suppose $m : \widehat{\Gamma} \to \{0, 1, 2, \ldots\}$ is a Borel function that satisfies the consistency inequality, and that $H = [h_{i,j}]$ is a filter relative to m and α^* such that the Ruelle operator S_H is a pure isometry on $\bigoplus_i L^2(\sigma_i)$. Let $\widetilde{m}$ be defined from m by the consistency equation (as in Eq. (8)), and let G be a complementary filter to H. Define $\mathcal{V}_0 = \bigoplus_i L^2(\sigma_i)$ and $\mathcal{W}_0 = \bigoplus_k L^2(\widetilde{\sigma}_k)$. For $n \geqslant 1$, inductively set $\mathcal{W}_n = \mathcal{D}(\mathcal{W}_{n-1})$, and set $\mathcal{H} = \mathcal{V}_0 \oplus \bigoplus_{n=0}^{\infty} \mathcal{W}_n$. Define a representation π of Γ, acting in $\mathcal{H}$, by*

$$\bigl[\pi_\gamma(f)\bigr](\omega) = \omega(\gamma) f(\omega).$$

Finally, define an operator T on $\mathcal{H}$ by

$$\bigl[T(f)\bigr]_a = \begin{cases} S_H(f_{\mathcal{V}_0}) + S_G(f_{\mathcal{W}_0}), & a = \mathcal{V}_0, \\ \mathcal{D}^{-1}(f_{\mathcal{W}_{n+1}}), & a = \mathcal{W}_n,\ n \geqslant 0, \end{cases} \tag{11}$$

where we represent an element f of $\mathcal{H}$ by $\{f_{\mathcal{V}_0}, f_{\mathcal{W}_0}, f_{\mathcal{W}_1}, \ldots\}$. Then

(1) *T is a unitary operator on $\mathcal{H}$.*
(2) *$T \pi_\gamma T^{-1} = \pi_{\alpha(\gamma)}$ for all $\gamma \in \Gamma$.*
(3) *If $\mathcal{V}_j$ is defined to be $T^{-j}(\mathcal{V}_0)$, then the collection $\{\mathcal{V}_j\}$ is a GMRA relative to π and δ, where $\delta = T^{-1}$.*
(4) *The multiplicity function associated to the core subspace $\mathcal{V}_0$ is the given function m, and the given H is a filter constructed from the GMRA $\{\mathcal{V}_j\}$.*

Proof. To prove the first claim, note that by Proposition 7, $\mathcal{D}^{-1}$ is an isometry from $\mathcal{W}_{n+1}$ onto $\mathcal{W}_n$. By Lemma 5, we also have that the definition of T above gives an isometry from $\mathcal{V}_0 \oplus \mathcal{W}_0$ onto $\mathcal{V}_0$. The second claim follows immediately from the definitions, since the operators S_H, S_G and $\mathcal{D}^{-1}$ all change the argument of the function from ω to $\alpha^*(\omega)$.

Next, we show that the collection $\{\mathcal{V}_j\}$ is a GMRA. The fact that $\mathcal{V}_j \subseteq \mathcal{V}_{j+1}$ follows from $T(\mathcal{V}_0) \subset \mathcal{V}_0$, which is immediate from the definition of T. That $\mathcal{V}_{j+1} = \delta(\mathcal{V}_j)$ follows immediately from the definition of $\delta = T^{-1}$. The trivial intersection property follows from our assumption that S_H is a pure isometry, and the dense union from the fact noted in the previous paragraph that $T^{-1}\mathcal{V}_0 = \mathcal{V}_0 \oplus \mathcal{W}_0$ and $T^{-1}\mathcal{W}_n = \mathcal{W}_{n+1}$.

As a component of the direct sum space, $\mathcal{V}_0$ is clearly invariant under the multiplication operators $\omega(\gamma)$ that define the representation π. The given function m is clearly the multiplicity function of that representation. To establish that H is a corresponding filter, we note that we can take J to be the identity for this $\mathcal{V}_0$, and calculate

$$
\begin{aligned}
J\delta^{-1}J^{-1}(C_i)(\omega) &= T(C_i)(\omega) \\
&= S_H(C_i)(\omega) \\
&= \bigoplus_j h_{i,j}(\omega)\chi_{\sigma_i}\big(\alpha^*(\omega)\big) \\
&= \bigoplus_j h_{i,j}(\omega),
\end{aligned}
$$

where the last equality follows from the fact that by the filter equation, $h_{i,j}$ is supported on $\alpha^{*-1}(\sigma_i)$. $\quad\square$

Remark 9. We will denote the GMRA $\{\mathcal{V}_j\}$ constructed above by $\{V_j^{m,H,G}\}$ and refer to it as the **canonical GMRA** having these parameters.

