Representing matrices of almost completely decomposable groups

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Abstract

Almost completely decomposable groups can be described in terms of integral matrices and in terms of anti-representations in finite modules over proper quotient rings of the ring of integers. The anti-representations are described by the so-called representating matrices. Representing matrices and their interrelationship with the integral matrices describing a group are studied in general. It is shown that the integral and representing matrices may be assumed to have a special form. Two applications demonstrate the usefulness of the results.

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1. Introduction

An almost completely decomposable group $X$ is, by definition, a finite extension of a finite rank completely decomposable group $A$. Setting $e = \exp(X/A)$, let

$$\tilde{\varphi}: A \to \tilde{A} = A/eA, \ a \mapsto \tilde{a} = a + eA$$

be the canonical epimorphism. Let $A = \bigoplus_{j=1}^{r} \langle v_j \rangle^d_j$ where $\mathcal{V} = \{v_1, \ldots, v_r\}$ is an $e$-basis of $A$, i.e., $\text{hgt}_p(v_j) \in \{0, \infty\}$ for all primes $p$ dividing $e$, $j = 1, \ldots, r$, and let $\mathcal{V} = \{x_{1} + A, \ldots, x_{k} + A\}$ be a generating set of $X/A$. The generating set $\tilde{\mathcal{V}} = \{\tilde{v}_1, \ldots, \tilde{v}_r\}$ of the $\mathbb{Z}/e\mathbb{Z}$-module $\tilde{A}$ is an induced decomposition basis. The generators $\mathcal{E}_i$ of $e\tilde{X} = eX/eA$
may be written as linear combinations of the induced decomposition basis:

\[ e_x = \sum_{j=1}^{r} \sum_{i=1}^{k} \varepsilon_{ij} e_y, \]

where \( \varepsilon_{ij} \in \mathbb{Z}/e\mathbb{Z} \). The matrix \( m_X = [\varepsilon_{ij}] \in M_{k \times r}(\mathbb{Z}/e\mathbb{Z}) \) is a representing matrix of \( X \) over \( A \) relative to \( e \)-decomposition basis \( V \).

A representing matrix describes a group \( X \) as extension of \( A \) by \( X/A \) relative to an \( e \)-decomposition basis of \( A \) and a generating set of \( X/A \). Every almost completely decomposable group contains a canonical fully invariant completely decomposable subgroup of finite index, namely Burkhardt's regulator \( R(X) \). Thus an almost completely decomposable group \( X \) is the extension of its regulator \( A = R(X) \) by the finite group \( X/A \), its regulator quotient. This point of view is of prime importance.

In [9] it is shown that two groups \( X \) and \( Y \) with \( eX, eY \subseteq A = R(X) = R(Y) \subseteq X, Y \) are isomorphic if and only if \( eX \) and \( eY \) are in the same orbit of subgroups of \( A \) under the operation induced by the automorphism group of \( A \).

Recall [16] that the type isomorphisms are

\[ \text{TypAut} \hat{A} = \{ \xi \in \text{Aut} \hat{A} : \forall \tau \in T_{\text{nr}(\hat{A})}, \hat{A}(\tau)\xi = \hat{A}(\tau) \}. \]

The groups \( X \) and \( Y \) are near-isomorphic if and only if there exists \( \xi \in \text{TypAut} \hat{A} \) such that \( eX \xi = eY \) ([16]). Hence the classification problems for almost completely decomposable groups up to isomorphism can be rephrased as equivalence problems for representing matrices. Namely, the two representing matrices \( m \) and \( m' \) describe isomorphic groups if and only if there are invertible matrices \( U \) and \( V \) such that \( m' = UmV \), where \( U \) is an automorphism of \( eX \cong X/A \) and \( V \) is an automorphism of \( \hat{A} \) that is induced by an automorphism of \( A \), while the two representing matrices \( m \) and \( m' \) describe near-isomorphic groups if and only if there are invertible matrices \( U \) and \( V \) such that \( m' = UmV \), where \( U \) is an automorphism of \( eX \cong X/A \) and \( V \) is a type automorphism of \( \hat{A} \). The classification up to near-isomorphism, compared with the classification up to isomorphism, avoids two serious difficulties, namely the global problem [14], and a number theoretic problem [6,9,16]. The known classifications of classes of almost completely decomposable groups are up to near-isomorphism and in all but the crq-case [17] special forms of the representing matrices play an essential role [4,5,7,8,10,16,17,19].

