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Residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries and lepton mixing

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ABSTRACT

We consider two novel scenarios of residual symmetries of the lepton mass matrices. Firstly we assume a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry G_ℓ for the charged-lepton mass matrix and a $\mathbb{Z}_2$ symmetry G_ν for the light neutrino mass matrix. With this setting, the moduli of the elements of one column of the lepton mixing matrix are fixed up to a reordering. One may interchange the roles of G_ℓ and G_ν in this scenario, thereby constraining a row, instead of a column, of the mixing matrix. Secondly we assume a residual symmetry group $G_\ell \cong \mathbb{Z}_m$ $(m > 2)$ which is generated by a matrix with a doubly-degenerate eigenvalue. Then, with $G_\nu \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ the moduli of the elements of a row of the lepton mixing matrix get fixed. Using the library of small groups we have performed a search for groups which may embed G_ℓ and G_ν in each of these two scenarios. We have found only two phenomenologically viable possibilities, one of them constraining a column and the other one a row of the mixing matrix.

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1. Introduction

A group-theoretical philosophy for explaining the phenomenological values of the lepton mixing parameters has emerged during the last few years [1–16]. In that philosophy, those values follow from the distinct Abelian symmetry groups—G_ℓ and G_ν—under which the lepton mass matrices—M_ℓ and M_ν, respectively—are invariant.[1] Those matrices are defined by the mass terms

$$\mathcal{L}_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2} \nu_L^T M_\nu C^{-1} \nu_L + \text{H.c.}, \tag{1}$$

where $\ell_{L,R}$ are the left- and right-handed charged-lepton fields, ν_L are the light neutrino fields, and C is the charge-conjugation matrix in Dirac space. (We assume the neutrinos to be Majorana particles.) Let $H_\ell \equiv M_\ell M_\ell^\dagger$; if the mass matrices are diagonalized as $U_\ell^\dagger H_\ell U_\ell = D_\ell \equiv \text{diag}(m_e^2, m_\mu^2, m_\tau^2)$ and $U_\nu^T M_\nu U_\nu = D_\nu \equiv \text{diag}(m_1, m_2, m_3)$, then the lepton mixing matrix is given by $U_{\text{PMNS}} = U_\ell^\dagger U_\nu$. ($m_{1,2,3}$ denote the three neutrino masses.) Let the symmetry group G_ℓ be generated by a matrix L such that $L^{-1} H_\ell L = H_\ell$.[2] If we choose a basis in which L is diagonal and if we assume that the diagonal matrix elements of L are all distinct, then this invariance forces H_ℓ to be diagonal. Thus, in that basis $U_\ell = \mathbb{1}_3$ (up to a permutation of the charged leptons) and $U_{\text{PMNS}} = U_\nu$. ($\mathbb{1}_3$ denotes the 3×3 unit matrix.) In the same basis, let a generator N of G_ν be a unitary 3×3 matrix of order two and with two different eigenvalues, i.e. $N^2 = \mathbb{1}_3$ but $N \neq \pm\mathbb{1}_3$. Such a matrix can always be written as

$$N = \gamma\left(\mathbb{1}_3 - 2uu^\dagger\right), \tag{2}$$

where $\gamma = \pm 1$ and $u = (u_1, u_2, u_3)^T$ is a normalized column vector, *viz.* $u^\dagger u = |u_1|^2 + |u_2|^2 + |u_3|^2 = 1$. Invariance of M_ν under N means that $N^T M_\nu N = M_\nu$. Then, it follows from $Nu = -\gamma u$ that $N^*(M_\nu u) = N^T(M_\nu u) = (N^T M_\nu N)(Nu) = -\gamma(M_\nu u)$. But, the eigenvalue $-\gamma$ of N^* is non-degenerate; therefore, $M_\nu u \propto u^*$. Since $M_\nu U_\nu = U_\nu^* D_\nu$ and the neutrino masses are non-degenerate, u must be one of the columns of $U_\nu = U_{\text{PMNS}}$. It thence follows that $|u_{1,2,3}|$ are, up to a reordering of the charged leptons, the moduli of the matrix elements of a column (one may still choose which column) of U_{PMNS}.

The above-mentioned philosophy assumes that there is a *finite* discrete group G which has both G_ℓ and G_ν as subgroups.[3] It tries

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[1] The possibilities for the experimental investigation of the implications of residual symmetries are discussed in Refs. [17–20]. Furthermore, residual symmetries have also been considered in the quark sector [21,22].