In Section 5 we will construct canonical GMRAs related to classical examples, as well as new ones. First, we establish in the next section conditions under which two GMRAs are the same. While our construction procedure requires the choice of a complementary filter G, we will see that the equivalence classes depend only on the two parameters m and H.

4. A classifying set for GMRAs

Let $\{V_j\}$ be a GMRA in a Hilbert space $\mathcal{H}$, relative to a representation π of Γ and a unitary operator δ, and let $\{V_j'\}$ be a GMRA in a Hilbert space $\mathcal{H}'$, relative to a representation π' of Γ and a unitary operator δ'.

Definition 10. We say that the GMRAs $\{V_j\}$ and $\{V_j'\}$ are **equivalent** if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ that satisfies:

(1) $U(V_j) = V_j'$ for all j.
(2) $U \circ \pi_\gamma = \pi_\gamma' \circ U$ for all $\gamma \in \Gamma$.
(3) $U \circ \delta = \delta' \circ U$.

For classical examples in $L^2(\mathbb{R}^d)$, the Fourier transform $\mathcal{F}$ gives an equivalence between any GMRA $\{V_j\}$ and $\{\widehat{V}_j\}$. Further, if an operator U gives an equivalence between $\{V_j\}$ and $\{V_j'\}$, two GMRAs for dilation by A and translation by $\mathbb{Z}^d$ in $L^2(\mathbb{R}^d)$, then $\widehat{U} = \mathcal{F} \circ U \circ \mathcal{F}^{-1}$ is multiplication by a function u with absolute value 1, and such that $u(A^{*j}\omega) = u(\omega)$ for all

integers j [8]. Thus equivalence between GMRAs for the same dilation in $L^2(\mathbb{R}^d)$ generalizes the notion of different MSF wavelets attached to the same wavelet set.

Recall that we consider only GMRAs with a finite multiplicity function m and with the associated measure μ absolutely continuous with respect to Haar measure. Our first aim is to prove that every such GMRA is equivalent to one of the canonical GMRAs constructed in the preceding section. We will then describe the equivalence relation among these GMRAs in terms of the parameters m, H and G. We will need the following lemma.

Lemma 11. *The GMRAs $\{V_j\}$ and $\{V_j'\}$ are equivalent if and only if there exists a unitary operator P mapping V_0 onto V_0' that satisfies*:

(1) $P \circ \pi_\gamma = \pi_\gamma' \circ P$ *for all $\gamma \in \Gamma$.*
(2) $P \circ \delta^{-1} = \delta'^{-1} \circ P$.

Proof. We first assume that the conditions above are satisfied and show that $\{V_j\}$ and $\{V_j'\}$ are equivalent. For each $n \geqslant 0$, define an operator $Q_n : W_n \to W_n'$ by

$$Q_n = \delta'^{n+1} \circ P \circ \delta^{-(n+1)}.$$

Now, define $U = P \oplus \bigoplus_{n=0}^\infty Q_n$ on $\mathcal{H} = V_0 \oplus \bigoplus_{n=0}^\infty W_n$. One checks directly that U satisfies the required conditions.

For the converse, assume that $\{V_j\}$ and $\{V_j'\}$ are equivalent, with $U : \mathcal{H} \to \mathcal{H}'$ implementing the equivalence. Define $P = U|_{V_0}$. By the definition of equivalence, P maps V_0 to V_0', and conditions (1) and (2) follow. $\quad\square$

Theorem 12. *Let $\{V_j\}$ be a GMRA. Let m be its (finite) associated multiplicity function, and let $H = [h_{i,j}]$ be a filter constructed from the GMRA using the map J. Let G be a complementary filter to H. Then the GMRA $\{V_j\}$ is equivalent to the canonical GMRA $\{V_j^{m,H,G}\}$.*

Proof. Define $P : V_0 \to \bigoplus L^2(\sigma_i)$ by $P = J$. Condition (1) of Lemma 11 follows immediately. The fact that $J \circ \delta^{-1} \circ J^{-1} = S_H$ proves the second condition of that lemma. $\quad\square$

Theorem 13. *The canonical GMRAs $\{V_j^{m,H,G}\}$ and $\{V_j^{m',H',G'}\}$ are equivalent if and only if $m = m'$, and there exists a matrix-valued function A on $\widehat{\Gamma}$ such that*

(1) $A(\omega) = \begin{pmatrix} A_1(\omega) & 0 \\ 0 & 0 \end{pmatrix}$, *where $A_1(\omega)$ is a unitary matrix of dimension $m(\omega)$.*
(2) $H(\omega)A^t(\omega) = A^t(\alpha^*(\omega))H'(\omega)$.

