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Quadratic and nonlinear programming problems solving and analysis in fully fuzzy environment

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Abstract This paper presents a ***comprehensive*** methodology for solving and analyzing quadratic and nonlinear programming problems in fully fuzzy environment. The solution approach is based on the Arithmetic Fuzzy Logic-based Representations, previously founded on normalized fuzzy matrices. The suggested approach is generalized for the fully fuzzy case of the general forms of quadratic and nonlinear modeling and optimization problems of both the *unconstrained* and *constrained* fuzzy optimization problems. The constrained problems are extended by incorporating the suggested fuzzy logic-based representations assuming complete fuzziness of all the optimization formulation parameters. The robustness of the optimal fuzzy solutions is then analyzed using the recently newly developed ***system consolidity index***. Four examples of quadratic and nonlinear programming optimization problems are investigated to illustrate the efficacy of the developed formulations. Moreover, consolidity patterns for the illustrative examples are sketched to show the ability of the optimal solution to withstand any system and input parameters changes effects. It is demonstrated that the geometric analysis of the consolidity charts of each region can be carried out based on specifying the type of consolidity region shape (such as elliptical or circular), slope or angle in degrees of the centerline of the geometric, the location of the centroid of the geometric shape, area of the geometric shape, lengths of principals diagonals of the shape, and the diversity ratio of consolidity points. The overall results demonstrate the consistency and effectiveness of the developed approach for incorporation and implementation for fuzzy quadratic and nonlinear optimization problems. Finally, it is concluded that the presented concept could provide a comprehensive methodology for various quadratic and nonlinear systems' modeling and optimization in fully fuzzy environments.

1. Introduction

The fuzzy systems in general can be designed to supplement the interpretation of uncertainties for real world random phenomenon. The fuzzy decision techniques allow collecting subjective data on what analyst perceive as relevant risk factors, and their relative importance, and to relatively build *individual*

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or *group* models for risk assessment. In dealing with these types of problems, several relevant techniques can be applied such as fuzzy mathematical programming, stochastic programming, fuzzy neural networks, fuzzy genetic algorithms, fuzzy particle swarm techniques, and fuzzy ant colony approach. Usually the available techniques can handle either *single objective* or *multi-objectives* formulations [1–4].

Fuzzy logic optimization is an extension of global optimization techniques operating in fuzzy environment. The classifications of the common fuzzy logic optimization techniques are elucidated in Fig. 1, [5–8]. In general, these methods represent an extension of global optimization techniques in fuzzy environment. Examples of the common fuzzy logic optimization approaches reported in the literature are fuzzy mathematical programming, fuzzy evolutionary algorithms, and fuzzy operations research techniques.

Most of these fuzzy optimization problems formulations are based on the characteristics of fuzzy goal, fuzzy constraints and fuzzy coefficients. In fuzzy environment, mathematical optimization models have to take into consideration of both flexible constraints and vague objective function. Many fuzzy optimization problems are formulated based on this conjunction.

In real life problems, various variables could have different fuzzy membership functions, fuzzy intervals and fuzzy matrices [9–12]. A normalization step has to be applied in order to unify these membership functions in one combined (compromising) function for each problem that can be applied for all situations as presented by Gabr and Dorrah [13–19]. There are many other areas in which fuzzy modeling and optimization can be used including the following: traffic systems, robotics, computers, industrial processes, biology and medicine, projects management and business. This list is by no means exhaustive. Virtually any computer decision-making system has the potential to benefit from the application of fuzzy logic for decision making under uncertainty.

2. The proposed methodology

2.1. The consolidity index

In this paper, a comprehensive methodology is presented for solving and analyzing general classes of *non-constrained* and *constrained* quadratic and nonlinear programming optimization problems in open fully fuzzy environment.[1] The approach used is by applying the arithmetic and visual fuzzy-based representation developed on the basis of normalized fuzzy matrices [13–19]. The robustness of the optimal fuzzy solutions will be then tested by the ***system consolidity index*** as defined in Appendix A [19–25].

Consolidity (the act and quality of consolidation) is a measured by the systems output reactions versus combined input/system parameters reaction when subjected to varying environments and events [1–3]. Moreover, consolidity can govern the ability of systems to withstand changes when subjected to incurring events or varying environments. In fact, consolidity is the scaling factor of managing system changes.

2.2. The consolidity chart

The analysis of the ***consolidity chart*** (or patterns chart) will be based on constructing the best geometric region that appropriately embodies all the various consolidity points obtained through the overall output fuzziness magnitude $|F_O|$ at the y-axis versus the overall combined input and system fuzziness magnitude $|F_{I+S}|$ at the x-axis. The definition of both $|F_O|$ and $|F_{I+S}|$ are given in Appendix A [22]. Such geometric region could follow many shapes such as the elliptical, circular or other forms. Furthermore, it can be analyzed for its geometric features as presented in the following table:

Symbol	Description
R	Type of consolidity region shape (elliptical, circular, or others)
Region class	Types of region classes are as follows: (i) consolidated, (ii) neutrally consolidated, (iii) unconsolidated, (iv) quasi-consolidated, (v) quasi-unconsolidated, or (vi) mixed-consolidated
S	Slope or angle in degrees of the S (degrees) = $\tan^{-1}$ (overall consolidity index)
$C = (x, y)$	Coordinates of the centroid of the geometric shape R
A	Area of the geometric shape of R in pu^2
l_1	Length of major diagonal of region (pu)
l_2	Length of minor diagonal of region (pu)
l_2/l_1	Diversity ratio of consolidity points (unitless)

Two case studies of the consolidity chart regions of elliptical and circular types are shown in Fig. 2. The analysis of the two cases can be summarized as follows:

Symbol	Meaning	Case I	Case II
R	Shape type	Elliptical	Circular
Region class	Region location	Unconsolidated	Consolidated
S	Slope	63.05°, or $\tan^{-1}$ (1.9667)	21.80°, or $\tan^{-1}$ (0.4000)
$C = (x, y)$	Centroid	(3.0,6.0)	(5.0,2.0)
A	Area (pu^2)	11.5	6.6
l_1	Length of major diagonal (*pu*)	5.75	2.90
l_2	Length of minor diagonal (*pu*)	2.55	2.90
l_2/l_1	Diversity ratio	0.4435	1.0000

In the above analysis, the position of the centroid $C = (x, y)$ (upward or downward) within main centerline depends mainly on the nature of the affected input fuzzy influences which are particular for each specific application. Higher values of such centroids mean higher fuzzy input effects or influences. In addition, a better system from the consolidity chart point of view is the one with smaller slope, smaller area A and smaller diversity ratio l_2/l_1.

[1] An "***open Fully Fuzzy Environment***" is defined as that all fuzzy levels can freely change all over the*positive* and *negative* values of the environment. A subclass of this environment is *bounded fuzzy environment* where all fuzzy levels can only change within restricted positive and negative ranges of the environment.

