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Lattice Covering by Semicrosses of Arm Length 2

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Using algebraic and graph theoretical methods we provide an algorithm to determine the integer lattice covering constant of a certain type of n-dimensional cubistic polyhedron. In particular, we verify a special case of a conjecture of S. K. Stein on covering finite abelian groups by cyclic subsets.

1. Introduction

Let $e_1, \ldots, e_n$ be the co-ordinate unit vectors in the n-dimensional space. Consider an n-dimensional unit cube the edges of which are parallel to $e_1, \ldots, e_n$. The union of translates of this unit cube by the vectors

$$je_i, \qquad 1 \leqslant i \leqslant n, \quad 0 \leqslant j \leqslant k$$

is called a (k, n) *semicross*. The (k, n) semicross is the union of $kn+1$ n-dimensional unit cubes, a corner cube and n attached arms of length k. A family of translates of this (k, n) semicross is a *covering* if its union is the n-space. If the translation vectors have only integer co-ordinates and they form a lattice we speak of an *integer lattice covering*.

Integer lattice coverings by translates of a (k, n) semicross are in an intimate connection with a covering problem of finite abelian groups. Let G be a finite abelian group written additively. We say that the subset $\{g_1, \ldots, g_n\}$ covers G by the *multiplier set* $\{1, 2, \ldots, k\}$ if the union of the elements

$$jg_i, \qquad 1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant k$$

contains $G \setminus \{0\}$. Let $f(k, n)$ be the order of the largest abelian group which can be covered in this way. Clearly, $f(k, n) \leqslant kn+1$. The importance of $f(k, n)$ is that $(kn+1)/f(k, n)$ is the density of the optimal integer lattice covering of n-dimensional space by (k, n) semicrosses. For the details see [2].

S. K. Stein conjectured that to determine $f(k, n)$ we may restrict our investigations to the case of cyclic groups. Hickerson [1] verified this conjecture in the special case that $f(k, n) = kn+1$; that is, when each non-zero element is covered only once. However, he showed that the conjecture does not extend to all multiplier sets, since

$$\{-5, -3, -2, -1, 1, 2, 3, 5, 7\}$$

covers $C(2) \oplus C(14)$ with 3 elements but not $C(28)$. In this paper we verify his conjecture in the special case $k = 2$ and arbitrary n.

2. Preliminaries

We consider a finite abelian group G and the multiplier set $\{1, 2\}$. Then we construct a covering set $\{g_1, \ldots, g_n\}$ for G with minimal n. To construct such a covering set we define a directed graph the vertices of which are the elements of G and the directed edges of which are $(g, 2g)$, $g \in G$. (We borrow this idea from [3].) If $\{g_1, \ldots, g_n\}$ is a covering set of G then the vertices of the edges

$$(g_1, 2g_1), \ldots, (g_n, 2g_n)$$

occupy all the non-zero vertices of the graph.

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When $|G|$ is odd define a relation between elements in G by setting $g \sim g'$ if there is an integer m such that $g = 2^m g'$. This is an equivalence relation which breaks G into equivalence classes that are cycles of the form $(g, 2g, 4g, \ldots)$. If each cycle other than the one containing 0 has even cardinality then G can be covered with $(|G|-1)/2$ elements. A cycle of odd length m can be covered with $(m+1)/2$ elements. Thus G can be covered by $(|G|-1)/2 + y/2$ elements, where y is the number of the cycles of odd length.

The directed graph always contains a loop on the zero node since $2 \cdot 0 = 0$. When $|G| = 2^t$ then this is the only cycle in the graph; that is, the graph is essentially a directed tree. In this case we must construct an optimal covering set for a directed tree. The nodes whose in-degrees are zero must belong to each covering set. After deleting these elements and their doubles from the graph we have a smaller tree and we may repeat the whole process again.

Finally, we will describe the structure of the graph in the general case, and we see that among the abelian groups of a given order the cyclic one admits one of the covering sets of smallest order.

3. The Case When $|G|$ is Odd

By the fundamental theorem of finite abelian groups, any finite abelian group is a direct sum of cyclic groups of orders that are powers of primes. We start with the cyclic group of order p^α, where p is an odd prime. It will be convenient to use $C(p^\alpha)$, which is the additive group of the ring of the integers modulo p^α.