Proof. Suppose first that $m = m'$ and that there exists a matrix-valued function A satisfying the conditions. Let τ_r be the subset of $\widehat{\Gamma}$ on which $m(\omega) = m'(\omega) = r$. Then both subspaces $V_0^{m,H,G}$ and $V_0^{m',H',G'}$ are equal to

$$\bigoplus_i L^2(\sigma_i) \equiv \bigoplus_r L^2(\tau_r, \mathbb{C}^r).$$

Define $P : V_0^{m,H,G} \to V_0^{m',H',G'}$ by $[P(f)](\omega) = A(\omega)f(\omega)$. It follows directly that P satisfies the conditions of Lemma 11, and hence $\{V_j^{m,H,G}\}$ and $\{V_j^{m',H',G'}\}$ are equivalent.

Conversely, suppose an operator P exists and satisfies the conditions of Lemma 11. The first condition on P implies that the two representations of Γ on $V_0^{m,H,G}$ and $V_0^{m',H',G'}$ are unitarily equivalent, whence m must equal m', and $V_0^{m,H,G} = V_0^{m',H',G'} = \bigoplus_i L^2(\sigma_i) = \bigoplus_r L^2(\tau_r, \mathbb{C}^r)$. It is known (e.g. [8]) that any unitary operator P on the direct sum of vector-valued L^2 spaces that commutes with all the multiplication operators $\gamma(\omega)$, is itself a multiplication operator of the form

$$[P(f)](\omega) = A(\omega)f(\omega),$$

where $A(\omega) = \begin{pmatrix} A_1(\omega) & 0 \\ 0 & 0 \end{pmatrix}$, and $A_1(\omega)$ is a unitary matrix whose dimension is $r = m(\omega)$ for $\omega \in \tau_r$. The second condition of Lemma 11 then implies that A satisfies condition (2) of the theorem. □

Corollary 14. *Let m be a multiplicity function and let H be a filter relative to m and α^* for which S_H is a pure isometry. If G and G' are any two complementary filters to H, then the GMRAs $\{V_j^{m,H,G}\}$ and $\{V_j^{m,H,G'}\}$ are equivalent.*

The preceding theorem introduces a notion of equivalence among filters that we will use to build a set of classifying parameters for the equivalence classes of GMRAs. In the following definition, we use our knowledge of the form of A to rewrite the equivalence using the conjugate transpose A^*.

Definition 15. Let m be a multiplicity function. Filters H and H' relative to m and α^* are called **equivalent** if there exists a matrix-valued function A on $\widehat{\Gamma}$, with $A(\omega) = \begin{pmatrix} A_1(\omega) & 0 \\ 0 & 0 \end{pmatrix}$, where $A_1(\omega)$ is a unitary matrix of dimension $m(\omega)$, and such that

$$H'(\omega) = A\big(\alpha^*(\omega)\big)H(\omega)A^*(\omega)$$

for almost all $\omega \in \widehat{\Gamma}$.

Remark 16. If H and H' are two filters constructed from the same GMRA using different Stone's Theorem operators J and J', then H and H' are equivalent according to this definition. Here the matrix-valued function A comes from the multiplication operator $J'J^{-1}$.

Lemma 17. *Let H be a filter relative to m and α^*, and let A be a matrix-valued function of the form described in the preceding theorem. Define the matrix-valued function H' by*

$$H'(\omega) = A\big(\alpha^*(\omega)\big)H(\omega)A^*(\omega).$$

Then H' is a filter relative to m and α^*, i.e., H' satisfies the filter equation.

Proof. We note that if we write $H_1(\omega)$ for the upper left $m(\alpha^*(\omega)) \times m(\omega)$ block of $H(\omega)$, and let $\Lambda(\omega)$ be N times the $m(\omega) \times m(\omega)$ identity, then the filter equation (4) can be rewritten as

$$\sum_{\alpha^*(\zeta)=\omega} H_1(\zeta) H_1^*(\zeta) = \Lambda(\omega). \tag{12}$$