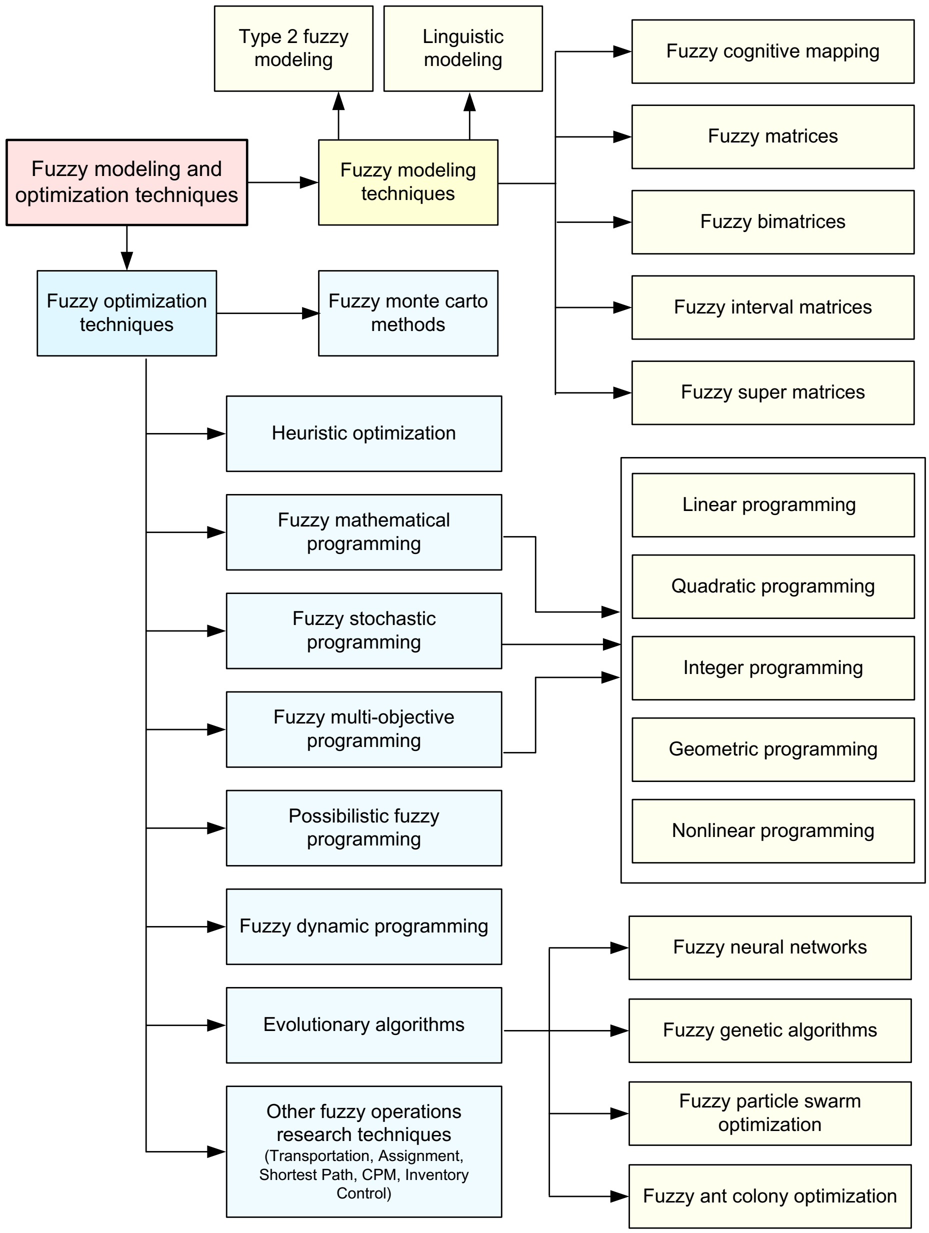

Figure 1 Various classifications of fuzzy modeling and optimization techniques.

The ***physical significance*** of the consolidity region is that it marks the boundary of all system interactive behavior resulting from all exhaustive ***internal*** and ***external*** influences. For instance, at a specific effect, the corresponding consolidity region describes all the plausible points of normalized input–output (*fuzzy* or *non-fuzzy*) interactions of such specific system.

The features of the consolidity charts will be the basis of the analysis of various optimization problems solutions given in the following sections.

3. Fuzzy quadratic programming problem

3.1. The fuzzy methodology development

A typical quadratic programming model is defined as follows [4]:

$$\text{Maximize } z = PX + X^T QX \tag{1}$$

subject to

$$DX \leqslant eX \geqslant 0 \tag{2}$$

where

$$X = (x_1, x_2, \ldots, x_n)^T \tag{3}$$

$$P = (p_1, p_2, \ldots, p_n) \tag{4}$$

$$e = (e_1, e_2, \ldots, e_m)^T \tag{5}$$

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{bmatrix} \tag{6}$$

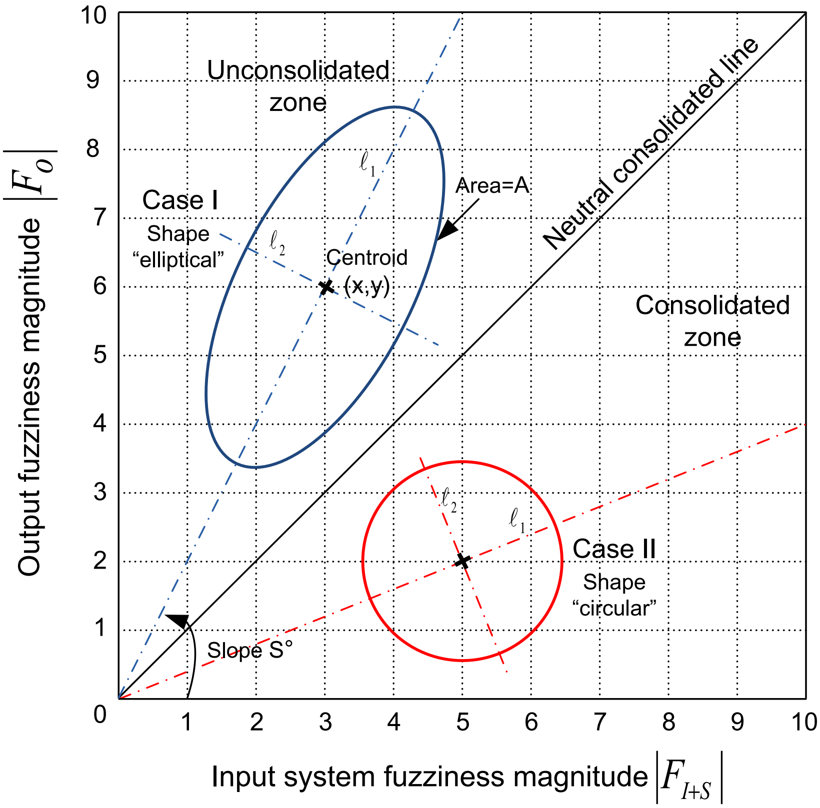

Figure 2 A sketch illustrating two case studies of the consolidity chart regions (Case I: Elliptical, Case II: Circular).

and

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}. \quad (7)$$

The function X^TQX defines a quadratic form, such that $(\cdot)^T$ indicates the transpose of $(\cdot)$. The matrix Q is assumed symmetric and negative-definite. This means that z is strictly concave. The constraints are linear which guarantees a convex solution space. The solution to this problem is based on the Kuhn–Tucker (KKT) necessary conditions. Because z is strictly concave and the solution space is convex, these conditions are also sufficient for a global optimum.

The above quadratic problem solution reduces to the compact form [4]

$$\begin{aligned} &\text{Maximize} \quad z = c_1x_1 + c_2x_2 + \cdots + c_{n'}x_{n'} \\ &\text{subject to} \quad a_{j_1}x_1 + a_{j_2}x_2 + \cdots + a_{j_n}x_{n'} \leqslant b_j \\ &\qquad\qquad j = 1, 2, \ldots, m' \\ &\qquad\qquad x_i \geqslant 0, i = 1, 2, \ldots, n' \end{aligned} \quad (8)$$

where n' comprises all problem basic variables, Lagrange multipliers, slacks, etc.

Now, incorporating the fuzzy logic arithmetic representation to the above problem of (8), we have the following fuzzy logic-based linear programming formulation:

$$\begin{aligned} &\text{Maximize } z = \underline{c}'^T\underline{x} \\ &\text{subject to } A'\underline{x} \leqslant \underline{b}' \\ &\text{and} \quad \underline{x} \geqslant \underline{0} \end{aligned} \quad (9)$$

where $(\cdot)'$ indicates the fuzzy logic arithmetic representation of $(\cdot)$, as illustrated in the following examples:

$$c_j = (c_j, \ell_{c_j}) \quad (10)$$
$$b_i = (b_i, \ell_{b_i}) \quad (11)$$

and

$$a_{ij} = (a_{ij}, \ell_{a_{ij}}) \quad (12)$$

where $\ell_{(\cdot)}$ denotes the fuzzy level of $(\cdot)$.

These fuzzy levels represent the ambiguity and uncertainty that could be found in the model parameters. They act similarly to conventional fuzzy numbers where fuzzy sets operation such as union and interaction as well as the notion of α-cuts, resolution, and the extension principle are all applicable [7]. In general, the normalized fuzzy level concept approach applied in this work is a linearized form of conventional fuzzy numbers.[2]

It has been previously elaborated that such normalized fuzzy levels concept is identical to that of the conventional fuzzy numbers for addition operations and gives average weighted fuzziness interval results of the subtraction operations. Moreover, it yields similar results of multiplications and divisions operations after ignoring the second order relative variations terms. However, the suggested approach offers

[2] The reason of introducing the approach of normalized fuzzy level had arisen from the inherent inconsistency in the fuzzy number operation. For instance if X and Y are fuzzy numbers with defined fuzzy intervals, then the fuzzy theory could lead to that: $fuzziness(X - X) \neq 0$, and $fuzziness(Y + X - X) \neq fuzziness(Y)$. Such inconsistency was solved in the normalized fuzzy level approach such that: $fuzziness(X - X) = 0$, and $fuzziness(Y + X - X) = fuzziness(Y)$.

