On the oscillation of certain second-order nonlinear differential equations

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Abstract
In this paper, oscillation criteria for the nonlinear second-order ordinary differential equation

\[(r(t)\Psi(x(t))\left|{x'(t)}\right|^{p-2}x'(t))'+c(t)f(x(t))=0\]

are given. The results extend the integral averaging technique and include earlier results. Our methodology is somewhat different from that of previous authors.
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1. Introduction

This paper is concerned with the second-order nonlinear differential equation

\[(r(t)\Psi(x(t))\left|{x'(t)}\right|^{p-2}x'(t))'+c(t)f(x(t))=0, \quad t \geq t_0 \geq 0,\]

where \(r \in C([t_0, \infty), R^+), c \in C([t_0, \infty), R), \Psi \in C(R, R), \) and \(f \in C^1(R, R)\) such that \(\Psi(x) > 0, x f(x) > 0 \) for \(x \neq 0,\) and \(p > 1\) is a constant.

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This equation can be considered as a natural generalization of the equation

\[ (r(t)\Psi(x(t))|x'(t)|^{p-2}x'(t))' + c(t)|x(t)|^{p-2}x(t) = 0 \]  

or the half-linear equation

\[ (r(t)|x'(t)|^{p-2}x'(t))' + c(t)|x(t)|^{p-2}x(t) = 0, \]

which have been the subject of intensive studies in recent years.

By a solution of (E), we mean a function \( x: [T_x, \infty) \to \mathbb{R}, T_x \geq t_0 \), such that \( x \) and \( r\Psi(x)|x'|^{p-2}x' \) are continuously differentiable and satisfy Eq. (E) for \( t \geq T_x \). A solution is said to be global if it exists on the whole interval \([t_0, \infty)\). On the other hand, a solution \( x \) of (E) which exists on some interval \((T_x, \infty)\), \( T_x \geq t_0 \) is called proper if \( \sup\{|x(t)| : t \geq T\} > 0 \) for all \( T \geq T_x \).

The existence and uniqueness of solutions of (HL) subject to the initial condition has been investigated by Kusano and Kitano [11]. The existence of proper solutions for the nonlinear second-order equation was investigated by Kiguradze and Chanturia [10]. They established sufficient conditions for all global solutions to be proper. So we shall suppose that Eq. (E) has proper solutions and our attention will be restricted to those solutions only.

A nontrivial solution of (E) is called oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Equation (E) is called oscillatory if all its solutions are oscillatory.

It is well known that Eq. (HL) and the linear equation

\[ (r(t)x'(t))' + c(t)x(t) = 0 \]  

have some properties in common. For such results the reader is referred to [3, 15, 18, 21]. If our attention is directed to the case where the equation is oscillatory, there are many works on the oscillatory behaviour of solutions of Eq. (HL) (see [2, 4, 6, 8, 12, 16–19]).

Investigation of Eq. (E) in this work is motivated by the most recent contributions in the sphere of weighted averages. Namely, among numerous papers dealing with averaging techniques in the study of second-order nonlinear differential equations, concerning oscillatory and nonoscillatory nonlinear oscillation, the majority involve the function \((t - s)^{\alpha}\) for \(\alpha > 1\) integer or real, as the weighted functions. Therefore, it is natural to ask if it is possible to use a more extensive class of functions as the weighting functions. An affirmative answer to this question was given for the first time by Philos [20] who used averaging functions from a general class of parameter functions and obtained new oscillation criteria for Eq. (L) with \(r(t) \equiv 1\). Recently, Li [14] for Eq. (HL) with \(r(t) \equiv 1\); Agarwal and Grace [1], Manojlovic [19] for Eq. (HL), and Ayanlar and Tiryaki [2] for Eq. (HL,Ψ) proceeded further in this direction and established general oscillation criteria.