We must show that if H satisfies Eq. (12), then so does H'. We have

$$\sum_{\alpha^*(\zeta)=\omega} H_1'(\zeta) H_1'^{*}(\zeta) = \sum_{\alpha^*(\zeta)=\omega} A_1(\omega) H_1(\zeta) A_1^*(\zeta) A_1(\zeta) H_1^*(\zeta) A_1^*(\omega)$$

$$= A_1(\omega) \Lambda(\omega) A_1^*(\omega)$$

$$= \Lambda(\omega). \quad \square$$

Let H be a filter relative to m and α^*. In [2] it was shown that the operator S_H fails to be a pure isometry if and only if it has an eigenvector, i.e., if and only if there exists an element $F \in \bigoplus L^2(\sigma_i)$ and a complex number λ for which $H^t(\omega) F(\alpha^*(\omega)) = \lambda F(\omega)$, where $|\lambda| = 1 = \|F(\omega)\|$ for almost all ω. Motivated by this result, we make the following definition:

Definition 18. A filter H is called an **eigenfilter** if there exists a constant λ with $|\lambda| = 1$ such that for almost all ω, $H_{1,1}(\omega) = \lambda$ and $H_{1,j}(\omega) = 0$ for $j > 1$.

Using this definition, we have the following restatement of the result from [2]:

Proposition 19. S_H *fails to be a pure isometry if and only if H is equivalent to an eigenfilter.*

Proof. If there exists a matrix-valued function A such that

$$H'(\omega) A(\omega) = A\big(\alpha^*(\omega)\big) H(\omega),$$

where $H'(\omega)$ is an eigenfilter, then, computing the first rows of both sides, we see that the first row of A is the desired eigenvector F.

Conversely, if S_H has an eigenvector F, build a unitary-valued matrix $A(\omega)$ having $F(\omega)$ as its first row. Set $H'(\omega) = A(\alpha^*(\omega)) H(\omega) A^*(\omega)$. By the previous lemma, H' is a filter relative to m and α^*. Moreover, one can see that $H'_{1,1} = \lambda$. Because H' is a filter, it follows that the elements $H'_{1,j}(\omega)$ are all 0 for $j > 1$. Hence, H' has the desired form. $\quad \square$

Now, let S be the set of all pairs (m, H), where m is a multiplicity function and H is a filter relative to m and α^*. Let S_0 be the subset of S comprising those pairs (m, H) for which H is equivalent to an eigenfilter, and let $S_1 = S \setminus S_0$.

Finally let $E = S_1/\equiv$ be the set of equivalence classes of S_1 with respect to the equivalence relation $(m_1, H_1) \equiv (m_2, H_2)$ if $m_1 = m_2$ and H_1 is equivalent to H_2.

Theorem 20. *The set E is a classifying set for the equivalence classes of GMRAs* (*with finite multiplicity functions and associated measures absolutely continuous with respect to Haar measure*), *in the sense that there is a 1–1 correspondence between E and the classes of GMRAs, and this correspondence can be described explicitly.*

Proof. Given an element $s \in E$, let (m, H) be a representative of the equivalence class s. Let G be a filter complementary to H, and define $\kappa(s)$ to be the equivalence class of the GMRA $\{V^{m,H,G}\}$. By Theorem 13, the map κ is both well defined and one-to-one, and by Theorem 12, it is onto. $\quad\square$

5. Examples

We will now use the technique outlined in Section 3 to construct examples of canonical GMRAs, and apply the ideas of Section 4 to discuss their equivalence. We work first in the classical setting of MRAs (so $m \equiv 1$) with single wavelets (so $\widetilde{m} \equiv 1$) for dilation by 2 in $L^2(\mathbb{R})$. Since m is determined for MRAs, their equivalence depends only on the filter H.

Example 21. Any MRA for dilation by 2 in $L^2(\mathbb{R})$ with $m = \widetilde{m} \equiv 1$ has canonical Hilbert space

$$L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \oplus \left(\bigoplus_{j=1}^{\infty} L^2(2^j\mathbb{T}) \right) = V_0^{m,H,G} \oplus W_0^{m,H,G} \oplus \left(\bigoplus_{j=1}^{\infty} W_j^{m,H,G} \right) \qquad (13)$$

with $\pi_n(\bigoplus f_l) = e_n \cdot f_l$, where $e_n(x) = e^{2\pi i n x}$, and

$$\delta^{-1}\left(f_{V_0} \oplus f_{W_0} \oplus \left(\bigoplus_{j=1}^{\infty} f_{W_j} \right) \right)(\omega) = \left(h(\omega) f_{V_0}(2\omega) + g(\omega) f_{W_0}(2\omega) \right) \oplus \sqrt{2} f_{W_1}(2\omega)$$

$$\oplus \left(\bigoplus_{j=2}^{\infty} \sqrt{2} f_{W_j}(2\omega) \right). \qquad (14)$$

Equivalence for two different MRAs with single wavelets is equivalent to the existence of a period 1 function a such that $|a(\omega)| = 1$ and $h'(\omega) = a(2\omega)h(\omega)\overline{a(\omega)}$, where h and h' are filters constructed from the two MRAs. Thus, in particular, equivalence requires that $|h| = |h'|$. However, this is not sufficient, as we will see below. Determining which filters give equivalent MRAs requires determining exactly which functions on the 1-torus are coboundaries where cohomological equivalence is given by Definition 15.