Table 1 Modified dual Simplex tableau of the fuzzy logic-based linear programming.

x_1		x_2		...
Value	Fuzzy level	Value	Fuzzy level	
a_{11}	$\ell_{a_{11}}$	a_{12}	$\ell_{a_{12}}$	...
a_{21}	$\ell_{a_{21}}$	a_{22}	$\ell_{a_{22}}$	...
$\vdots$	$\vdots$			...
a_{i1}	ℓ_{i1}	a_{i2}	ℓ_{i2}	...
$\vdots$				$\vdots$
c_1	ℓ_{c1}	c_2	ℓ_{c2}	...

x_j				RHS		
Value	Fuzzy level			Value	Fuzzy level	
a_{1j}	$\ell_{a_{1j}}$	...	=	b_1	ℓ_{b_1}	Row 1
a_{2j}	$\ell_{a_{2j}}$	...	=	b_2	ℓ_{b_2}	Row 2
		...	=	$\vdots$	$\vdots$	
a_{ij}	$\ell_{a_{ij}}$	...	=	b_i	ℓ_{b_i}	Row i
			=	$\vdots$		
c_i	ℓ_{c_i}	...	=	0	0	z

additional advantages of linearity, reversibility, simplicity, and applicability [18,19].

The methodology is based on two parts. The first is the solution of the original (quadratic or nonlinear) optimization problem in exact forms which is presented in detail. The second part is the calculation of the corresponding fuzzy level at each step of computation which is straightforward following the fuzzy operation rules of $(+,-,*,/)$. Detailed methodology for consolidity calculations was published recently in reference [22]. During implementation procedure in this paper, the exact fractional values of fuzzy levels are preserved all over the calculations and are rounded to integer values only for presentation at the final results.

Following the above representation, the Simplex tableau can be expressed as shown in Table 1. The corresponding modified fuzzy logic-based algorithm is a direct extension to the Simplex Method algorithm.. It follows that for each iteration, we have for the corresponding fuzzy levels:

Pivot row:

$$\ell_{a_{kj}} = (\ell_{a_{kj}} - \ell_{a_{km}}) \tag{13}$$

Pivot columns:

$$\text{All fuzzy logic levels} = 0 \tag{14}$$

Other coefficients:

$$\ell_{a_{ij}} = \ell\{a_{ij} - a_{im} \cdot a_{ki}/a_{km}\} \tag{15}$$

and

$$\ell_{c_i} = \ell\{c_i - c_m a_{ki}/a_{km}\}, \tag{16}$$

where $\ell\{\cdot\}$ is an operator that denotes the fuzzy level of $\{\cdot\}$.

The calculation of the Simplex tableau of Table 1 can then be expressed using the pivoting rules into two steps:

Step 1:

Let

$$a'_{ij} = a_{im}a_{ki}/a_{km} \tag{17}$$

then

$$\ell\{a'_{ij}\} = \ell\{a_{im} \cdot a_{ki}/a_{km}\} = \ell\{a_{im}\} + \ell\{a_{ki}\} - \{a_{km}\}. \tag{18}$$

Step 2:

It follows then that

$$\ell_{a_{ij}} = \ell\left\{a_{ij} - a'_{ij}\right\} = \frac{a_{ij}\ell_{a_{ij}} - a'_{ij}\ell_{a'_{ij}}}{a_{ij} - a'_{ij}}. \tag{19}$$

A simple function on spreadsheet model can be easily programmed to calculate such Simplex method iteration directly and obtaining the corresponding fuzzy level for each cell at each step of calculations. In general, such fuzzy levels could be fractional all over the calculations steps. As far as (19) gives the new fuzzy level as a weighted average of the two other bounded levels, it can be proven that the new values of these fuzzy level will also be bounded as far as the solutions of the Simplex algorithm are bounded. Mathematical proof of the boundedness of the fuzzy levels during solution iterations is left for future research.

3.2. Illustrative example 1

Consider the quadratic programing optimization problem [4,16]:

Maximize

$$z = p_1 \cdot x_1 + p_2 \cdot x_2 + q_{11} \cdot x_1^2 + q_{12} \cdot x_1 \cdot x_2 + q_{21} \cdot x_1 \cdot x_2 + q_{22} \cdot x_2^2 \tag{20}$$

subject to

$$d_1x_1 + d_2x_2 \leqslant e_1 \tag{21}$$

$$x_1, x_2 \geqslant 0.$$

The problem can be expressed in matrix form as follows:

$$\text{Maximize } z = [p_1, p_2] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [x_1, x_2] \cdot \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{22}$$

subject to

$$[d_1, d_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leqslant e_1 \quad x_1, x_2 \geqslant 0. \tag{23}$$

The Kuhn–Tucker conditions are given as [4]

$$\begin{bmatrix} -2q_{11} & -2q_{12} & d_1 & -1 & 0 & 0 \\ -2q_{21} & -2q_{22} & d_2 & 0 & -1 & 0 \\ d_1 & d_2 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \mu_1 \\ \mu_2 \\ s_1 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ e_1 \end{bmatrix}. \tag{24}$$

Eq. (24) can be formulated as two-phase linear programming for the following numerical values given in Table 2, describing the input of the numerical example variables values and their corresponding fuzzy levels.

The developed approach is applied for simplicity using spreadsheet representation with Visual Basic Applications (VBA) programming and obtaining the corresponding fuzzy level for each cell at each step of calculations. However, the approach is general and can be applied to other unlimited forms of representations and other known programming software. Using the two phase Simplex technique, the final output results of different scenarios are elucidated in Table 3. The results of different scenarios of this table provide a good insight to the effect of various input fuzziness on the fuzziness of the optimized outputs.

Table 2 Input variables values of quadratic fuzzy optimization of ***Illustrative example 1***.

Ser	Parameter	Value	Fuzzy levels scenario no						
			I	II	III	IV	V	VI	VII
1	p_1	4	−3	−2	−1	1	2	3	1
2	p_2	6	−3	−2	−1	1	2	3	3
3	q_{11}	−2	3	2	1	−1	−2	−3	2
4	q_{12}	−1	−3	−2	−1	1	2	3	1
5	q_{21}	−1	−3	−2	−1	1	2	3	2
6	q_{22}	−2	3	2	1	−1	−2	−3	1
7	d_1	1	1	1	1	−1	−1	−1	1
8	d_2	2	1	1	1	−1	−1	−1	1
9	e_1	2	−1	−1	−1	1	1	1	−1

The numerical application of the fuzzy logic-based quadratic programming demonstrates that the proposed technique is highly pragmatic and easy to be applied for analyzing fuzziness in various numerical calculations.

The problem ***consolidity index*** is also shown in the Table 3. The input for the consolidity analysis was the overall fuzziness of all input parameters of Table 3 while the output was taken as the fuzziness of the performance index z. The consolidity chart of the problem described by plotting the overall output fuzziness factor $|F_O|$ versus input fuzziness factor $|F_{(S+I)}|$ is shown in Fig. 3.

Applying the geometric analysis to the consolidity chart of ***Illustrative example 1*** elucidated in Fig. 3, we obtain the following results:

Symbol	Meaning	Results
R	Shape type	Elliptical
Region Class	Region location	Quasi-unconsolidated
S	Slope	56.99°, or $\tan^{-1}$ (1.5392)
$C=(x,y)$	Centroid	(2.4,3.7)
A	Area (pu^2)	12.4
l_1	Length of major diagonal (pu)	5.8
l_2	Length of minor diagonal (pu)	2.7
l_2/l_1	Diversity ratio	0.4655

The results elucidate that the consolidity region has a *moderate* overall consolidity index and relatively *high* diversity ratio. Furthermore, both the area of the consolidity region R and the diversity ratio are *moderate* supporting the moderate diversity of calculated consolidity points.

In real life systems the overall system consolidity will be normally bounded indicating the robustness of the proposed calculations scheme.[3] These are due to the internal compensating effects of different fuzziness in the parameters incorporated within these real life systems.

[3] It is remarked that the typical ranges of the consolidity indices $F_{O/(I+S)}$ based on previous real life applications are as follows: ***very low*** (< 0.5), ***low*** (0.5–1.5), ***moderate*** (1.5–5), ***high*** (5–15), and ***very high*** (> 15) [20–24]. A good practical consolidated system should have the value of consolidity index $F_{O/(I+S)} = 1.5$.

4. Fuzzy unconstrained nonlinear programming problem

In this section, we demonstrate the applicability of the proposed Arithmetic Fuzzy Logic-based Representation approach to fully fuzzy nonlinear optimization problems by two representative illustrative examples [4].

4.1. Illustrative example 2

Consider the nonlinear function F expressed as [4, 17]

$$F = f(x,y,z) = ax^2 + bxy + cy^2 + dyz + ez^2 - fx - gy - hz + i \quad (25)$$

such that all coefficients are fuzzy parameters.

The critical point of f can be determined using the first-order necessary conditions as follows:

$$\frac{\partial f(x,y,z)}{\partial x} = 2ax + by - f = 0 \quad (26)$$

$$\frac{\partial f(x,y,z)}{\partial y} = bx + 2cy + dz - g = 0 \quad (27)$$

and

$$\frac{\partial f(x,y,z)}{\partial z} = dy + 2ez - h = 0. \quad (28)$$

The conditions can be expressed in the following matrix equation

$$\begin{bmatrix} 2a & b & 0 \\ b & 2c & d \\ 0 & d & 2e \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}. \quad (29)$$

The problem is solved for the numerical parameters shown in Table 4 for different scenarios of input fuzzy levels.