Note that, Wong and Agarwal [23] established oscillation criteria for the more general equation

\[ (r(t)|x'|^{p-2}x')' + Q(t, x) = P(t, x, x'). \]  

These authors [24, 25] and Hsu and Yeh [8] also considered some special cases of this equation by using a technique which is an extension of the methods used in the works of Graef and Spikes [5], Kwang and Wong [13] for differential equations. Hong in [7]
generalized to (E) criteria for the oscillation of Eq. (HL) due to Hsu and Yeh [8]. But these results, for example, for Eq. (HL) require that
\[ \int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} c(t) \, dt < \infty. \] (2)

Our purpose here is to develop oscillation theory for Eq. (E) without any restriction on the sign of \( c(t) \). We extend and improve some earlier oscillation criteria by allowing more general means along the lines given in [22]. Our methodology is somewhat different from that of previous authors. We believe that our approach is simpler and also provides a more unified account of the study of Kamenev-type oscillation theorems [9]. We will also show that we do not need any restriction on the functions \( r \) and \( c \) of form (2).

2. Preliminaries

In order to discuss our main results we introduce the general mean and we shall present some properties, which will be used in the proof of our results.

Let \( D = \{(t, s) : t_0 \leq s \leq t \} \) denote a subset of \( \mathbb{R}^2 \) and let \( D_1 = \{(t, s) : t_0 \leq s < t \} \).

Consider a kernel function \( k(t, s) \), which is defined, continuous, and sufficiently smooth on \( D \), so that the following conditions are satisfied:

(K1) \( k(t, t) = 0 \) and \( k(t, s) > 0 \) for \( (t, s) \in D_1 \).

(K2) \( \frac{\partial k}{\partial s}(t, s) \leq 0, \quad -\frac{\partial k}{\partial s}(t, s) = \lambda(t, s)\left(k(t, s)\right)^{1/q} \) for \( (t, s) \in D_1 \). Where \( 1/p + 1/q = 1 \).

(K3) \( \frac{\partial^2 k}{\partial s \partial t}(t, s) = \frac{\partial^2 k}{\partial t \partial s}(t, s) \) for \( (t, s) \in D \).

(K4) \( \frac{\partial}{\partial t} \left( \lambda(t, s)\left(k(t, s)\right)^{-1/p} \right) \leq 0 \) for \( (t, s) \in D_1 \).

(K5) For each \( s \geq t_0, \lim_{t \to \infty} k(t, s) = \infty \), and there exist positive constants \( k_0, K_0 \) such that
\[ 0 < k_0 \leq \lim_{t \to \infty} k(t, s) \leq K_0 < \infty. \]

A kernel function \( k(t, s) \) satisfying (K1)–(K4) satisfies the following lemma, which is proved as in [22].

Lemma 2.1. Let \( k(t, s) \) be a continuous kernel function on \( D \) satisfying (K1)–(K4). If \( h \in C[0, \infty) \) and \( h(s) \geq 0 \) then
\[
\frac{1}{k(t,t_0)} \int_{t_0}^{t} k(t,s) h(s) \, ds
\]
is nondecreasing in \( t \).

Let \( \rho \in C^1[t_0, \infty) \) and \( \rho(t) > 0 \) on \([t_0, \infty)\). We take the integral operator \( A^\rho_t \), which is defined in [22], in terms of \( k(t,s) \) and \( \rho(s) \) as
\[
A^\rho_t (h; t) = \int_{\tau}^{t} k(t,s) h(s) \rho(s) \, ds, \quad t \geq \tau \geq t_0,
\]
where \( h \in C[t_0, \infty) \). It is easily seen that \( A^\rho_t \) is linear and positive, and in fact satisfies the following:
\[
A^\rho_t (\alpha_1 h_1 + \alpha_2 h_2; t) = \alpha_1 A^\rho_t (h_1; t) + \alpha_2 A^\rho_t (h_2; t),
\]
\( (4) \)
\[
A^\rho_t (h; t) \geq 0 \text{ whenever } h \geq 0,
\]
\( (5) \)
\[
A^\rho_t (h'; t) = -k(t, \tau) h(\tau) \rho(\tau) - A^\rho_t \left( \left[ -\lambda k^{-1/p} + \frac{\rho'}{\rho} \right] h; t \right)
\geq -k(t, \tau) h(\tau) \rho(\tau) - A^\rho_t \left( \left| \left[ \frac{\lambda k^{-1/p} + \rho'}{\rho} \right] h \right|; t \right).
\]
\( (6) \)