For the Shannon MRA, with $\widehat{V_0} = L^2([-\frac{1}{2}, \frac{1}{2}])$, we have $h = \sqrt{2}\chi_{[-\frac{1}{4}, \frac{1}{4}]}$ and $g = \sqrt{2}\chi_{\pm[\frac{1}{4}, \frac{1}{2}]}$ in the above formula. By mapping $W_j^{m,H,G} \mapsto L^2(\pm 2^j[\frac{1}{2}, 1])$, we can map this canonical GMRA to the Fourier transform of the Shannon GMRA.

For the Haar MRA, with V_0 spanned by translates of $\chi_{[0,1]}$, we have $h = \frac{1}{\sqrt{2}}(1 + e_{-1})$, $g = \frac{1}{\sqrt{2}}(e_{-1} - 1)$ in the above formula. Here there is no obvious mapping between the canonical GMRA and either the original or its Fourier transform. However, we know all three are equivalent by Theorem 12.

In either the Shannon or Haar examples, we can switch the roles of h and g to get a new MRA that cannot be realized in $L^2(\mathbb{R})$ (since iterating the refinement equation (7) leads to a scaling function that must be identically 0). The canonical Hilbert space will still be given by Eq. (13), and the operators π_n will be as above. However, in the dilation formula (14), we will now have h given by the old g, and g by the old h. Proposition 19 shows that we still have S_h a pure isometry, so that the canonical construction does produce a GMRA. Looking at the $V_{-j}^{m,h,g}$ and $W_{-j}^{m,h,g}$ that

result in the case of the reversed Shannon GMRA shows how this example differs from Shannon MRA itself:

$$\delta^{-1} : V_0^{m,h,g} = L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \mapsto L^2\left(\pm\left[\frac{1}{4}, \frac{1}{2}\right]\right) \mapsto L^2\left(\pm\left[\frac{3}{8}, \frac{1}{2}\right]\right) \mapsto \cdots,$$

$$\delta^{-1} : W_0^{m,h,g} = L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \mapsto L^2\left(\left[-\frac{1}{4}, \frac{1}{4}\right]\right) \mapsto L^2\left(\pm\left[\frac{1}{4}, \frac{3}{8}\right]\right) \mapsto \cdots.$$

Since the absolute values of the filters in the three examples discussed here are all different on sets of positive measure, the three are seen to be inequivalent MRAs. To see that for MRAs with wavelets, the filters having equal absolute value almost everywhere is not sufficient for equivalence, consider the MRA built from $h' = -h$, where h is the filter for the Haar example. A simple Fourier analysis argument shows that there is no solution to $h'(\omega) = a(2\omega)h(\omega)\overline{a(\omega)}$, so this MRA must be inequivalent to the Haar MRA. We note that it has the same canonical Hilbert space as Haar, and the same subspaces V_j, but its dilation on V_0 is the negative of the Haar dilation. This negative sign causes problems in the iteration of the refinement equation, so this example cannot be realized in $L^2(\mathbb{R})$.

A fourth example in this setting begins with the Cohen filters $h = \frac{1}{\sqrt{2}}(1 + e_{-3})$ and $g = \frac{1}{\sqrt{2}}(1 - e_{-3})$. The infinite product construction which follows from the refinement equation in $L^2(\mathbb{R})$ yields the functions $\phi = \frac{1}{3}\chi_{[0,3)}$ and $\psi = \frac{1}{3}(\chi_{[0,\frac{3}{2})} - \chi_{[\frac{3}{2},3)})$, which fail to be an orthonormal scaling function and orthonormal wavelet, respectively, since neither has orthonormal translates. However, it can be shown that the negative dilate space (for dilation by 2) of the Cohen Parseval wavelet coincides with that of the Haar orthonormal wavelet. Hence, the Cohen GMRA equals the Haar MRA.

We may apply Theorem 8 to the Cohen filters and multiplicity functions $m \equiv 1$, $\widetilde{m} \equiv 1$. The canonical Hilbert space will be that given by Eq. (13), on which the integers act by multiplication by exponentials. We see that the spaces V_j for the canonical Cohen GMRA are the same as those for the canonical Haar MRA whenever $j \geqslant 0$. However, since $\frac{1}{\sqrt{2}}(1 + e_{-3})$ and $\frac{1}{\sqrt{2}}(1 + e_{-1})$ have different moduli, the two filters must be inequivalent. Therefore the two canonical GMRAs must be inequivalent.