It can easily be checked that the above solution is a local minimum verifying the second-order sufficient conditions. The input for the consolidity analysis was the overall fuzziness of all input parameters of Table 4 while the output was taken as the fuzziness of the performance function F. The consolidity chart of the problem described by plotting the overall output fuzziness factor $|F_O|$ versus input fuzziness factor $|F_{(S+I)}|$ is shown in Fig. 4.

Applying the geometric analysis to the consolidity chart of ***Illustrative example 2*** shown in Fig. 4, we get the following results:

Symbol	Meaning	Results
R	Shape type	Elliptical
Region class	Region location	Unconsolidated
S	Slope	73.67°, or $\tan^{-1}$ (3.4136)
$C=(x,y)$	Centroid	(2.22, 7.35)
A	Area (pu^2)	53.2
l_1	Length of major diagonal (pu)	12.45
l_2	Length of minor diagonal (pu)	5.40
l_2/l_1	Diversity ratio	0.4337

Table 3 Output results of quadratic fuzzy optimization of ***Illustrative example 1***.

Parameter	Optimal value	Fuzzy levels scenario no						
		I	II	III	IV	V	VI	VII
x_1	0.3333	−6	−3	−3	3	3	6	−3
x_2	0.8333	−1	−1	−1	1	1	1	−2
λ_1	1.0000	−1	0	1	−1	0	1	1
z	4.1666	−6	−4	−2	2	4	6	2
Calculated consolidity index[a] $F_{O/(I+S)}$		1.5429	1.5652	1.6364	1.6364	1.5652	1.5429	1.2857

[a] Average value of consolidity index $F_{O/(I+S)} = \mathbf{1.5392}$.

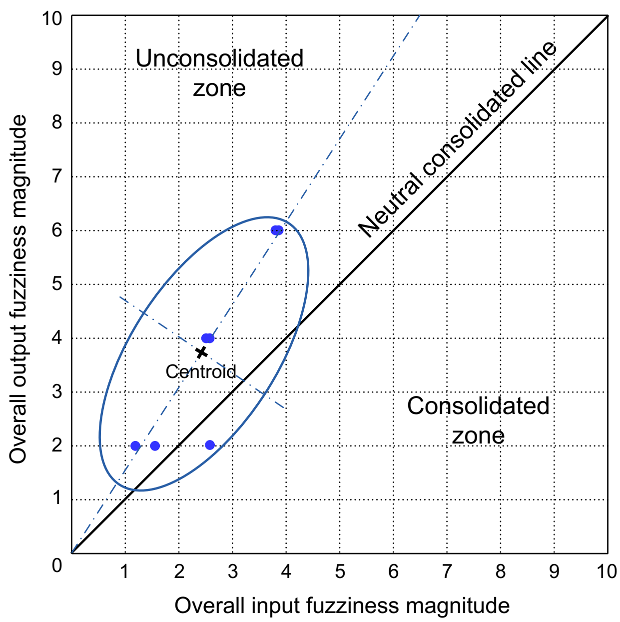

Figure 3 Consolidity region (quasi-unconsolidated class) of the optimal solution fuzziness results of ***Illustrative example 1***.

The results indicate that the consolidity region has a *moderate* overall consolidity index. Moreover, the area of the consolidity region R is extremely large and the diversity ratio is relatively *above moderate* levels supporting the *very high* diversity of calculated consolidity points.

4.2. Illustrative example 3

Determine the local minimum of the fuzzy function expressed as [4,17]:

$$f(x,y) = ay + bxe^{cy} + dx^3 \tag{30}$$

where a, b, c and d are fuzzy parameters.

The local minimum of $f(x,y)$ satisfies the first order necessary conditions, expressed as follows:

$$\text{(i)}\ \frac{\partial f(x,y)}{\partial x} = 0 \quad be^{cy} + 3\ d\ x^2 = 0 \tag{31}$$

$$\text{(ii)}\ \frac{\partial f(x,y)}{\partial y} = 0 \quad a + bcxe^{cy} = 0. \tag{32}$$

From (i) and (ii) we have

$$x^3 = \frac{a}{3cd}. \tag{33}$$

This gives

$$x = \sqrt[3]{\left(\frac{a}{3cd}\right)} \tag{34}$$

and

$$y = \frac{1}{c} \cdot \ln\left(\frac{a}{-bcx}\right). \tag{35}$$

The numerical solution of the illustrative example 3 is shown in Table 5 for different input fuzziness level scenarios.

The input for the consolidity analysis was the overall fuzziness of all input parameters of Table 5 while the output was taken as the fuzziness of the performance index $f(x,y)$. The consolidity chart of the problem described by plotting the overall output fuzziness factor $|F_O|$ versus input fuzziness factor $|F_{(S+I)}|$ is shown in Fig. 5.

Applying the geometric analysis to the consolidity chart of ***Illustrative example 3*** sketched in Fig. 5, yields the following results:

Symbol	Meaning	Results
R	Shape type	Elliptical
Region class	Region location	Quasi-unconsolidated
S	Slope	62.29° or $\tan^{-1}$ (1.9039)
$C = (x,y)$	Centroid	(2.5, 4.8)
A	Area (pu^2)	22.5
l_1	Length of major diagonal (pu)	8.9
l_2	Length of minor diagonal (pu)	3.2
l_2/l_1	Diversity ratio	0.3596

The results show that the consolidity region has a *moderate* overall consolidity index and relatively *moderate* diversity ratio. Moreover, the area of the consolidity region R is *moderate*supporting the *moderate* diversity of calculated consolidity points.

Table 4 Results of the nonlinear fuzzy optimization of ***Illustrative example 2***.

Parameter	Value	Fuzzy levels scenario no								
		I	II	III	IV	V	VI	VII	VIII	IX
a	2	1	3	−3	−2	4	2	1	1	2
b	1	2	3	3	−4	3	4	1	2	2
c	1	1	−3	−3	−3	2	3	2	2	1
d	1	1	1	1	3	1	1	2	2	1
e	1	2	−2	−2	−2	1	4	1	1	3
f	6	3	1	1	2	1	2	1	2	3
g	7	2	4	4	4	2	1	2	3	1
h	8	1	1	3	3	1	2	1	3	2
i	9	1	2	3	4	2	1	2	1	1
x	1.636	2	−1	4	3	−2	1	0	1	1
y	−0.545	4	8	−4	−1	5	0	1	2	1
z	4.273	−1	4	5	5	0	−2	0	−2	-1
F	2.595	9	9	12	−6	10	2	3	−3	2
Calculated consolidity index[a] $F_{O/(I+S)}$		5.6218	5.2762	5.9064	2.2698	5.5511	1.2456	2.2390	1.2852	1.3271

[a] Average value of consolidity index $F_{O/(I+S)} = \mathbf{3.4136}$.

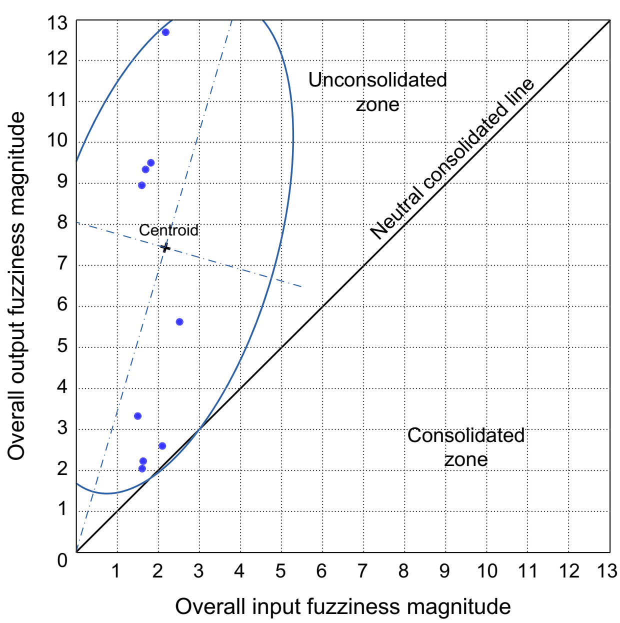

Figure 4 Consolidity region (unconsolidated class) of the optimal solution fuzziness results of ***Illustrative example 2***.

5. Fuzzy constrained nonlinear programming problem using Lagrangean technique

5.1. The fuzzy methodology development

Consider the general classical nonlinear optimization formulation, expressed as [4]:

$$\text{Minimize } Z = f(X) \tag{36}$$

subject to

$$g(X) = 0 \tag{37}$$

where $X = (x_1, x_2, \ldots, x_n)$ and $g = (g_1, g_2, \ldots, g_m)^T$. The functions $f(X)$ and $g_i(X)$, $i = 1, 2, \ldots, m$ are twice continuously differentiable.