Here \( h_1, h_2, h \in C[t_0, \infty) \) and \( \alpha_1, \alpha_2 \) are real numbers.

For an arbitrary positive \( \xi \in C^1[t_0, \infty) \), define the kernel function
\[
k(t,s) = \left( \int_{s}^{t} d\tau \frac{1}{\xi(\tau)} \right)^m,
\]
with \( \int_{t_0}^{\infty} (1/\xi(\tau)) \, d\tau = \infty \). For example, an important particular case is \( \xi(\tau) = \tau^n \), where \( n \leq 1 \) is real. When \( \xi(\tau) = 1 \) we have \( k(t, s) = (t - s)^m \), and when \( \xi(\tau) = \tau \) we have \( k(t, s) = (\ln(t/s))^m \). It is easily verified that the kernel function (7) satisfies (K_1)–(K_5).

### 3. Main results

We are now able to state the main results.

**Theorem 3.1.** Let
\[
\frac{f'(x)}{(\Psi(x)f(x))^{(p-2)/2(p-1)}} \geq \beta > 0, \quad x \neq 0.
\]
Assume that \( k(t,s) \) satisfies conditions (K_1) and (K_2), and \( A^\rho_t \) is defined by (3). If there exists a positive function \( \rho \in C^1[t_0, \infty) \) such that
\[
\limsup_{t \to \infty} \frac{1}{k(t,t_0)} A_{t_0}^\rho \left( c - \frac{r}{\beta p-1} \left[ \frac{1}{p} \left| \lambda k^{-1/p} + \frac{\rho'}{\rho} \right| \right]^p \right) = \infty.
\]
Then Eq. (E) is oscillatory.
Proof. Let $x(t)$ be a nonoscillatory solution of (E) that, in view of the assumption that $xf(x) > 0$ whenever $x(t) \neq 0$, can be assumed to be positive on $[t_0, \infty)$. Define
\[ w(t) = \frac{r(t)\Psi(x(t))|x'(t)|^{p-2}x'(t)}{f(x(t))}, \quad t \geq t_0. \] (10)
By Eq. (E), we find that
\[ w'(t) = -c(t) - \frac{1}{(r(t))^{q-1}} \left( \frac{f'(x(t))}{\Psi(x(t))} \right)^{(2-q)/(q-1)} |w(t)|^q. \] (11)
Because of condition (8), Eq. (11) implies that $w(t)$ satisfies the differential inequality
\[ w'(t) \leq -c(t) - \frac{\beta}{(r(t))^{q-1}} |w(t)|^q. \] (12)
Applying the operator $A_\rho^\tau$ to (12) and using (6), we obtain
\[ A_\rho^\tau(c) \leq k(t, \tau)\rho(\tau)w(\tau) + A_\rho^\tau\left( \frac{\lambda k^{-1/p} + \rho' \beta}{\rho} \right)^p |w|^{(1/p)} - \frac{\beta}{r^{q-1}} |w|^q. \] (13)
For given $t$ and $s$, set
\[ F(u) := \frac{1}{q\beta} \left[ \lambda k^{-1/p} + \frac{\rho'}{\rho} \right]^p \frac{1}{u^{(1/p)}}, \quad u > 0. \]
$F(u)$ obtains its maximum at
\[ u = \left\{ \frac{\lambda k^{-1/p} + \rho' \beta}{\rho} \right\}^{1/(q-1)} \]
and
\[ F(u) \leq F_{\text{max}} = r \left( \lambda k^{-1/p} + \frac{\rho'}{\rho} \right)^p \frac{1}{\beta^{p-1}}. \] (14)
Then we get, by using (14),
\[ A_\rho^\tau(c) \leq k(t, \tau)\rho(\tau)w(\tau) + A_\rho^\tau\left( \frac{1}{p} \left( \frac{1}{p^{(1/p)}} \right)^p \frac{1}{\beta^{p-1}} \right)^p. \] (15)
Set $\tau = t_0$ and divide (15) through by $k(t, t_0)$, then we have
\[ \frac{1}{k(t, t_0)} A_\rho^{t_0} \left( c - \frac{r}{\beta^{p-1}} \left( \frac{1}{p} \left( \frac{1}{p^{(1/p)}} \right)^p \frac{1}{\beta^{p-1}} \right)^p \right) \leq \rho(t_0)w(t_0). \] (16)
Take lim sup in (16) as $t \to \infty$, condition (9) gives the desired contradiction in (16). Thus, the existence of nonoscillatory solution $x(t)$ is ruled out, so Eq. (E) is oscillatory. \[\Box\]