Lastly, we remark that while the Cohen wavelet $\psi = \frac{1}{3}(\chi_{[0,\frac{3}{2})} - \chi_{[\frac{3}{2},3)})$ is only a Parseval wavelet in $L^2(\mathbb{R})$, the element $(0, \chi_{[-\frac{1}{2}, \frac{1}{2})}, 0, 0, 0, \ldots)$ is an orthonormal wavelet for the canonical Hilbert space (13), with respect to π_n and δ defined by Eq. (14) using the Cohen filters. Dutkay et al. [9,17] also produced an orthonormal wavelet from the Cohen filter, using a "superwavelet" construction. The associated GMRA in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ can be seen to be equivalent to our canonical Cohen GMRA by defining the map P in Lemma 11 in the natural way to take the nth translate of the Dutkay scaling function to $e^{2\pi inx}$ in the canonical V_0. Of course, this example cannot be realized in $L^2(\mathbb{R})$.

Next, we consider two non-MRA examples for dilation by 2 in $L^2(\mathbb{R})$: the Journé GMRA, and the example for the Journé multiplicity function with low-pass filter of rank $a = 2$ described in [4, Example 13]. As is noted there, a GMRA cannot be constructed for this second example using the infinite product construction. However, by Proposition 19, the construction of this paper can be carried out to give such a GMRA.

Example 22. Let m be the multiplicity function corresponding to the Journé wavelet:

$$m(x) = \begin{cases} 2 & \text{if } x \in [-\frac{1}{7}, \frac{1}{7}), \\ 1 & \text{if } x \in \pm[\frac{1}{7}, \frac{2}{7}) \cup \pm[\frac{3}{7}, \frac{1}{2}), \\ 0 & \text{otherwise,} \end{cases}$$

so $\sigma_1 = [-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}]$ and $\sigma_2 = [-\frac{1}{7}, \frac{1}{7}]$. Since we know the Journé GMRA has an associated single orthonormal wavelet, $\widetilde{m} \equiv 1$.

Filters that give rise to the Journé wavelet via the infinite product construction are described in [13,1,3]. In particular, we may take

$$H = \begin{pmatrix} \sqrt{2}\chi_{[-\frac{2}{7}, -\frac{1}{4}] \cup [-\frac{1}{7}, \frac{1}{7}) \cup [\frac{1}{4}, \frac{2}{7}]} & 0 \\ \sqrt{2}\chi_{[-\frac{1}{2}, -\frac{3}{7}] \cup [\frac{3}{7}, \frac{1}{2}]} & 0 \end{pmatrix}$$

and

$$G = \begin{pmatrix} \sqrt{2}\chi_{[-\frac{1}{4}, -\frac{1}{7}] \cup [\frac{1}{7}, \frac{1}{4}]} & \sqrt{2}\chi_{[-\frac{1}{7}, \frac{1}{7}]} \end{pmatrix}.$$

Here $V_0^{m,H,G} = L^2(\sigma_1) \oplus L^2(\sigma_2)$, and $W_j^{m,H,G} = L^2(2^j \mathbb{T})$, $j \geqslant 0$. This canonical GMRA can be mapped to the usual Journé GMRA by integrally translating σ_1 and σ_2 to the scaling set to form V_0, and $\mathbb{T} \equiv [-\frac{1}{2}, \frac{1}{2}]$ to the wavelet set to form W_0.

In [4], an alternative filter H' for the same multiplicity function, but which satisfies the low-pass condition of rank $a = 2$ is constructed:

$$h'_{1,1} = \sqrt{2}\chi_{[-\frac{2}{7}, -\frac{1}{4}) \cup [-\frac{1}{7}, \frac{1}{7}) \cup [\frac{1}{4}, \frac{2}{7})}, \qquad h'_{1,2} = h'_{2,1} = 0, \quad \text{and} \quad h'_{2,2} = \sqrt{2}\chi_{[-\frac{1}{14}, \frac{1}{14})}.$$

By partitioning $\mathbb{R}/\mathbb{Z}$ as described in [13,1], we can build the following complementary filter G':

$$g'_{1,1} = \sqrt{2}\chi_{[-\frac{1}{2}, -\frac{3}{7}) \cup [-\frac{1}{4}, -\frac{1}{7}) \cup [\frac{1}{7}, \frac{1}{4}) \cup [\frac{3}{7}, \frac{1}{2})}, \qquad g'_{1,2} = \sqrt{2}\chi_{[-\frac{1}{7}, -\frac{1}{14}) \cup [\frac{1}{14}, \frac{1}{7})}.$$