Define

$$L(X, \lambda) = f(X) - \lambda \cdot g(X) \tag{38}$$

such that L designates the Lagrangean function of the problem and the parameters λ are the Lagrange multipliers.

The equations

$$\frac{\partial L}{\partial \lambda} = 0, \quad \frac{\partial L}{\partial X} = 0 \tag{39}$$

Table 5 Results of the nonlinear fuzzy optimization of ***Illustrative example 3***.

Parameter	Value	Fuzzy levels scenario no								
		I	II	III	IV	V	VI	VII	VIII	IX
a	6	1	3	−1	−3	5	4	2	2	5
b	−1	−2	−5	−2	2	4	5	1	4	4
c	1	4	4	1	−2	2	4	3	1	−1
d	2	−3	3	−2	−1	3	2	−3	2	2
x	1	0	−1	0	0	1	−1	1	0	1
y	ln(6)	−5	−6	−1	0	3	−6	−4	−2	1
$f(x, y)$	6.7506	−2	−5	−2	−4	8	−4	−3	−1	7
Calculated consolidity index[a] $F_{O/(I+S)}$		2.4017	1.7905	2.2857	1.2221	2.2617	1.1412	3.3702	0.8432	1.8186

[a] Average value of consolidity index $F_{O/(I+S)} = \mathbf{1.9039}$.

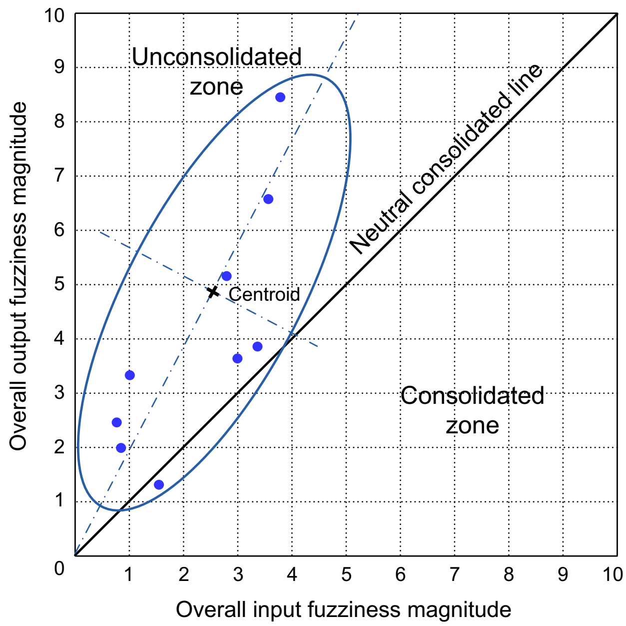

Figure 5 Consolidity region (quasi-unconsolidated class) of the optimal solution fuzziness results of ***Illustrative example 3***.

provide the necessary conditions for determining stationary points of $f(X)$ subject to $g(X)=0$. The sufficiency conditions for the Lagrangean method can be stated as follows [4]. Define

$$H^B = \begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix}_{(m+n)\times(m+n)} \tag{40}$$

where

$$P = \begin{pmatrix} \nabla_{g_1}(X) \\ \vdots \\ \nabla_{g_m}(X) \end{pmatrix}_{m\times n} \tag{41}$$

and

$$Q = \left\| \frac{\partial^2 L(X,\lambda)}{\partial x_i \partial x_j} \right\|_{n\times n} \quad \text{for all } i \text{ and } j. \tag{42}$$

The matrix H^B is the bordered ***Hessian matrix***.

Given the stationary point (X_0, λ_0) for the Lagrangean function $L(X,\lambda)$ and the bordered Hessian matrix H^B evaluated at (X_0, λ_0), then X_0 is:

(i) A maximum point if, starting with the principal major determinant of order $(2m+1)$, the last $(n-m)$ principal minor determinants of H^B form an alternating sign pattern starting with $(-1)^{m-1}$.

(ii) A minimum point if, starting with the principal minor determinant of order $(2m+1)$, the last $(n-m)$ principal minor determinants of H^B have the sign $(-1)^m$.

These conditions are sufficient, but not necessary, for identifying an extreme point. This means that a stationary point may be an extreme point without satisfying these conditions.

5.2. Illustrative example 4

Let us consider the following nonlinear constrained fuzzy optimization problem [4,17]:

$$\text{Minimize } f(X) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 \tag{43}$$

subject to

$$g_1(X) = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - b_1 = 0 \tag{44}$$

and

$$g_2(X) = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - b_2 = 0 \tag{45}$$

Accordingly, the Lagrangean function of the problem as defined in (38) can be expressed as

$$L(X,\lambda) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - \lambda_1(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - b_1) - \lambda_2(a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - b_2) \tag{46}$$

This yields the following necessary conditions:

$$\frac{\partial L}{\partial x_1} = 2\cdot c_1\cdot x_1 - \lambda_1 a_{11} - \lambda_2\cdot a_{21} = 0 \tag{47}$$

$$\frac{\partial L}{\partial x_2} = 2\cdot c_2\cdot x_2 - \lambda_1\cdot a_{12} - \lambda_2\cdot a_{22} = 0 \tag{48}$$

$$\frac{\partial L}{\partial x_3} = 2\cdot c_3\cdot x_3 - \lambda_1\cdot a_{13} - \lambda_2\cdot a_{23} = 0 \tag{49}$$

$$\frac{\partial L}{\partial \lambda_1} = -(a_{11}\cdot x_1 + a_{12}\cdot x_2 + a_{13}\cdot x_3 - b_1) = 0 \tag{50}$$

and

$$\frac{\partial L}{\partial \lambda_2} = -(a_{21}\cdot x_1 + a_{22}\cdot x_2 + a_{23}\cdot x_3 - b_2) = 0. \tag{51}$$

Table 6 Results of the nonlinear fuzzy optimization of ***Illustrative example 4***.

Parameter	c_1	c_2	c_3	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	b_1	b_2	$f(\cdot)$	$F_{O/(I+S)}$ [a]
Value	1	1	1	1	1	3	5	2	1	2	5	0.8478	
Fuzzy levels	1	−1	−1	1	1	−1	−2	−2	−2	−2	−1	1.8796	1.6011
	−3	3	4	4	−3	4	3	3	3	−3	3	−2.6706	1.2285
	−2	2	3	3	−2	3	2	2	2	−2	2	−1.8055	1.1864
	−1	1	2	1	−1	3	2	1	1	−1	1	−2.3885	2.0347
	1	−1	−2	1	1	−1	−2	−1	−1	−2	−1	1.4342	1.3195
	2	−2	−2	2	2	−2	−2	−2	−2	2	−2	1.4002	1.2387
	3	−3	−3	2	2	−1	−3	−3	−3	−3	3	1.9621	1.3273
	2	−2	−2	3	2	3	1	−1	3	2	2	2.9744	2.1378
	1	−2	−1	1	2	−2	−3	−2	−3	2	−2	2.3590	1.6441

[a] Average calculated value of $F_{O/(I+S)} = \mathbf{1.5242}$.

The solution to these simultaneous equations of the illustrative example 4 yields the results shown in Table 6. The table also provides the consolidity analysis of the problem. The solution also includes: $X_0 = (x_1, x_2, x_3) = (0.8043, 0.3478, 0.2826)$ and $\lambda = (\lambda_1, \lambda_2) = (0.0870, 0.3043)$.

To show that the given point is a minimum, consider the Hessian matrix:

$$H^B = \left(\begin{array}{cc|ccc} 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{21} & a_{22} & a_{23} \\ \hline a_{11} & a_{21} & 2c_1 & 0 & 0 \\ a_{12} & a_{22} & 0 & 2c_2 & 0 \\ a_{13} & a_{23} & 0 & 0 & 2c_3 \end{array}\right) \quad (52)$$

Because $n = 3$ and $m = 2$, $n - m = 1$, and we need to check the determinant of H^B only, which must have the sign of $(-1)^2$ for the stationary point X_0 to be a minimum. Because determinant of $H^B = 460 > 0$, X^0 is a minimum point. It can be easily checked that the above solution is a local minimum verifying the second-order sufficient conditions.

The input for the consolidity analysis was the overall fuzziness of all input parameters of Table 5 while the output was taken as the fuzziness of the performance index $f(x, y)$. The consolidity pattern of the problem described by plotting the overall output fuzziness factor $|F_O|$ versus input fuzziness factor $\left|F_{(S+I)}\right|$ is presented in Fig. 6.