[2] In our search in Section 2.1, G_ℓ is generated by *two* matrices L_1 and L_2 instead of just one.

[3] If G is not assumed to be finite (and small), then G_ℓ and G_ν will be largely arbitrary and the philosophy will have little predictive power.

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to find a suitable G such that the ensuing $|u_{1,2,3}|$ agree with the phenomenological values of the moduli of the matrix elements of one of the columns of U_{PMNS}. This has been done in Ref. [2] under the assumption that G is a subgroup of $SU(3)$ of order smaller than 512. In Ref. [11] a more complete search has been undertaken, wherein G was assumed to be a subgroup of $U(3)$ of order less than 1536. Both Refs. [2] and [11] assume G to possess a faithful three-dimensional irreducible representation. In Ref. [11] it was moreover assumed that G fully determines U_{PMNS}, because its subgroup G_ν is generated by *two* commuting matrices N and N', both of the form in Eq. (2) but with two mutually orthogonal vectors u and u', respectively. (Thus, $G_\nu \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $G_\nu \cong \mathbb{Z}_2$.) A variant of this philosophy has been employed in Refs. [7,16,21], where the neutrino mass terms have been assumed to be of the Dirac type and, correspondingly, the matrix N has been assumed to generate a group $G_\nu \cong \mathbb{Z}_n$ with $n > 2$.

In this Letter we report on two group searches that we have undertaken and which might hold promise of relevant results. In the first search—in Section 2.1—we have assumed that $G_\ell \cong \mathbb{Z}_2 \times Z_2$ (instead of the usual choice $G_\ell \cong \mathbb{Z}_m$ with $m > 2$) and $G_\nu \cong \mathbb{Z}_2$. In the second search—in Section 2.2—we have assumed that $G_\nu \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and that $G_\ell \cong \mathbb{Z}_m$ but with a doubly-degenerate eigenvalue, in such a way that a *row* (instead of a column) of U_{PMNS} gets fixed. In Section 3 the results of our searches are confronted with the phenomenological values. Section 4 contains the conclusions of this work.

2. Group searches

2.1. First search: $G_\ell \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $G_\nu \cong \mathbb{Z}_2$

We consider in this section a scenario in which the lepton flavor symmetry group G is broken to two residual symmetry subgroups $G_\ell \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G_\nu \cong \mathbb{Z}_2$. The symmetry group G_ℓ holds in the charged-lepton sector while G_ν holds in the neutrino sector. We require the embedding group G to be *finite* and to have *a faithful three-dimensional irreducible representation* $D(G)$. We assume that $-\mathbb{1}_3 \notin D(G_\ell)$ and also $-\mathbb{1}_3 \notin D(G_\nu)$. Furthermore, there must be a mismatch between the residual symmetries G_ℓ and G_ν, i.e. we require that $G_\nu \not\subset G_\ell$.

To summarize, we have searched for groups G which fulfill the following conditions:

1. G is finite.
2. G has a faithful three-dimensional irreducible representation $D(G)$.
3. G has two subgroups, $G_\ell \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G_\nu \cong \mathbb{Z}_2$, which have a trivial intersection, i.e. $G_\ell \cap G_\nu = \{e\}$.
4. Neither $D(G_\ell)$ nor $D(G_\nu)$ contain the matrix $-\mathbb{1}_3$.

Since we are interested in groups G which have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, $\mathrm{ord}(G)$ must be divisible by four. Since we require G to have a three-dimensional irreducible representation, $\mathrm{ord}(G)$ must be divisible by three. Thus, we only need to consider groups of order divisible by 12.

Since G is finite, there is a basis in which $D(G)$ consists of unitary matrices. Since $G_\ell \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is Abelian, a basis can be chosen in which $D(G_\ell)$ is formed by diagonal matrices. Thus, $D(G_\ell)$ comprehends $\mathbb{1}_3$ and

$$L_1 = \alpha\,\mathrm{diag}(+1,\,-1,\,-1), \tag{3a}$$

$$L_2 = \beta\,\mathrm{diag}(-1,\,+1,\,-1), \tag{3b}$$

$$L_1 L_2 = L_2 L_1 \equiv L_3 = \alpha\beta\,\mathrm{diag}(-1,\,-1,\,+1), \tag{3c}$$

where both α and β may be either $+1$ or -1. The residual symmetry G_ℓ means that $L_1^{-1} H_\ell L_1 = L_2^{-1} H_\ell L_2 = H_\ell$. Therefore, in the basis where L_1 and L_2 are as in Eqs. (3), H_ℓ must be diagonal.