Remark 3.1. Although many studies on Eq. (HL) has been carried out for the case $\rho'(t) \geq 0$, the authors do not know of any study not imposing a restriction on the sign of $\rho'(t)$, except for Agarwal and Grace’s study [1] involving $k(t, s) = (t - s)^m$ for some constant $m > 1$. We believe that this is an important aspect of Theorem 3.1.
Remark 3.2. For \( r(t) \equiv 1, \Psi(x(t)) = 1, \rho(t) = 1, \) and \( k(t,s) = t-s, \) we may derive Corollary 3.2 in [21] from Theorem 3.1.

Remark 3.3. Taking \( k(t,s) = (t-s)^\lambda \) for some constant \( \lambda > 1, \) and \( \rho(t) = 1 \) in case of Eq. (HL), Theorem 3.1 reduces to the oscillation criterion [17].

Remark 3.4. For \( k(t,s) = (t-s)^m \) from Theorem 3.1, we obtain Theorem 2 in [1].

Remark 3.5. Letting \( \rho'(t) \geq 0 \) for Eqs. (HL\( \Psi \)) and (HL), we obtain Theorem 1 in [2] and Theorem 4 in [19], respectively, from Theorem 3.1.

A close look at the proof of Theorem 3.1 reveals that condition (9) may be replaced by the conditions

\[
\limsup_{t \to \infty} \frac{1}{k(t,t_0)} A^p_{\tau_0}(c) = \infty \tag{17}
\]

and

\[
\limsup_{t \to \infty} \frac{1}{k(t,t_0)} A^p_{\tau_0} \left( r \left| \lambda k^{-1/p} + \frac{\rho'}{\rho} \right| \right) < \infty. \tag{18}
\]

This leads to the following result.

**Corollary 3.1.** Let the conditions of Theorem 3.1 be satisfied except that condition (9) is replaced by (17) and (18). Then Eq. (E) is oscillatory.

**Theorem 3.2.** Let condition (8) hold. Suppose that \( k(t,s) \) satisfies conditions (K1)–(K5) and \( A^p_{\tau} \) is defined by (3). If there exist functions \( \phi_1, \phi_2 \in C[t_0, \infty) \) and a positive function \( \rho \in C^1[t_0, \infty) \) such that for \( \tau \geq t_0, \)

\[
\limsup_{t \to \infty} \frac{1}{k(t,t_0)} A^p_{\tau}(c) \geq \phi_2(\tau) \tag{19}
\]

and

\[
\lim_{t \to \infty} \frac{1}{k(t,t_0)} A^p_{\tau} \left( r \left| \lambda k^{-1/p} + \frac{\rho'}{\rho} \right| \right) \leq \phi_1(\tau), \tag{20}
\]

where \( \phi_1 \) and \( \phi_2 \) satisfy

\[
\lim_{t \to \infty} \frac{1}{k(t,t_0)} A^p_{\tau} \left( \frac{\rho^{-q} \phi_2 - \frac{1}{p^p \beta^{p-1}} \phi_1}{r_{q-1}} \right) = \infty, \tag{21}
\]

[\( \phi(t) \) is \( \max(\phi(t), 0) \), then Eq. (E) is oscillatory.]