Applying the geometric analysis to the consolidity chart of ***Illustrative example 4*** shown in Fig. 6 for $a_{11} = 1$, gives the following results:

Symbol	Meaning	Results
R	Shape type	Circular
Region class	Region location	Quasi-consolidated
S	Slope	56.73°, or $\tan^{-1}$ (1.5242)
$C = (x, y)$	Centroid	(1.35, 2.20)
A	Area (pu^2)	2.3
l_1	Length of major diagonal (pu)	1.7
l_2	Length of minor diagonal (pu)	1.7
l_2/l_1	Diversity ratio	1.0

The results show that the consolidity region has a *moderate* overall consolidity index and relatively *high* diversity ratio. Though the diversity ratio is *very high* due to the circular shape of the region, the corresponding area of the consolidity region R is *very small* supporting the low diversity of calculated consolidity points.

6. Fuzzy constrained nonlinear programming problem using Jacobian technique

For the nonlinear optimization problem described in (36) and (37), define [4]

$$X = (Y, Z) \quad (53)$$

such that

$$Y = (y_1, y_2, \ldots, y_m), \quad Z = (z_1, z_2, \ldots, z_{n-m}) \quad (54)$$

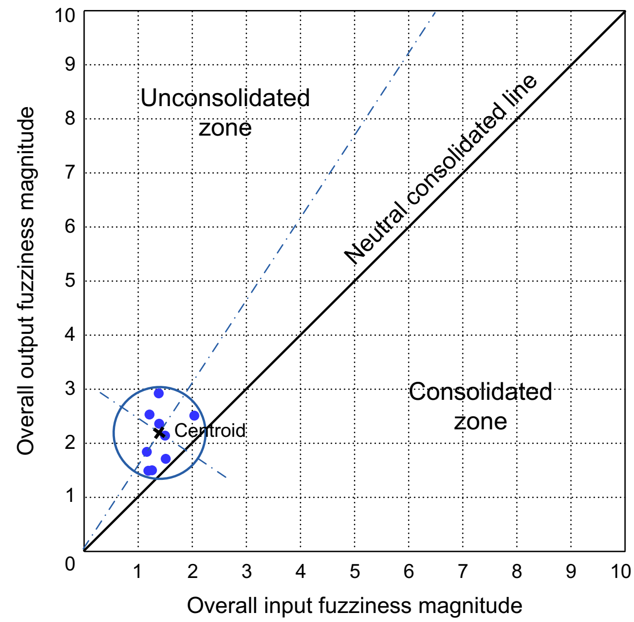

Figure 6 Consolidity region (quasi-unconsolidated class) of the optimal solution fuzziness results for $a_{11} = 1$ of ***Illustrative example 4***.

The vectors Y and Z are called the dependent and independent variables, respectively. Rewriting the gradient vectors of f and g in terms of Y and Z, we get

$$\nabla f(Y, Z) = (\nabla_y f, \nabla_z f) \quad (55)$$

and

$$\nabla g(Y, Z) = (\nabla_y g, \nabla_z g) \quad (56)$$

Define

$$J = \nabla_y g = \begin{pmatrix} \nabla_y g_1 \\ \vdots \\ \nabla_y g_m \end{pmatrix} \quad (57)$$

and

$$C = \nabla_z g = \begin{pmatrix} \nabla_z g_1 \\ \vdots \\ \nabla_z g_m \end{pmatrix} \quad (58)$$

where $J_{m\times n}$ denotes the Jacobian matrix and $C_{m\times n-m}$ the control matrix. The Jacobian J is assumed nonsingular. This is always possible because the given m equations are independent by definition. The components of the vector Y must thus be selected such that the matrix J is nonsingular.

The original set of equations in $\partial f(X)$ and ∂X can be written as follows:

$$\partial f(Y, Z) = \nabla_y f \cdot \partial Y + \nabla_z f \cdot \partial Z \quad (59)$$

and

$$J \cdot \partial Y = -C \cdot \partial Z. \quad (60)$$

Because J is nonsingular, its inverse J^{-1} exists. Hence,

$$\partial Y = -J^{-1} \cdot C \cdot \partial Z. \quad (61)$$

Substituting for ∂Y in (59) of $\partial f(X)$ gives ∂f as a function of ∂Z, that is,

$$\partial f(Y,Z) = (\nabla_z f - \nabla_y f . J^{-1} \cdot C)\partial Z \tag{62}$$

From this equation, the constrained derivative with respect to the independent vector Z is given by

$$\nabla_c f = \frac{\partial_0 f(Y,Z)}{\partial_0 Z} = \nabla_z f - \nabla_y f \cdot J^{-1} \cdot C \tag{63}$$

where $\nabla_c f$ is the constrained gradient vector of f with respect to Z. Thus $\nabla_c f(Y,Z)$ must be null at the stationary points.

The Hessian matrix will correspond to the independent vector Z, and the elements of the Hessian matrix must be the constrained second derivatives. To show how this is obtained, let

$$\nabla_c f = \nabla_z f - W \cdot C \tag{64}$$

It thus follows that the ith row of the (constrained) Hessian matrix is $\partial \nabla_c f / \partial z_i$. The parameter W is a function of Y and Y is a function of Z. Thus, the partial derivative of $\nabla_c f$ with respect to z_i is based on the following chain rule:

$$\frac{\partial w_j}{\partial z_i} = \frac{\partial w_j}{\partial y_j}\frac{\partial y_j}{\partial z_i}. \tag{65}$$

The ***illustrative example 4*** described in (43)–(45) with its input parameters values given in Table 6 is now solved using the Jacobian Method. To determine the constrained extreme points, let [4]:

$$Y = (x_1, x_2) \text{ and } Z = x_3. \tag{66}$$

The equations for determining the stationary points are thus given as [4]:

$$\begin{aligned} \nabla_c f &= 0 \\ g_1(X) &= 0 \\ g_2(X) &= 0 \end{aligned} \tag{67}$$

or

$$\begin{pmatrix} \alpha & \beta & \gamma \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ b_1 \\ b_2 \end{pmatrix} \tag{68}$$

such as

$$\alpha = 2c_1 d^{-1}(-a_{22}a_{13} + a_{12}a_{23}),$$
$$\beta = 2c_2 d^{-1}(a_{21}a_{13} - a_{11}a_{23}) \quad \text{and} \quad \gamma = 2c_3.$$

For this numerical example, we have

$$\alpha = \frac{10}{3}, \quad \beta = -\frac{28}{3} \quad \text{and} \quad \gamma = 2.$$

The solution is $X^0 = (0.8043, 0.3478, 0.2826)$ and $\partial_0^2 f / \partial_0 x_3^2 = \frac{460}{9} > 0$. Hence X^0 is the minimum point and the objective function $f(X) = 0.8478$.

It is pointed out that both the ***Lagrangean Function method*** and the ***Jacobian technique*** have given identical results for all the parameters solutions and their corresponding fuzzy levels. The only difference was for the fuzzy levels of the sufficiency conditions given by the Hessian Matrix H^B and the derivative $\partial_0^2 f / \partial_0 x_3^2$ as they indicate different formulas.

In order to analyze more the proposed fuzzy logic based formulation, six different scenarios of the same ***illustrative example 4*** were designed as shown in Table 7. The results of solving the scenarios using both the Lagrangean Function method and the Jacobian technique are shown in the same table. These identical results demonstrate the consistency and robustness of the developed approach for incorporation with classical nonlinear optimization problems for different selected levels of fuzziness.

The results indicated, on the other hand, that the sufficiency condition of the ***Jacobian technique*** ($\partial_0^2 f / \partial_0 x_3^2$) is more susceptible to variations of input parameters' fuzziness than the ***Lagrangean Function approach*** (H^B).

In order to obtain a more in-depth analysis of the effect of changing system parameters on the shape and features of the consolidity region is investigated. Several scenarios of changing parameter a_{11} are studied using the same fuzziness levels of Table 6 for the ***Illustrative example 4***. The corresponding consolidity charts are shown in Fig. 7, and the results of consolidity chart analysis are summarized in Table 8. These results reveal appreciable shifts in consolidity index from moderate value of 1.3734 to a higher value of 5.0613. Furthermore, the areas also varied from 2.3 to 14.1 pu^2, but still are within the small and moderate values. The vertical value of the centroid also moves upward from 1.75 to 5.6 pu, while changes in the horizontal value of the centroid are limited between 1.05 and 2.20 pu.

7. Additional comments on the proposed approach

The proposed fuzzy logic-based quadratic and nonlinear programming optimization has many advantages over other reported techniques such as the stochastic programming' chance constraints programming, and the perturbation techniques, these are:

Table 7 Results comparison of the nonlinear fuzzy optimization of ***Illustrative example 4***.