In this article we study representing matrices in general and establish some special forms that can always be achieved (Theorem 3.6). We also make a connection with the so-called standard description developed in [2]. The results suggest that the representing matrix is best viewed as a product of two matrices (Theorem 3.7). We are able to clarify why the case of a homocyclic regulator quotient is particularly accessible.

Several of the ideas developed here have appeared before in some special form or fashion. The study of special classes of almost completely decomposable groups can now start with the general results developed in the present article, rather than developing similar results in each special case separately. As applications we reprove the
well-known result of Arnold–Lewis that a local almost completely decomposable group with two critical types is the direct sum of rank-one and rank-two groups (Proposition 4.2) and we prove a new result of a similar nature about almost completely decomposable groups with three critical types (Theorem 4.6).

A type is an isomorphism class of rational groups, and a rational group is an additive subgroup of the rationals containing \( \mathbb{Z} \). We frequently abuse notation and use \( \text{EFS} \) for a representative of the class \( \text{EFS} \). The groups \( G(\text{EFS}) \) are the usual (pure) type subgroups of \( G \). The critical typeset of a group \( G \) is \( T_{\text{cr}}(G) = \{ \tau : G(\tau)/G^2(\tau) \neq 0 \} \).

An almost completely decomposable group \( X \) is \( p \)-local for a prime \( p \) if \( X/\text{R}(X) \) is a (finite) \( p \)-group where \( \text{R}(X) \) is the regulator of \( X \).

All “groups” in this paper are abelian, and the torsion-free groups all have finite rank. The symbol \( \mathbb{M}_{k,r}(S) \) denotes the set of \( k \times r \) matrices with entries in the set \( S \). The set \( S \) is usually a ring, in particular the ring of integers \( \mathbb{Z} \) will occur and its quotient ring \( \mathbb{Z}/e\mathbb{Z} \), but \( S \) may also be an abelian group \( G \). When \( r = 1 \) we write \( s \in \mathbb{M}_{k,1}(S) \). Similarly, when \( k = 1 \) we write \( s \in \mathbb{M}_{1,r}(S) \). Frequently we will need to deal with submatrices of a matrix and we will use the following notation. Let \( M \in \mathbb{M}_{k,r}(S) \).

Then \( M[i] \) for \( 1 \leq i \leq k \) denotes the submatrix of \( M \) consisting of its \( i \)th row; \( M[j] \) for \( 1 \leq j \leq r \) denotes the submatrix of \( M \) consisting of its \( j \)th column; \( M[i,j] \) is the entry of \( M \) in the \( i \)th row and \( j \)th column; \( M[x] \) for \( x \subset \{1, \ldots, k\} \) denotes the submatrix of \( M \) formed by the rows with index in \( x \); \( M[\beta] \) for \( \beta \subset \{1, \ldots, r\} \) denotes the submatrix of \( M \) formed by the columns with index in \( \beta \); \( M[\alpha,\beta] \) for \( \alpha \subset \{1, \ldots, k\} \) and \( \beta \subset \{1, \ldots, r\} \) denotes the submatrix of \( M \) formed by deleting all rows with index not listed in \( \alpha \) and all columns with index not listed in \( \beta \).

For background on almost completely decomposable groups we refer the reader to the survey article [12] or the monograph [13].

2. Background and basic observations

Throughout \( X \) denotes an almost completely decomposable group. The natural and usual description (except for form) is the standard description established in [2]

\[
X = A + \mathcal{Z} N^{-1} a', \tag{2.1}
\]

where \( A \) is a completely decomposable group, \( \mathcal{Z} \) is the set of all \( 1 \times k \) integral matrices, \( N \) is a non-singular \( k \times k \) integral matrix, the structure matrix, \( a' \) is a \( k \times 1 \) matrix of elements in \( A \), and juxtaposition is matrix multiplication in the usual fashion. The standard description becomes transparent if we write \( N^{-1} = (1/\det N) \text{adj}(N) \) and introduce entries, \( a' = [a_1, \ldots, a_k] \), \( \text{adj}(N) = [n_{ij}] \). Performing the computations implicit in (2.1) we obtain

\[
X = A + \sum_{i=1}^{k} \mathcal{Z} \frac{1}{\det N} \sum_{j=1}^{k} n_{ij} a_j.
\]
This shows that every row of $N$ determines an element of $X$, namely

$$x_i := \frac{1}{\det N} \sum_{j=1}^{k} n_{ij} a_j \in \mathbb{Q} A, \quad i = 1, \ldots, k,$$

and these $k$ elements together with $A$ generate $X$. We refer to the set $\{x_1, \ldots, x_k\}$ as the generating system implicit in the standard description (2.1). It is also seen that the "standard description" is standard in the ordinary sense of the word since an almost completely decomposable group nearly always is given as a group generated by a completely decomposable group $A$ together with a finite number of elements in a divisible hull $\mathbb{Q} A$. The standard description is simply a compact matrix theoretic form of the usual description. We record for future use that

$$X A \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \cong [X : A] = \det N \iff \gcd^4(N, a^\dagger) = I. \quad (2.2)$$