**Proof.** We proceed as in the proof of Theorem 3.1 and return to inequality (15). Dividing (15) through by \( k(t,t_0), \) we obtain

\[
\frac{1}{k(t,t_0)} A^p_{\tau}(c) - \frac{1}{p^p \beta^{p-1}} \frac{1}{k(t,t_0)} A^p_{\tau} \left( r \left| \lambda k^{-1/p} + \frac{\rho'}{\rho} \right| \right) \leq \frac{k(t,\tau)}{k(t,t_0)} \rho(\tau) w(\tau). \tag{22}
\]
Take lim sup in (22) as \( t \to \infty \) and note from (19), (20), and (K₅) that
\[
\phi_2(\tau) - \frac{1}{p^p \beta^{p-1}} \phi_1(\tau) \leq K_0 \rho(\tau) w(\tau),
\]
from which it follows that
\[
K_0^{-q} \frac{r(\tau)}{(\rho(\tau)r(\tau))^q} \left[ \phi_2(\tau) - \frac{1}{p^p \beta^{p-1}} \phi_1(\tau) \right]^q \leq r(\tau) \left( \frac{|w(\tau)|}{r(\tau)} \right)^q.
\]
To reach a contradiction from the foregoing and condition (21), we need to show that
\[
\lim_{t \to \infty} \frac{1}{k(t,t_0)} A_\rho^0 \left( r \left( \frac{|w|}{r} \right)^q \right) < \infty.
\]
Returning to (13) and rearranging, we obtain
\[
A_\rho^0 (c) + A_\rho^0 \left( \frac{r}{\beta} \left( \frac{|w|}{r} \right)^q \right) \leq k(t,\tau) \rho(\tau) w(\tau).
\]
Set
\[
f(s) = \frac{1}{\beta} \left| \lambda(t,s)(k(t,s))^{-1/p} + \frac{\rho'(s)}{\rho(s)} \right|
\]
and
\[
g(s) = \frac{|w(s)|}{r(s)}.
\]
Using Young inequality, we get
\[
\frac{1}{\beta} \left| \lambda(t,s)(k(t,s))^{-1/p} + \frac{\rho'(s)}{\rho(s)} \right| \leq \frac{1}{p} \left( \left( \frac{\lambda(t,s)(k(t,s))^{-1/p} + \rho'(s)/\rho(s)}{\beta} \right)^p + \frac{1}{q} \left( \frac{|w(s)|}{r(s)} \right)^q \right).
\]
Substituting (25) into (24), we have
\[
\frac{\beta}{p} A_\rho^0 \left( r \left( \frac{|w|}{r} \right)^q \right) + A_\rho^0 \left( c - \frac{\beta r}{p} \left[ \frac{\beta k^{-1/p} + \rho'/\rho}{\beta} \right] \right) \leq k(t,\tau) \rho(\tau) w(\tau).
\]
Set \( \tau = t_0 \) in (26). Dividing (26) through by \( k(t, t_0) \), we note that by Lemma 2.1 and (26), the following limits exist and are finite:
\[
\lim_{t \to \infty} \frac{1}{k(t,t_0)} A_\rho^0 \left( r \left( \frac{|w|}{r} \right)^q \right), \quad \lim_{t \to \infty} \frac{1}{k(t,t_0)} A_\rho^0 \left( r \lambda k^{-1/p} + \frac{\rho'}{\rho} \right).
\]
Thus we can take lim sup in (26) as \( t \to \infty \) and obtain, by (19) and (20),
\[
\lim_{t \to \infty} \frac{\beta}{p} k(t, t_0) A_\rho^0 \left( r \left( \frac{|w|}{r} \right)^q \right) \leq \rho(t_0) w(t_0) - \phi_2(t_0) + \frac{\beta}{p \beta^p} \phi_1(t_0).
\]
This gives the desired contradiction to (21), and therefore, proves the theorem. \( \Box \)
Example 3.1. Consider the differential equation
\[(t^γ |x(t)|^{p+2} |x'(t)|^{p-2} x'(t))' + t^ν \cos^3 x(t) = 0\] (28)
for \(t \geq t_0\), where \(γ, p, ν\) are constants such that \(-3 \leq ν < -2\), \(p > 1\), \(3 + γ - p \leq ν + 2 < 0\) and \((p - γ - 3)p^2 \geq 3\sqrt{2}\).