Parameter	Value	Corresponding fuzzy level of different scenarios						
		I′	II′	III′	IV′	V′	VI′	VII′
c_1	1	−3	−2	−1	1	2	3	1
c_2	1	3	2	1	−1	−2	−3	1
c_3	1	−3	2	2	−2	−3	−4	2
$a_1 1$	1	3	2	1	−1	−2	−3	3
$a_1 2$	1	−3	−2	−1	1	2	3	1
$a_1 3$	3	−4	−1	−1	1	2	−4	3
$a_2 1$	5	3	2	1	−1	−2	−3	−1
$a_2 2$	2	3	2	1	−1	−2	−3	−1
$a_2 3$	1	3	2	1	−1	−2	−3	3
b_1	2	−1	−2	−1	1	2	4	2
b_2	5	3	2	1	−1	−2	−3	−3
x_1	0.8043	1	1	0	0	−1	−1	−3
x_2	0.3478	−5	−3	−2	2	3	5	2
x_3	0.2826	1	−1	−2	2	3	6	1
λ_1	0.0870	4	−4	3	−3	−3	−2	3
λ_2	0.3043	−6	4	−2	2	4	5	0
$f(X)$	0.8478	−2	−1	−1	1	1	2	−2
H^B	460	−1	1	0	0	−1	1	1
$\partial_0^2 f / \partial_0 x_3^2$	51.11	7	6	3	−3	−6	−7	12[a]

[a] Very high value for fuzzy level.

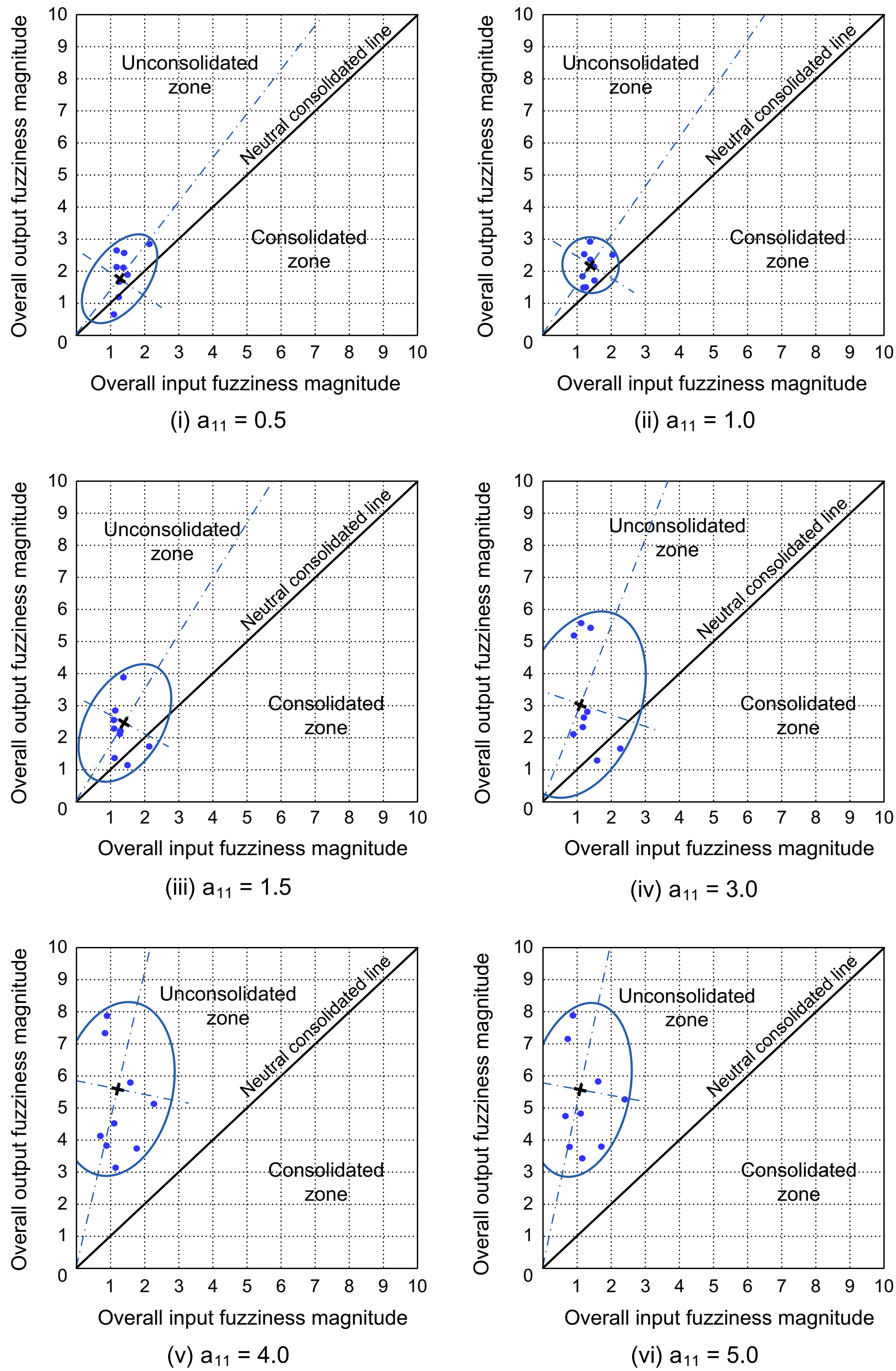

Figure 7 Consolidity regions (unconsolidated or quasi-consolidated classes) of the optimal solution fuzziness results of selected values of parameter a_{11} of ***Illustrative example 4***.

Table 8 Results of consolidity regions analysis of the optimal solution corresponding to different values of parameter a_{11} of ***Illustrative example 4***.

Symbol	Results of changes of parameter a_{11}					
	0.5	1.0	1.5	3.0	4.0	5.0
R Region class	Elliptical Quasi-unconsolidated	Circular Quasi-unconsolidated	Elliptical Quasi-unconsolidated	Elliptical Quasi-unconsolidated	Elliptical Unconsolidated	Elliptical Unconsolidated
S	53.94° or $\tan^{-1}$ (1.3734)	56.73° or $\tan^{-1}$ (1.5242)	59.97° or $\tan^{-1}$ (1.7300)	74.95° or $\tan^{-1}$ (3.7198)	77.71° or $\tan^{-1}$ (4.5911)	78.82° or $\tan^{-1}$ (5.0613)
$C = (x, y)$	(1.20, 1.75)	(1.35, 2.20)	(1.40, 2.45)	(1.15, 4.40)	(2.20, 5.60)	(1.05, 5.50)
A (pu^2)	4.2	2.3	7.1	12.4	14.1	12.7
l_1 (pu)	3.1	1.7	3.9	4.9	5.5	5.5
l_2 (pu)	1.7	1.7	2.3	3.2	3.3	2.9
l_2/l_1	0.5484	1.0000	0.5897	0.6531	0.6000	0.5273

(i) The suggested approach can be incorporated simply in spreadsheet models and obtaining the corresponding fuzzy level for each cell at each step of calculations. Such spreadsheet representation can accommodate an extended number of parameters and variables with present scales of fuzzy levels. No special sophisticated software is needed for algorithms implementation; only macros built using normal Visual Basic Applications (VBA) package.

(ii) The proposed approach is general in such a way that all problem's coefficients and parameters can associate with their values corresponding fuzzy logic levels all over the steps of the problem solution, and satisfying the conditions of the normalized fuzzy matrices.

(iii) The presented consolidity analysis of the results through representable consolidity charts represents an effective way for examining the ability of the optimal solution for withstanding changes due to input parameters changes effects that takes place "***on and above***" normal situations and stands [26–29].

(iv) For future implementations, all the basic fuzzy operations, fuzzy functions and matrices operations, as well as fuzzy optimization operations could be transferred as built-in function in special computational Toolbox in Matlab or to be created as special functions inside other likes software languages [22]. The building of such library will strengthen the capability of the consolidity chart to effectively handle various types of optimization regardless of their dimensionality, types and complexities.

8. Conclusions

It was illustrated using an illustrative example the effectiveness of incorporating the consolidity chart in the solving and analysis of the quadratic and nonlinear programming problems in a fully fuzzy environment. It was also shown that optimal solution robustness against change can be easily checked at the final solution using the newly developed notion of the ***system consolidity index***.

Consolidity results charts of the fuzzy optimal solution were sketched for each illustrative example revealing the degree of susceptibility of the optimal solution for withstanding changes due to any system or input parameters changes effects. These results demonstrated the consistency and effectiveness of the developed approach for incorporation with quadratic and nonlinear optimization problems solving and analysis.

It was also demonstrated that the geometric analysis of the consolidity charts of each region can be carried out based on specifying the type of consolidity region shape (such as elliptical or circular), slope or angle in degrees of the centerline of the geometric, the location of the centroid of the geometric shape, area of the geometric shape, lengths of principals diagonals of the shape, and the diversity ratio of consolidity points.

It is pointed out that the suggested approach opens the door toward more future extensions of the proposed approach to other fuzzy global optimization techniques and for solving other classes of mathematical programming problems, such as geometric programming, goal programming, integer and mixed Integer programming, nonlinear programming, transportation problems, assignment models, critical path methods, and optimal scheduling problems. As the consolidity computations are based on matrix manipulations, the approach is extendable to optimization problems of high dimensional forms.