The spaces $V_0^{m,H',G'}$ and $W_j^{m,H',G'}$ for $j \geqslant 0$ are the same as those for the canonical GMRA corresponding to the standard Journé filters. However, the different filters in the rank 2 example will change the dilation, and thus change the spaces $V_j^{m,H',G'}$ and $W_j^{m,H',G'}$ for $j < 0$. For example, we have $V_{-1}^{m,H',G'} = L^2([-\frac{1}{7}, \frac{1}{7}] \cup \pm[\frac{1}{4}, \frac{2}{7}]) \oplus L^2([-\frac{1}{14}, \frac{1}{14}])$, while the standard $V_{-1}^{m,H,G} = L^2([-\frac{1}{7}, \frac{1}{7}] \cup \pm[\frac{1}{4}, \frac{2}{7}] \cup \pm[-\frac{3}{7}, \frac{1}{2}]) \oplus 0$. The fact that all the $V_{-j}^{m,H',G'}$ allow nonzero second components with support overlapping that of the first component suggests the impossibility of mapping the rank 2 example into $L^2(\mathbb{R})$ as we mapped the standard example. Indeed, since iterating the refinement equation would lead to a scaling function with a degenerate multiplicity function [4], the rank 2 example cannot be realized in $L^2(\mathbb{R})$. Thus, these two examples must not be equivalent.

For our next example, we consider dilation by 3, both in $L^2(\mathbb{R})$ and in the Dutkay/Jorgensen enlarged Cantor fractal space [16].

Example 23. The MRA Haar 2-wavelet for dilation by 3 in $L^2(\mathbb{R})$ has canonical Hilbert space

$$L^2(\mathbb{T}) \oplus \left(L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \right) \oplus \left(\bigoplus_{j=1}^{\infty} L^2(3^j \mathbb{T}) \oplus L^2(3^j \mathbb{T}) \right).$$

The canonical $\delta^{-1} = S_h \oplus (S_{g_1} \oplus S_{g_2}) \oplus (\bigoplus_{j=1}^{\infty} \mathcal{D}^{-j})$, where

$$h = \frac{1}{\sqrt{3}}(1 + e_1 + e_2), \qquad g_1 = \frac{1}{\sqrt{2}}(e_1 - e_2) \quad \text{and} \quad g_2 = \frac{1}{\sqrt{6}}(-2 + e_1 + e_2),$$

and $\mathcal{D}^{-j}(f_1 \oplus f_2)(\omega) = \sqrt{3}^j (f_1 \oplus f_2)(3^j \omega)$.

The Cantor set MRA has the same canonical GMRA except with

$$h = \frac{1}{\sqrt{2}}(1 + e_2), \qquad g_1 = e_1 \quad \text{and} \quad g_2 = \frac{1}{\sqrt{2}}(1 - e_2).$$

These two examples must be inequivalent since their h's have different absolute values. The latter cannot be realized in $L^2(\mathbb{R})$, since $h(0) = \sqrt{2}$, so that the iterated refinement equation (7) would again force the scaling function to be identically 0.

Our final example uses a group Γ different from $\mathbb{Z}^d$.

Example 24. Let $\Gamma_j = \bigoplus_{i=j}^{\infty} [\mathbb{Z}_2]_i = \bigoplus_{i=j}^{\infty} \{1, -1\}_i$, embedded as a subgroup of $D = \Gamma_{-\infty} = \bigoplus_{i=-\infty}^{\infty} [\mathbb{Z}_2]_i$ by $\Gamma_j = \bigoplus_{i=-\infty}^{j-1} \{1\}_i \oplus \bigoplus_{i=j}^{\infty} [\mathbb{Z}_2]_i$. Let α be defined on Γ_0 by $\alpha(\gamma)_n = \gamma_{n-1}$ for $n > 0$ and $\alpha(\gamma)_0 = 1$. Let $\mathcal{H} = l^2(D)$, and let π be the restriction to Γ_0 of the regular representation of D. Define S on D by $[S(d)]_n = d_{n-1}$, and note that $S(\gamma) \equiv \alpha(\gamma)$ for $\gamma \in \Gamma_0$. Define δ on $\mathcal{H} = l^2(D)$ by $[\delta(f)](d) = f(S(d))$, and note that $\delta^{-1}\pi_\gamma\delta = \pi_{\alpha(\gamma)}$.