The elements which are relatively prime to p^α form a cyclic multiplicative group. In other words, there is a primitive root modulo p^α, say g. The permutation

$$c \to gc, \qquad c \in C(p^\alpha)\setminus\{0\}$$

consists of α cycles:

$$(1, g, g^2, \ldots, g^{(p-1)p^{\alpha-1}}),$$
$$(p, pg, pg^2, \ldots, pg^{(p-1)p^{\alpha-2}}),$$
$$(p^2, p^2g, p^2g^2, \ldots, p^2g^{(p-1)p^{\alpha-3}}),$$
$$\ldots$$
$$(p^{\alpha-1}, p^{\alpha-1}g, p^{\alpha-1}g^2, \ldots, p^{\alpha-1}g^{(p-1)}).$$

We call the cycles of the permutation $c \to 2g$, $c \in C(p^\alpha)\setminus\{0\}$ *binary* cycles to distinguish them from the cycles formed by the powers of g.

Let $2 \equiv g^t \pmod{p^\alpha}$. (We suggest that the reader draws the corresponding graph when $G = C(27)$.) The length of the cycle $(1, 2, 2^2, \ldots)$ is r if r is the least positive integer for which $2^r \equiv g^{tr} \equiv 1 \pmod{p^\alpha}$; that is, for which $tr \equiv 0 \pmod{(p-1)p^{\alpha-1}}$. If d is the greatest common divisor of t and $(p-1)p^{\alpha-1}$, then $r = (p-1)p^{\alpha-1}/d$. Since 2 is relatively prime to p, r is the length of every cycle consisting of elements prime to p.

The length of the cycle $(p, 2p, 2^2p, \ldots)$ is r' if r' is the least positive integer with $2^{r'} \equiv g^{tr'} \equiv 1 \pmod{p^{\alpha-1}}$; that is, with $tr' \equiv 0 \pmod{(p-1)p^{\alpha-2}}$. If d' is the greatest common divisor of t and $(p-1)p^{\alpha-2}$, then $r' = (p-1)p^{\alpha-2}/d'$. This r' is the length of every cycle consisting of elements which are divisible by p but not by p^2. Similar computations hold for binary cycles of the form $(p^s, 2p^s, 4p^s, \ldots)$.

Note that either $d' = d$ or $pd' = d$, and so either $pr' = r$ or $r' = r$. Let $r_1 < r_2 < \cdots < r_s$ be all the different numbers among the lengths of the cycles. Now,

$$r_2 = pr_1, \ldots, r_s = p^{s-1}r_1.$$

This means that either all $r_1, \ldots, r_s$ are odd or all are even.

Let G and H be finite abelian groups of odd orders. The length of the cycle containing the element $g+h$ of the direct sum $G \oplus H$ is the least common multiple of the lengths of the cycles containing g and h in G and H respectively.

Now we show that the number of the cycles in $C(p^{\alpha+\beta})$ is not greater than the number of cycles in $C(p^{\alpha}) \oplus C(p^{\beta})$. To do this, let $r_1, \ldots, r_s$ be all the different numbers among the lengths of the cycles in $C(p^{\alpha+\beta})$ and let $l_1, \ldots, l_s$ be the corresponding multiplicities. Similarly, let $r_1, \ldots, r_u$ and $r_1, \ldots, r_v$ be all the different numbers among the lengths of the cycles in $C(p^{\alpha})$ and $C(p^{\beta})$ respectively. We may suppose that $u \leqslant v$. The lengths of the cycles in $C(p^{\alpha}) \oplus C(p^{\beta})$ are the least common multiples of r_i and r_j, $1 \leqslant i \leqslant u$, $1 \leqslant j \leqslant v$. All the different number among them are $r_1, \ldots, r_v$. Suppose that $m_1, \ldots, m_v$ are the corresponding multiplicities. Note that $m_1 \geqslant l_1, \ldots, m_v \geqslant l_v$ and $v \leqslant s$. If $v = s$, then clearly

$$\sum_{i=1}^{v} m_i \geqslant \sum_{i=1}^{s} l_s,$$

what we wanted to prove. If $v \leqslant s$, then from

$$\sum_{i=1}^{v} r_i m_i = \sum_{i=1}^{s} r_i l_i$$

it follows that

$$r_v \sum_{i=1}^{v} (m_i - l_i) \geqslant \sum_{i=1}^{v} r_i(m_i - l_i) = \sum_{i=v+1}^{s} r_i l_i \geqslant r_{v+1} \sum_{i=v+1}^{s} l_i$$

and so

$$\sum_{i=1}^{v} (m_i - l_i) \geqslant \sum_{i=v+1}^{s} l_i,$$

that is,

$$\sum_{i=1}^{v} m_i \geqslant \sum_{i=1}^{s} l_i$$

as we claimed.