In the same basis, the generator N of $D(G_\nu)$ is a unitary 3×3 matrix of order two, i.e. a matrix of the form in Eq. (2), where $u = (u_1, u_2, u_3)^T$ is a normalized column vector. Then, up to a reordering, the $|u_k|$ ($k = 1, 2, 3$) are the moduli of the matrix elements of one column of U_{PMNS}. Given the matrices L_1, L_2, and N *in an arbitrary basis*, one may compute the $|u_k|^2$ without the need to diagonalize L_1 and L_2; indeed,

$$|u_k|^2 = \frac{1}{4}\left[1 + \frac{\mathrm{tr}(L_k N)}{\mathrm{tr}(L_k)\,\mathrm{tr}(N)}\right]. \tag{4}$$

Eq. (4) is easily verified in the basis where Eqs. (2) and (3) hold; since it is written in terms of traces, it holds in any other basis—even in one where $D(G)$ is not formed by unitary matrices. One may thus compute the moduli of the matrix elements of one column of U_{PMNS} just from the knowledge of L_1, L_2, and N in an arbitrary basis.[4]

The computer algebra system GAP [24] has access to SmallGroups [25], a library of all the groups (up to isomorphisms) of order smaller than 2000—excluding the 49 487 365 422 groups of order 1024. Since there are 408 641 062 groups of order $1536 = 12 \times 128$, we have restricted our search to the 1 336 749 groups of order $12n$ for $n \leqslant 127$. We have furthermore excluded groups G which are direct products of the form

$$G \cong \mathbb{Z}_m \times G' \quad (m \geqslant 2), \tag{5}$$

because such groups do not provide any restrictions beyond those already following from the smaller group G'.

Going through these groups, by constructing their character tables, we have sieved out the groups which have a faithful three-dimensional irreducible representation. We have used the GAP package SONATA [26] to find all the subgroups of the groups under investigation. For those groups which have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup and a $\mathbb{Z}_2$ subgroup with trivial intersection, we have explicitly constructed all the non-equivalent faithful three-dimensional irreducible representations D and we have computed all the candidates for pairs $(D(G_\ell), D(G_\nu))$. When neither $D(G_\ell)$ nor $D(G_\nu)$ contained $-\mathbb{1}_3$, we have computed the corresponding $|u_k|^2$ through Eq. (4). The results can be found in Table 1.

In Table 1 (and in the second column of Table 3) one observes that, whenever G_ℓ and G_ν together generate a group D_n with even n, this leads to $(|u_1|^2, |u_2|^2, |u_3|^2) = (0, \sin^2\frac{2\pi}{m}, \cos^2\frac{2\pi}{m})$ with $m = 2n$ and, possibly, smaller (integer) values of m.[5] The group D_n may be defined as consisting of the matrices

$$X(p) = \begin{pmatrix} -\cos(p\alpha_n) & -\sin(p\alpha_n) \\ -\sin(p\alpha_n) & \cos(p\alpha_n) \end{pmatrix} \quad \text{and}$$

$$Y(p) = \begin{pmatrix} \cos(p\alpha_n) & -\sin(p\alpha_n) \\ \sin(p\alpha_n) & \cos(p\alpha_n) \end{pmatrix}, \tag{6}$$

where $\alpha_n \equiv 2\pi/n$ and $p = 0, 1, 2, \ldots, n-1$. For even n, this group has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup formed by $\mathbb{1}_2$, $Y(n/2)$, $X(n/2)$, and $X(0)$. The group D_n is a subgroup of $SO(3)$ through its *reducible* triplet representation

[4] The computation of mixing-matrix elements from invariant traces was pioneered in Ref. [23].

[5] The group D_{14} is of particular interest, especially for quark mixing, because it nicely fits Cabibbo mixing [27], as can be seen in the second line before the last of Table 1.

Table 1
In the first column, the groups resulting from the search described in Section 2.1; the symbol $[g, j]$ denotes the j-th group of order g in the SmallGroups Library. In the second column, the corresponding values for the $|u_k|^2$ ($k = 1, 2, 3$). In the third column, the symbol $\langle\langle G_\ell, G_\nu\rangle\rangle$ denotes the group generated by $D(G_\ell)$ and $D(G_\nu)$, i.e. the smallest finite group having G_ℓ and G_ν as subgroups. A characterization of the occurring groups can be found in Table 3.