**Proof.** Observe first that right multiplication by an invertible integral matrix $L$ is an automorphism of the free abelian group $\mathbb{Z}$, so that $\mathbb{Z} L = \mathbb{Z}$. Now let $S = \text{diag}(d_1, \ldots, d_k)$ be the Smith Normal Form of $N$ and $P, Q$ invertible integral matrices such that $S = PNQ$. Then

$$\frac{\mathbb{Z}}{\mathbb{Z} N} = \frac{\mathbb{Z}}{\mathbb{Z} P^{-1} SQ^{-1}} = \frac{\mathbb{Z} Q^{-1}}{\mathbb{Z} S} \cong \frac{\mathbb{Z}}{\mathbb{Z} d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} d_k \mathbb{Z}}.$$  

This shows (assuming $\det N > 0$ without loss of generality) that

$$| \frac{\mathbb{Z}}{\mathbb{Z} N} | = \det S = \det N. \quad (2.3)$$

Next, consider the map

$$\phi: \frac{\mathbb{Z}}{\mathbb{Z} N} \to \frac{X}{A}: \bar{x} \phi = \bar{x} N^{-1} a^\dagger + A \quad \text{where } \bar{x} \in \mathbb{Z}.$$  

This is obviously an epimorphism and $\mathbb{Z} N \subset \text{Ker } \phi$. It follows that

$$\frac{X}{A} \cong \frac{\mathbb{Z}}{\mathbb{Z} N} \quad \text{if and only if } \quad [X : A] = [\bar{\mathbb{Z}} : \bar{\mathbb{Z}} N]. \quad (2.4)$$

If $D = \gcd^4(N, a^\dagger)$, and $N = DM$, $a^\dagger = Db^\dagger$, then $\mathbb{Z} N \subset \mathbb{Z} M \subset \text{Ker } \phi$ as $\bar{x} M \phi = \bar{x} M N^{-1} a^\dagger + A = \bar{x} D^{-1} a^\dagger + A = \bar{x} b^\dagger + A = 0$. Now $\mathbb{Z} / \mathbb{Z} N \cong X / A$ implies that Ker $\phi = \mathbb{Z} N$ which means that $D$ must be invertible, i.e., $\gcd^4(N, a^\dagger) = I$. Showing that Ker $\phi = \mathbb{Z} N$ if $\gcd^4(N, a^\dagger) = I$ requires properties of greatest common divisors that can be found in [2] or [13]. We obtained that

$$\frac{X}{A} \cong \frac{\mathbb{Z}}{\mathbb{Z} N} \quad \text{if and only if } \quad \gcd^4(N, a^\dagger) = I. \quad (2.5)$$

Results (2.3)–(2.5) together establish the claim.  \[\square\]
It may and will be assumed routinely that $\text{gcd}^A(N,a) = I$ since $N = DM$ and $a = Db$, imply that $N^{-1}d = M^{-1}b$, i.e., common divisors can be cancelled without changing the standard description. Furthermore, if $S = PNQ$ is the Smith Normal Form of $N$, then $\mathbb{Z}N^{-1}d = \mathbb{Z}QS^{-1}Pa = \mathbb{Z}S^{-1}(Pa)$, and it may be assumed further, without loss of generality, that $N = S = \text{diag}(d_1, \ldots, d_k)$ with $1 \leq d_{i-1}|d_i$ for $i = 2, \ldots, k$. In this case there is a clearer description of the quotient $X/A$, namely, with $a' = [a_1, \ldots, a_k]^T$, $a_i = m_{i1}v_1 + \cdots + m_{ir}v_r$

and obtain a coordinate matrix $M = [m_{ij}]$ such that $a' = Mv'$. The decomposition basis $\mathcal{v}'$ can be chosen such that $M$ is an integral matrix, $M \in \mathbb{N}_{k \times r}(\mathbb{Z})$ (see e.g. [2]). Since $X$ contains $A$ as a subgroup of finite index, there is a positive integer $e$ such that $eX \subseteq A$. Given such an $e$, the decomposition basis $\mathcal{v}'$ of $A$ may be chosen to be an $e$--basis. This can be done in such a way that the coordinate matrix $M$ of $a'$ with respect to $\mathcal{v}'$ remains integral.