Actually, this argument shows that if G is a finite abelian p-group of odd order n, then the number of cycles in $C(n)$ is not greater than the number of cycles in G. Furthermore, if H is an abelian group of odd order, then the number of cycles in $C(n) \oplus H$ is not greater than the number of cycles in $G \oplus H$.

Finally, we show that if G is an abelian group of order n, then the number of cycles of odd lengths in $C(n)$ is not greater than the number of cycles of odd lengths in G. Indeed, G is a direct sum of the p_1-, $\ldots$, p_s-groups $G_1, \ldots, G_s$, where $p_1, \ldots, p_s$ are the distinct prime factors of n. We have already seen that the lengths of the cycles of G_i are either all odd or all even depending only on the prime p_i. Suppose that the lengths of the cycles in $H = G_1 \oplus \cdots \oplus G_t$ are odd and the lengths of the cycles in $K = G_{t+1} \oplus \cdots \oplus G_s$ are even. We replace the prime components in H and K by cyclic groups to obtain H' and K', and the number of cycles does not increase. Clearly, $H' \oplus K'$ is isomorphic to $C(n)$ and the number of cycles of odd lengths in $H' \oplus K'$ is the number of cycles in $H' \oplus \{0\}$. Thus we have verified the conjecture when $|G|$ is odd.

4 The Case $|G| = 2^t$

Let G be an abelian group of order 2^t. By the fundamental theorem of finite abelian groups G is the direct sum of cyclic groups of orders that are powers of two. Now, we describe the structure of the graph the directed edges of which are $(g, 2g)$, $g \in G$. There is a loop on the zero element. (We suggest that the reader works out the cases

$G = C(16)$ and $G = C(2) \oplus C(8)$.) Consider the descending chain of subgroups G, $2G$, $4G$, $8G, \ldots$. The edges of the graph the initial points of which are in $G \setminus 2G$ terminate in $2G$. The edges the initial points of which are in $2G \setminus 4G$ terminate in $4G$, etc. The elements of $G \setminus 2G$, $4G \setminus 8G$, $16G \setminus 32G, \ldots$ form an optimal covering set.

When G is cyclic then $2\,|2G| = |G|$ and if a node from $2G$ is the terminal element of an edge then it is the terminal element of two edges.

When G is the direct sum of s cyclic groups, then $2^s\,|2G| = |G|$, and if a node from $2G$ is a terminal element of an edge then it is a terminal element of at least two edges. Consequently, on the average an element of the covering set is responsible for covering at most 3/2 or precisely 3/2 elements, depending on whether the group is non-cyclic or cyclic. (Note that the role of the zero element is exceptional and so we need to distinguish two cases dealing with this node.) Thus among the abelian groups of order 2^t the cycle one admits one of the smallest covering sets.

5. The General Case

Finite abelian groups are of form $G \oplus H$, where $|G| = 2^t$ and $|H|$ is odd. In the graph corresponding to $G \oplus H$ the elements of G form a tree with a loop on the zero element. Let T be this tree without the loop and call the zero element the root of T. In the graph corresponding to $G \oplus H$ the elements of H form cycles. Using these two graphs we can construct the whole graph corresponding to $G \oplus H$; namely, consider $|H|$ copies of T and identify their roots with the elements of the cycles. (Here working out the special case $G = C(8) \oplus C(3)$ may help the reader to follow the argument.) To verify this construction, consider a cycle $(h_0, h_1, \ldots, h_{l-1})$ of H and l copies of the tree T. The nodes of the new graph are elements of the direct sum $G \oplus H$. Consider the tree the root of which coincides with h_i. Suppose that the nodes in the jth level in T are labelled by $g_1, \ldots, g_t$. Now assign

$$g_1 + h_s, \ldots, g_t + h_s$$

to these nodes, where $0 \leq s \leq l-1$ and $s \equiv -j \pmod{l}$.

The arguments we have already used to construct optimal covering sets can now be applied. (We should still bear in mind that the root of a tree is always exceptional, and so it is reasonable to distinguish two cases depending on whether or not the covering set constructed for the tree covers elements of the cycle.) The final conclusion of this consideration is that among the finite abelian groups of a given order the cyclic one admits one of the optimal covering set.

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