G	$(\lvert u_1\rvert^2, \lvert u_2\rvert^2, \lvert u_3\rvert^2)$	$\langle\langle G_\ell, G_\nu\rangle\rangle$
[24, 12]; [96, 64]; [168, 42]; [216, 95]; [384, 568]; [600, 179]; [648, 259]; [648, 260]; [648, 266]; [648, 563]; [864, 701]; [1080, 260]; [1176, 243]	$(0, \sin^2\frac{2\pi}{8}, \cos^2\frac{2\pi}{8}) = (0, 1/2, 1/2)$	[8, 3]
[216, 95]; [648, 259]; [648, 260]; [648, 266]; [648, 563]; [864, 701]	$(0, \sin^2\frac{2\pi}{12}, \cos^2\frac{2\pi}{12}) = (0, 1/4, 3/4)$	[12, 4]
[384, 568]	$(0, \sin^2\frac{2\pi}{16}, \cos^2\frac{2\pi}{16}) \approx (0, 0.1464, 0.8536)$	[16, 7]
[600, 179]	$(0, \sin^2\frac{2\pi}{10}, \cos^2\frac{2\pi}{10}) \approx (0, 0.3455, 0.6545)$; $(0, \sin^2\frac{2\pi}{20}, \cos^2\frac{2\pi}{20}) \approx (0, 0.0955, 0.9045)$	[20, 4]
[864, 701]	$(0, \sin^2\frac{2\pi}{24}, \cos^2\frac{2\pi}{24}) \approx (0, 0.0670, 0.9330)$	[24, 6]
[1176, 243]	$(0, \sin^2\frac{2\pi}{7}, \cos^2\frac{2\pi}{7}) \approx (0, 0.6113, 0.3887)$; $(0, \sin^2\frac{2\pi}{14}, \cos^2\frac{2\pi}{14}) \approx (0, 0.1883, 0.8117)$; $(0, \sin^2\frac{2\pi}{28}, \cos^2\frac{2\pi}{28}) \approx (0, 0.0495, 0.9505)$	[28, 3]
[24, 12]; [96, 64]; [168, 42]; [216, 95]; [384, 568]; [600, 179]; [648, 259]; [648, 260]; [648, 266]; [648, 563]; [864, 701]; [1080, 260]; [1176, 243]	$(1/4, 1/4, 1/2)$	[24, 12]
[60, 5]; [1080, 260]	$(1/4, \frac{3-\sqrt{5}}{8}, \frac{3+\sqrt{5}}{8}) \approx (0.25, 0.0955, 0.6545)$	[60, 5]

$$X(p) \to \widetilde{X}(p) \equiv \begin{pmatrix} -1 & 0_{1\times 2} \\ 0_{2\times 1} & X(p) \end{pmatrix},$$
$$Y(p) \to \widetilde{Y}(p) \equiv \begin{pmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & Y(p) \end{pmatrix}. \tag{7}$$

In this representation of D_n, its $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup is formed by

$$\{\mathbb{1}_3,\ \widetilde{Y}(n/2) = L_1,\ \widetilde{X}(n/2) = L_2,\ \widetilde{X}(0) = L_3\}, \tag{8}$$

where the matrices $L_{1,2,3}$ are as in Eqs. (3) with $\alpha = \beta = +1$. The G_ν subgroup is formed by

$$\{\mathbb{1}_3,\ \widetilde{X}(p)\}. \tag{9}$$

By using Eq. (4) one then obtains $|u_1|^2 = 0$ and $|u_2|^2 = \sin^2(p\alpha_n/2)$.

2.2. Second search: $G_\ell \cong \mathbb{Z}_n$, $G_\nu \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

One may interchange the roles of Klein's four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and of the cyclic group $\mathbb{Z}_2$ in Section 2.1. When one does that, the neutrino mass matrix M_ν is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$, *viz.* $N_1^T M_\nu N_1 = N_2^T M_\nu N_2 = M_\nu$ with $N_1^2 = N_2^2 = \mathbb{1}_3$. If we choose the basis where N_1 and N_2 are diagonal, then in that basis M_ν will be diagonal too. Let H_ℓ possess a residual $\mathbb{Z}_2$ symmetry, i.e. $L^{-1} H_\ell L = H_\ell$ with $L = \gamma(\mathbb{1}_3 - 2uu^\dagger)$ as in Eq. (2). Consequently, $LH_\ell = H_\ell L$ and therefore $L(H_\ell u) = H_\ell L u = -\gamma(H_\ell u)$. Then, since the eigenvalue $-\gamma$ of L is non-degenerate, $H_\ell u \propto u$. Now, the eigenvalues of H_ℓ, *viz.* the squares of the charged-lepton masses, are non-degenerate. Therefore, u must be a column of the unitary matrix U_ℓ diagonalizing H_ℓ. Since we are in the basis where M_ν is diagonal, $U_{\text{PMNS}} = U_\ell^\dagger$ up to a permutation of the rows of U_{PMNS}. We have thus found that in this case the residual symmetries constrain *a row*, rather than a column, of the mixing matrix U_{PMNS}. The possible restrictions on the moduli of the matrix elements of the row are of course precisely the same as those obtained in Section 2.1, see Table 1.