Then condition (8) is satisfied. Moreover, taking \(k(t, s) = (t - s)^2\) for \(t > s \geq τ \geq t_0\) and \(ρ(t) = t^2\), we have
\[
\limsup_{t \to \infty} \frac{1}{(t - τ)^2} \int_τ^t (t - s)^2 s^{γ+2} \cos s \, ds \geq -τ^{γ+2} \sin τ,
\]
\[
\lim_{t \to \infty} \frac{1}{(t - τ)^2} \int_τ^t (t - s)^2 s^{γ+2} \left(\frac{2}{t - s} + \frac{2}{s}\right)^p \, ds
\leq \lim_{t \to \infty} \frac{2^p t^p}{(t - τ)^2} \int_τ^t (t - s)^{2-p} s^{γ+2-p} \, ds \leq \frac{2^p}{p - γ - 3} τ^{3+γ-p}.
\]

Let
\[
Φ(s) = φ_2(s) - \frac{1}{3p-1-p^p} φ_1(s) = -s^{γ+2} \sin s - \frac{2^p}{3p-1-p^p} \frac{1}{p - γ - 3} τ^{3+γ-p}
\geq s^{γ+2} \left( -\sin s = \frac{2^p}{3p-1-p^p (p - γ - 3)} \right).
\]

Define
\[
ε := \frac{2^p}{3p-1-p^p (p - γ - 3)}.
\]

Consider an integer \(N\) such that
\[
2Nπ + \frac{5π}{4} \geq \max \{t_1, (1 - \sqrt{2})^{-1/(γ+2)}\}.
\]

Then, for all integers \(n \geq N\), we have \(Φ(s) \geq 1/\sqrt{2}\) for every \(s \in [2nπ + 5π/4, 2nπ + 7π/4]\), which implies
\[
\lim_{t \to \infty} \frac{1}{(t - t_0)^2} \int_{t_0}^t (t - s)^2 \left(\frac{1}{\sqrt{2}}\right)^q s^{γ+2-2q-γq} \, ds
\geq \sum_{n=N}^{∞} \left(\frac{1}{\sqrt{2}}\right)^q s^{γ+2-2q-γq} \, ds.
\]

Since \(3 + γ - p < 0\) and \(1/p + 1/q = 1\) we see that \(γ + 2 - 2q - γq \geq -1\). Therefore condition (21) is satisfied. On the other hand, condition (9) is not satisfied for \(ν < -3\). Hence by Theorem 3.2, Eq. (28) is oscillatory.
Remark 3.6. Letting $\rho'(t) \geq 0$ for Eqs. (HL$_\varphi$) and (HL), we obtain Theorem 5 in [2] and Theorem 5 in [19], respectively, from Theorem 3.2.

Remark 3.7. For $k(t, s) = (t - s)^m$ from Theorem 3.2, we get Theorem 3 in [1].

References