Other extensions are also recommended for handling existing artificial intelligent and expert systems-based techniques, risk assessment in economic models, etc., with applications to various operational engineering networks in different disciplines operating in fully fuzzy environments. Finally, work has to be extended for building special computational Toolbox in Matlab or special functions in other software languages for easily executing the various optimization procedures of the fuzzy optimization problems of different formulations.

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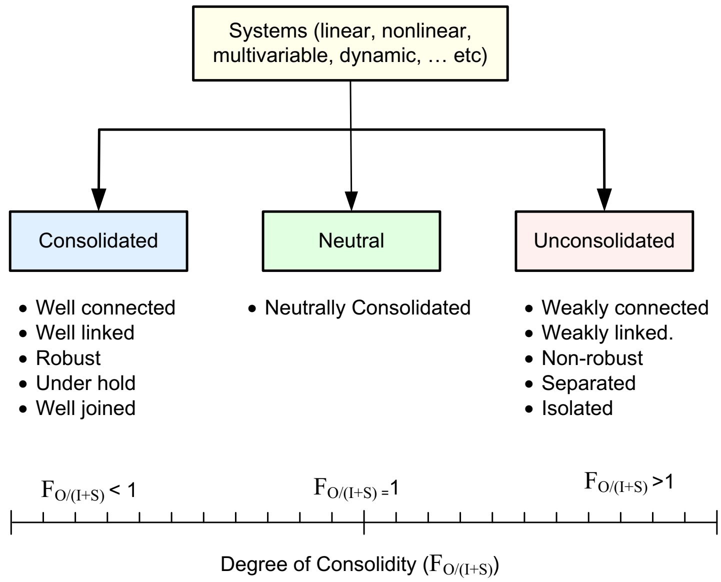

Figure A.1 Basic definition of system consolidity [20–24].

Appendix A. Basic definition of system consolidity index

Systems can be classified according to consolidity into three categories as follows,[4] see Fig. A.1 [21,22]:

(i) ***Consolidated Systems*** or well connected, under hold, under grasp, well linked, robust or well joined systems,
(ii) ***Neutrally Consolidated Systems***, and
(iii) ***Unconsolidated Systems*** or weakly connected, separated, non-robust or isolated systems.

A system operating at a certain stable original state in fully fuzzy environment is said to be ***consolidated*** if it is overall output is suppressed corresponding to their combined input and parameters effect, and vice versa for ***unconsolidated*** systems. Neutrally consolidated systems correspond to marginal or balanced reaction of output, versus combined input and system.

The ***system consolidity index*** is now presented in this section as given by [20–22]. This index measures the system overall output fuzziness behavior versus the combined input and system parameters variations. It describes the degree of how the systems react against input and system variation actions. Let us assume a general system operating in fully fuzzy environment, having the following elements:

Input Parameters:

$$\underline{I} = (V_{I_i}, \ell_{I_i}) \quad \text{(A.1)}$$

such that V_{I_i}, $i = 1, 2, \ldots, m$ describe the value of input component I_i, and ℓ_{I_i} indicates its corresponding fuzzy level.

System Parameters:

$$\underline{S} = (V_{S_j}, \ell_{S_j}) \quad \text{(A.2)}$$

such that V_{S_j}, $j = 1, 2, \ldots, n$ denote the value of system parameter S_j, and ℓ_{S_j} denotes its corresponding fuzzy level.

[4] Consolidity could be regarded as a general internal property of physical systems that can also be defined far from fuzzy logic or rough sets. Other consolidity indices, however, could be defined by researchers but the concept will still remain the same.

Output Parameters:

$$\underline{O} = (V_{O_i}, \ell_{O_i}) \quad \text{(A.3)}$$

such that V_{O_i}, $i = 1, 2, \ldots, k$ designate the value of output component O_i, and ℓ_{O_i} designates its corresponding fuzzy level.

We will apply in this investigation, the overall fuzzy levels notion, first for the combined input and system parameters, and second for output parameters. As the relation between combined input and system with output is close to (or of the like type) of the multiplicative relations, the multiplication fuzziness property is applied for combining the fuzziness of input and system parameters.

For the combined input and system parameters, we have for the weighted fuzzy level to be denoted as the combined Input and System Fuzziness Factor F_{I+S}, given as:

$$F_{1+S} = \frac{\sum_{i=1}^{m} V_{I_i} \cdot \ell_{I_i}}{\sum_{i=1}^{m} V_{I_i}} + \frac{\sum_{j=1}^{n} V_{S_j} \cdot \ell_{S_j}}{\sum_{j=1}^{n} V_{S_j}}. \quad \text{(A.4)}$$

Similarly, for the Output Fuzziness Factor F_O, we have

$$F_O = \frac{\sum_{i=1}^{k} V_{O_i} \cdot \ell_{O_i}}{\sum_{i=1}^{k} V_{O_i}}. \quad \text{(A.5)}$$

Let the positive ratio $|F_O/F_{I+S}|$ defines the **SystemConsolidity Index**, to be denoted as $F_{O/(I+S)}$. Based on $F_{O/(I+S)}$ the system consolidity state can then be classified as [21,22]:

(i) ***Consolidated*** *if* $F_{O/(I+S)} < 1$, to be referred to as *"Class C"*.
(ii) ***Neutrally Consolidated*** *if* $F_{O/(I+S)} \approx 1$, to be denoted by *"Class N"*.
(iii) ***Unconsolidated*** *if* $F_{O/(I+S)} > 1$, to be referred to as *"Class U"*.

For cases where the system consolidity indices lie at both consolidated and unconsolidated parts, the system consolidity will be designated as a ***Mixed-Consolidated*** class or "*Class M*". Other classes are ***Quasi-Consolidated*** "Class

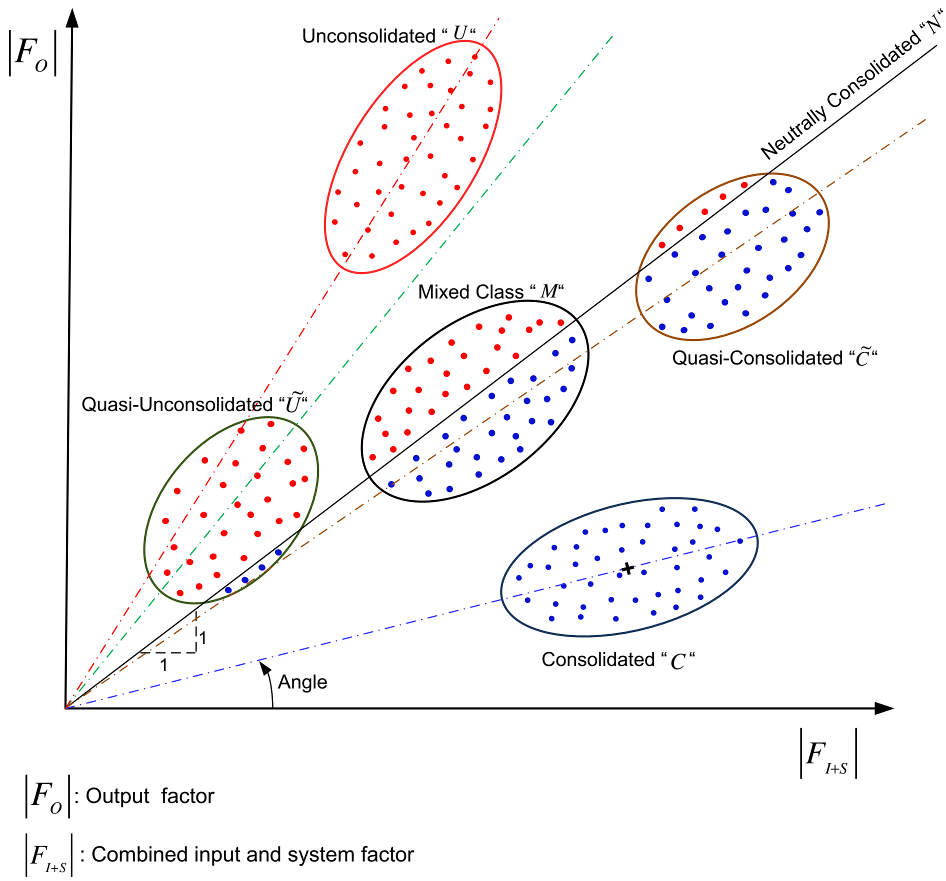

Figure A.2 Various classifications of consolidity regions [20–24].

$\tilde{C}$" (or ***Quasi-Unconsolidated*** "Class $\tilde{U}$") if the prevailing areas of the regions are Consolidated (or Unconsolidated). Finally, the various classifications of consolidity regions are elucidated in Fig. A.2 [20–24].

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