An important feature of the scenario just described is that the matrix L generating the residual symmetry group of H_ℓ has two degenerate eigenvalues and the third eigenvalue is different. The matrix L is, however, restricted by the condition $L^2 = \mathbb{1}_3$, since it generates a group $\mathbb{Z}_2$. We now lift this restriction and suppose instead that L generates a group $\mathbb{Z}_n$ with $n > 2$, i.e. $L^n = \mathbb{1}_3$. We thus assume that in the neutrino sector there is a residual symmetry $G_\nu \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, represented by $D(G_\nu)$ which, in an appropriate basis, is formed by $\mathbb{1}_3$ together with

$$N_1 = \alpha\, \mathrm{diag}(+1, -1, -1), \tag{10a}$$

$$N_2 = \beta\, \mathrm{diag}(-1, +1, -1), \tag{10b}$$

$$N_1 N_2 = N_2 N_1 \equiv N_3 = \alpha\beta\, \mathrm{diag}(-1, -1, +1). \tag{10c}$$

In this basis, M_ν is diagonal and therefore $U_{\text{PMNS}} = U_\ell^\dagger$ up to a permutation of rows. In the charged-lepton sector the residual symmetry is $\mathbb{Z}_n$, generated by a matrix L *with a degenerate eigenvalue* σ and another eigenvalue $\rho \neq \sigma$ (of course $\sigma^n = \rho^n = 1$). Let, in the basis where Eqs. (10) hold, $v = (v_1, v_2, v_3)^T$ denote the normalized eigenvector of L corresponding to the eigenvalue ρ. One may then write

$$L = \sigma \mathbb{1}_3 + (\rho - \sigma) v v^\dagger. \tag{11}$$

The $|v_k|$ are, up to a reordering, the moduli of the matrix elements of one row of U_{PMNS}. They may be computed in a basis-independent way through

$$|v_k|^2 = \frac{1}{2(\rho - \sigma)}\left[\rho - \frac{\mathrm{tr}(N_k L)}{\mathrm{tr}(N_k)}\right]. \tag{12}$$

Thus, we have searched for groups G which fulfill the following conditions:

1. G is finite.
2. G has a faithful three-dimensional irreducible representation $D(G)$.
3. G has two subgroups, $G_\ell \cong \mathbb{Z}_n$ ($n > 2$) and $G_\nu \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which have a trivial intersection, i.e. $G_\ell \cap G_\nu = \{e\}$.
4. $D(G_\nu)$ does not contain $-\mathbb{1}_3$.

Table 2
In the first column, the groups resulting from the search described in Section 2.2 and of order smaller than 1000. In the second column, the corresponding values of the $|v_k|^2$ $(k = 1, 2, 3)$. The group G_ℓ is shown in the third column and the smallest finite group having G_ℓ and G_ν as subgroups is listed in the fourth column. A characterization of the occurring groups can be found in Table 3.