We have $\Gamma_{j+1} \subseteq \Gamma_j$, and $\bigcap_{j=-\infty}^{\infty} \Gamma_j = \{e_D\}$, where $e_D = (\ldots, 1, 1, 1, \ldots)$ denotes the multiplicative identity element of $D = \Gamma_{-\infty}$, so that $l^2(\Gamma_{j+1}) \subseteq l^2(\Gamma_j)$ and $\bigcap_{j=-\infty}^{\infty} l^2(\Gamma_j) = l^2(\{e_D\})$. If we let $V_j = l^2(\Gamma_{-j})$, then $\{V_j\}$ is almost a GMRA. It fails only because constant multiples of the function $\chi_{\{e_D\}}$ belong to $\bigcap V_j$. We will make it into a GMRA by tensoring it with the dilation by 2 Haar GMRA. It is known (as in [18]) that the tensor product of two GMRAs gives a GMRA. By tensoring our almost GMRA with an actual one, we will preserve all the properties of the almost GMRA, and eliminate the non-trivial intersection.

Accordingly, let $\Gamma' = \mathbb{Z}$ act in $\mathcal{H}' = L^2(\mathbb{R})$ by $\pi'_n f(x) = f(x - n)$, and let $\delta' f(x) = \sqrt{2} f(2x)$. We have α' acting on Γ' by $\alpha'(n) = 2n$. Write $\{V'_j\}$ for the usual Haar GMRA that results from taking V'_0 to be the closed linear span of translates of $\chi_{[0,1]}$. Set $\mathcal{H}'' = \mathcal{H} \otimes \mathcal{H}'$, equipped with the representation $\pi \times \pi'$ of $\Gamma'' = \Gamma_0 \times \Gamma'$ and the operator $\delta \otimes \delta'$. We let $\alpha'' = \alpha \times \alpha'$ and note that α''^* acts on $\widehat{\Gamma''} = \prod_{i=0}^{\infty} [\mathbb{Z}_2]_i \times \mathbb{T}$ by $\alpha''^*((\omega_0, \omega_1, \omega_2, \ldots) \times x) = (\omega_1, \omega_2, \ldots) \times 2x$, where we parameterize $\mathbb{T}$ by $[-\frac{1}{2}, \frac{1}{2})$. We have $N = 4$, and $\ker(\alpha''^*) = (\{-1, 1\} \times \prod_{i=1}^{\infty} [\{1\}]_i) \times \{0, \frac{1}{2}\}$. To build the dilation described in Section 3, we can take the cross section $c((\omega_0, \omega_1, \omega_2, \ldots) \times x) = (1, \omega_0, \omega_1, \ldots) \times \frac{x}{2}$. We have $m \equiv 1$ and $\widetilde{m} \equiv 3$, so $\sigma_1 = \widetilde{\sigma}_1 = \widetilde{\sigma}_2 = \widetilde{\sigma}_3 = \prod_{i=0}^{\infty} [\mathbb{Z}_2]_i \times \mathbb{T}$.

We define our filter $H = h_1 \otimes h_2$, where h_1 is the filter on $\prod_{i=0}^{\infty} [\mathbb{Z}_2]_i$ given by $h_1 = \sqrt{2}\chi_{\{1\}_0 \times \prod_{i=1}^{\infty} [\mathbb{Z}_2]_i}$, and h_2 is the low-pass filter for the Haar GMRA described in Example 21,

that is, $h_2 = \frac{1}{\sqrt{2}}(1 + e_{-1})$. For our filter complementary to H, we define $g_1 = \sqrt{2}\chi_{\{-1\}_0 \times \prod_{i=1}^{\infty}[\mathbb{Z}_2]_i}$ and let g_2 be the high-pass filter for the Haar GMRA, $g_2 = \frac{1}{\sqrt{2}}(e_{-1} - 1)$. We then take our complementary filter G to be the matrix whose rows are $h_1 \otimes g_2$, $g_1 \otimes h_2$, and $g_1 \otimes g_2$.

For an alternative GMRA, we can replace h_1 by $h_1' = \chi_{\{1\}_0 \times \prod_{i=1}^{\infty}[\mathbb{Z}_2]_i} - \chi_{\{-1\}_0 \times \prod_{i=1}^{\infty}[\mathbb{Z}_2]_i}$ and g_1 by $g_1' = -\chi_{\{1\}_0 \times \prod_{i=1}^{\infty}[\mathbb{Z}_2]_i} + \chi_{\{-1\}_0 \times \prod_{i=1}^{\infty}[\mathbb{Z}_2]_i}$. These could be viewed as more fractal-like when combined with the h_2 and g_2 in the standard tensor product construction.

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