G	$(\lvert v_1\rvert^2, \lvert v_2\rvert^2, \lvert v_3\rvert^2)$	G_ℓ	$\langle\langle G_\ell, G_\nu\rangle\rangle$
[48, 30]; [192, 182]; [432, 260]	(0, 1/2, 1/2)	$\mathbb{Z}_4$	[16, 3]
[216, 95]; [648, 259]; [648, 260]; [648, 266]; [648, 563]; [864, 701]	(0, 1/2, 1/2)	$\mathbb{Z}_6$	[24, 10]
[96, 64]; [384, 568]; [864, 701]	(0, 1/2, 1/2)	$\mathbb{Z}_4$	[32, 11]
[96, 65]; [384, 571]; [864, 703]	(0, 1/2, 1/2)	$\mathbb{Z}_8$	[32, 5]
[648, 266]	(0, 1/2, 1/2)	$\mathbb{Z}_3$	[36, 12]
[432, 260]	(0, 1/2, 1/2)	$\mathbb{Z}_{12}$	[48, 21]
[192, 186]	(0, 1/2, 1/2)	$\mathbb{Z}_{16}$	[64, 29]
[648, 266]	(0, 1/2, 1/2)	$\mathbb{Z}_6$	[72, 30]
[648, 563]	(0, 1/2, 1/2)	$\mathbb{Z}_{18}$	[72, 10]
[864, 701]	(0, 1/2, 1/2)	$\mathbb{Z}_{12}$	[96, 54]
[864, 703]	(0, 1/2, 1/2)	$\mathbb{Z}_{24}$	[96, 48]
[600, 179]	(0, 1/2, 1/2)	$\mathbb{Z}_5$	[100, 14]
[648, 259]; [648, 260]	(0, 1/2, 1/2)	$\mathbb{Z}_9$	[108, 24]
[384, 568]	(0, 1/2, 1/2)	$\mathbb{Z}_8$	[128, 67]
[384, 581]	(0, 1/2, 1/2)	$\mathbb{Z}_{32}$	[128, 131]
[600, 179]	(0, 1/2, 1/2)	$\mathbb{Z}_{10}$	[200, 31]
[648, 259]; [648, 260]	(0, 1/2, 1/2)	$\mathbb{Z}_{18}$	[216, 58]
[216, 95]; [648, 259]; [648, 260]; [648, 266]; [648, 563]; [864, 701]	(1/4, 1/4, 1/2)	$\mathbb{Z}_6$	[72, 42]
[648, 563]	(1/4, 1/4, 1/2)	$\mathbb{Z}_{18}$	[216, 89]
[216, 95]; [648, 259]; [648, 260]; [648, 266]; [648, 563]; [864, 701]	(0, 1/4, 3/4)	$\mathbb{Z}_6$	[36, 12]
[648, 563]	(0, 1/4, 3/4)	$\mathbb{Z}_{18}$	[108, 24]
[864, 701]	$(0, \sin^2\frac{2\pi}{24}, \cos^2\frac{2\pi}{24}) \approx (0, 0.0670, 0.9330)$	$\mathbb{Z}_6$	[72, 28]

5. $D(G_\ell)$ is generated by a matrix L which has a twice degenerate eigenvalue σ and another eigenvalue ρ which differs from σ.
6. The group $\langle\langle G_\ell, G_\nu\rangle\rangle$ generated by $D(G_\ell)$ and $D(G_\nu)$ is non-Abelian.

Once again, we have excluded groups of the form $G \cong \mathbb{Z}_m \times G'$ with $m \geqslant 2$. For each group of order smaller than[6] 1000 fulfilling the above requirements, we have computed the corresponding $|v_k|^2$ by means of Eq. (12). The results can be found in Table 2.

3. The case (1/4, 1/4, 1/2)

One sees in Tables 1 and 2 that most predicted columns or rows of U_{PMNS} contain a zero matrix element. Such a situation is phenomenologically excluded[7] and therefore most of the data in those tables seem irrelevant for our purposes.

The remaining cases are more encouraging. The possibility $(|u_1|^2, |u_2|^2, |u_3|^2) = (\frac{3+\sqrt{5}}{8}, 1/4, \frac{3-\sqrt{5}}{8})$, in the last line of Table 1, was recently discovered and constitutes a viable prediction for the first column of U_{PMNS} [28]. On the other hand, the possibility $(|u_1|^2, |u_2|^2, |u_3|^2) = (1/4, 1/4, 1/2)$ gives a rather poor fit to the second column of U_{PMNS}.

Here we shall instead consider the case $(|v_1|^2, |v_2|^2, |v_3|^2) = (1/4, 1/4, 1/2)$ as a prediction for the third *row* of U_{PMNS}. Using the standard parametrization for U_{PMNS}, one then has

Table 3
List of the groups appearing in Tables 1 and 2. Details on those groups in the left column which are of order smaller than 512 can be found in Ref. [30]. The symbol $D^{(1)}_{18,6}$ denotes an $SU(3)$ subgroup of type D, cf. Ref. [31].

G	$\langle\langle G_\ell, G_\nu\rangle\rangle$
$[24, 12] \cong S_4 \cong \Delta(6 \times 2^2)$	$[8, 3] \cong D_4$
$[48, 30] \cong A_4 \rtimes \mathbb{Z}_4$	$[12, 4] \cong D_6$
$[60, 5] \cong A_5$	$[16, 3] \cong (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
$[96, 64] \cong \Delta(6 \times 4^2)$	$[16, 7] \cong D_8$
$[96, 65] \cong A_4 \rtimes \mathbb{Z}_8$	$[20, 4] \cong D_{10}$
$[168, 42] \cong \Sigma(168) \cong \mathrm{PSL}(2, 7)$	$[24, 6] \cong D_{12}$
$[192, 182] \cong ((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_4$	$[24, 10] \cong \mathbb{Z}_3 \times D_4$
$[192, 186] \cong A_4 \rtimes \mathbb{Z}_{16}$	$[24, 12] \cong S_4$
$[216, 95] \cong \Delta(6 \times 6^2)$	$[28, 3] \cong D_{14}$
$[384, 568] \cong \Delta(6 \times 8^2)$	$[32, 5] \cong (\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
$[384, 571] \cong ((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_8$	$[32, 11] \cong (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$
$[384, 581] \cong A_4 \rtimes \mathbb{Z}_{32}$	$[36, 12] \cong \mathbb{Z}_6 \times S_3$
$[432, 260] \cong ((\mathbb{Z}_6 \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_4$	$[48, 21] \cong \mathbb{Z}_3 \times ((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$
$[600, 179] \cong \Delta(6 \times 10^2)$	$[60, 5] \cong A_5$
$[648, 259] \cong D^{(1)}_{18,6} \cong (\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes S_3$	$[64, 29] \cong (\mathbb{Z}_{16} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
$[648, 260] \cong ((\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$[72, 10] \cong \mathbb{Z}_9 \times D_4$
$[648, 266] \cong ((\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$[72, 28] \cong \mathbb{Z}_3 \times D_{12}$
$[648, 563] \cong ((\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$[72, 30] \cong \mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$
$[864, 701] \cong \Delta(6 \times 12^2)$	$[72, 42] \cong \mathbb{Z}_3 \times S_4$
$[864, 703] \cong ((\mathbb{Z}_6 \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_8$	$[96, 48] \cong \mathbb{Z}_3 \times ((\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$
$[1080, 260] \cong \Sigma(360 \times 3)$	$[96, 54] \cong \mathbb{Z}_3 \times ((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2)$
$[1176, 243] \cong \Delta(6 \times 14^2)$	$[100, 14] \cong \mathbb{Z}_{10} \times D_5$
	$[108, 24] \cong \mathbb{Z}_{18} \times S_3$
	$[128, 67] \cong (\mathbb{Z}_8 \times \mathbb{Z}_8) \rtimes \mathbb{Z}_2$
	$[128, 131] \cong (\mathbb{Z}_{32} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
	$[200, 31] \cong \mathbb{Z}_5 \times ((\mathbb{Z}_{10} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$
	$[216, 58] \cong \mathbb{Z}_9 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$
	$[216, 89] \cong \mathbb{Z}_9 \times S_4$

$$c_{23}^2 c_{13}^2 = 1/2, \tag{13a}$$

$$s_{12}^2 s_{23}^2 + c_{12}^2 c_{23}^2 s_{13}^2 - 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos\delta = 1/4, \tag{13b}$$

$$c_{12}^2 s_{23}^2 + s_{12}^2 c_{23}^2 s_{13}^2 + 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos\delta = 1/4, \tag{13c}$$

where $s_i \equiv \sin\theta_i$ and $c_i \equiv \cos\theta_i$ for $i = 12, 13, 23$.

[6] We have stopped this search at a lower group order because the construction of the irreducible representations becomes, for large groups, extremely expensive in terms of computer time.

[7] One might consider the possibility where our predictions only hold as a first approximation and are corrected by other effects—for instance, suppressed terms in the Lagrangian and/or the renormalization-group evolution of the parameters of U_{PMNS}. We shall not entertain such possibilities here.

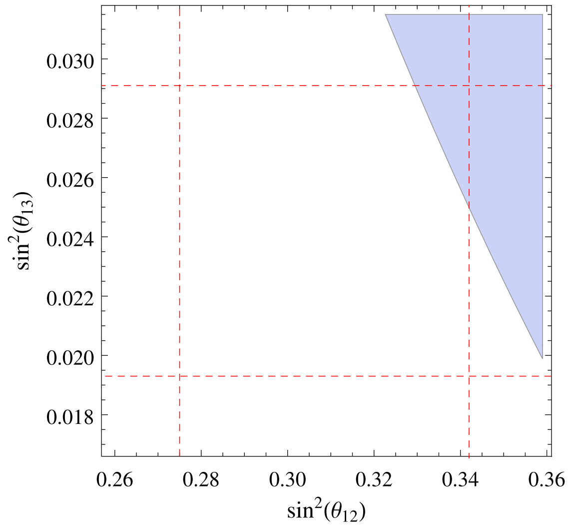

Fig. 1. The area in the $\sin^2\theta_{12}$–$\sin^2\theta_{13}$ plane allowed by our prediction in Eq. (17). The dotted lines represent the 2σ phenomenological bounds of Ref. [29] on those parameters; the shaded area extends to their 3σ bounds.

The sum of Eqs. (13b) and (13c) is equivalent to Eq. (13a). It makes a prediction for θ_{23} as a function of θ_{13}:

$$s_{23}^2 = \frac{1-2s_{13}^2}{2-2s_{13}^2}. \tag{14}$$

With $0.0169 \leqslant s_{13}^2 \leqslant 0.0315$ at 3σ level [29], this yields $0.4837 \leqslant s_{23}^2 \leqslant 0.4914$. This means that *the atmospheric mixing angle is maximal* for all practical purposes.

The difference between Eqs. (13b) and (13c) yields a prediction for $\cos\delta$:

$$4s_{12}c_{12}s_{23}c_{23}s_{13}\cos\delta = \left(s_{12}^2 - c_{12}^2\right)\left(s_{23}^2 - c_{23}^2 s_{13}^2\right). \tag{15}$$

Using Eq. (14), this gives

$$\cos\delta = -\frac{c_{12}^2 - s_{12}^2}{4s_{12}c_{12}}\,\frac{1-3s_{13}^2}{\sqrt{s_{13}^2 - 2s_{13}^4}}. \tag{16}$$

Since $c_{12} > s_{12}$, $\cos\delta$ is predicted to be negative.[8] Moreover, $|\cos\delta|$ is quite large; the bound $\cos^2\delta \leqslant 1$ gives

$$\sin(2\theta_{12}) \geqslant \frac{1-3s_{13}^2}{1-s_{13}^2} \approx 1 - 2s_{13}^2 - 2s_{13}^4 - 2s_{13}^6 - \cdots. \tag{17}$$

This implies that θ_{12} and θ_{13} cannot be both within their 1σ intervals of Ref. [29] and can only marginally be both within their 2σ intervals, see Fig. 1. Anyway, the angle δ should be close to either 0 or π, i.e. *CP* violation in lepton mixing is predicted to be small.

4. Conclusions

In this work, using the software GAP and the SmallGroups Library, we have looked for *finite* groups G which have *a faithful three-dimensional irreducible representation* $D(G)$ and have two subgroups, $\mathbb{Z}_n$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, with a trivial intersection. Moreover, $D(\mathbb{Z}_n)$ should have *a twice degenerate eigenvalue* and neither $D(\mathbb{Z}_n)$ (for $n = 2$) nor $D(\mathbb{Z}_2 \times \mathbb{Z}_2)$ should contain the matrix $-\mathbb{1}_3$. When $n = 2$ we have taken the search up to group order 1536 but for $n > 2$ we only reached group order 1000.

Applying the results of our search to the prediction of lepton mixing, we have noticed that almost all the groups that we have found lead to a zero mixing matrix element, which is phenomenologically disallowed. There are only two exceptions. In one of them, the groups $[60, 5] \cong A_5$ and $[1080, 260] \supset [60, 5]$ may lead to the first column of the lepton mixing matrix having elements with moduli squared $(0.6545, 0.25, 0.0955)$; this is viable and had already been found in a previous paper [28]. In the other exception, many groups—see Tables 1 and 2—may lead to either the second or the third row of the lepton mixing matrix having elements with moduli $(1/2, 1/2, 1/\sqrt{2})$; the consequences of this prediction are a very close to maximal atmospheric mixing angle and $|\cos\delta|$ straddling 1.

[8] This is not very meaningful because it just follows from our choice of fitting $(|v_1|^2, |v_2|^2, |v_3|^2) = (1/4, 1/4, 1/2)$ to the *third* row of U_{PMNS}. If we had opted to fit it to the *second* row instead, then the predicted value of $\cos\delta$ would be symmetric to the one in Eq. (16).

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