From now on we assume that $eX \subseteq A$, $\mathcal{v}' = \{v_1, \ldots, v_r\}$ is an $e$--basis of $A$, $A = \tau_1v_1 \oplus \cdots \oplus \tau_rv_r$, and $a' = Mv'$ for some coordinate matrix $M \in \mathbb{N}_{k \times r}(\mathbb{Z})$. If it so happens that $\text{gcd}^d (p, v_i) = 1$ (equivalently $\text{hgt}_p^d(v_i) = 0$) for all $i \in \{1, \ldots, r\}$ and all prime divisors of $e$, then $\text{gcd}^d(N, Mv') = \text{gcd}(N, M)$ [2, Proposition 5.5]. In the following, we will restrict ourselves to $p$--local almost completely decomposable groups, i.e., the case where $e$ is a $p$--power, say $e = p^d$. In this case a $p^d$--divisible critical type $\tau_i$ creates a $p$--divisible direct summand $\tau_iv_i$ of $X$ and these summands are uninteresting for most purposes. We therefore usually assume that the $p$--local groups under consideration are $p$--reduced, meaning that there are no non-trivial $p$--divisible subgroups. In this situation a $p^d$--basis $\mathcal{v}' = \{v_1, \ldots, v_r\}$ is the same as a $p$--basis and it means that $\text{gcd}^d(p, v_i) = 1$ for each $i$, or, equivalently, $1/p \not\in \tau_i$. It is easy to see that a $p$--basis in the absence of $p$--divisible subgroups is a $p$--independent set and this leads to a simple proof of [2, Proposition 5.5] that makes the computation of greatest common divisors possible. By $\text{gcd}(N, M)$ we denote the greatest common left divisor of the matrices $N, M$. It is a standard fact, apparently known since the dawn of matrix theory, that the greatest common divisor of two integral matrices (of compatible size) exists and can be effectively computed by elementary matrix operations [3,11] or [2]. Greatest common left divisors of integral matrices are only defined up to right invertible factors, hence greatest divisors can and will be assumed to have positive determinant.

**Lemma 2.1.** Suppose $\mathcal{v}' = \{v_1, \ldots, v_r\}$ is a $p$--independent set in a torsion-free group $A$, $v' = [v_1, \ldots, v_r]^T$, and $a' = Mv'$ for some matrix $M \in \mathbb{N}_{k \times r}(\mathbb{Z})$. Further let
Lemma 2.2. Let \( N \in \mathbb{M}_k(\mathbb{Z}) \) with \( \det N = p^n \). Then

\[
gcld(N, M) = I \iff \gcd^d(N, a^d) = I.
\]

More generally, \( \gcd^d(N, a^d) = \gcd(N, M) \).

**Proof.** It is immediately seen that \( \gcd^d(N, a^d) = I \) implies that \( \gcd(N, M) = I \). Suppose conversely that \( \gcd(N, M) = I \) and let \( D = \gcd^d(N, a^d) \). Write \( N = DN_D \) and \( a^d = Da_D \). Then \( \det D \) divides \( \det N \), so \( \det D \) is also a power of \( p \), say \( \det D = p^d \). Now

\[
p^d a^d_D = (\text{adj} D) D a^d_D = (\text{adj} D) a^d = (\text{adj} D) M^v \in (\mathbb{Z} v^i).
\]

(2.6)

By hypothesis the set \( \mathcal{V} \) is a \( p \)-independent set and therefore the subgroup \( \mathbb{Z} v^i \) generated by \( \mathcal{V} \) is \( p \)-pure in \( A \). Now (2.6) implies that \( a^d_D = L v^i \) for some \( L \in \mathbb{M}_{k \times r}(\mathbb{Z}) \). We conclude that \( M v^i = a^d = DL v^i \). The \( p \)-independence of \( \mathcal{V} \) implies in particular that \( \mathcal{V} \) is an independent set and therefore we can conclude that \( M = DL \). Since \( \gcd(N, M) = I \), \( D \) must be invertible and this shows that \( \gcd^d(N, a^d) = I \).

The general statement follows from the special one. Let \( D = \gcd(N, M) \). Then \( D \) is a common left divisor of \( N \) and \( a^d \) and \( \gcd^d(D^{-1}N, D^{-1}a^d) = I \) since \( \gcd(D^{-1}N, D^{-1}M) = I \), and this means that \( D = \gcd^d(N, a^d) \). \( \square \)

It is desirable that the matrices appearing in a standard description \( X = A + \mathbb{Z} N^{-1} M v^i \) be as simple as possible. One might hope to achieve some “normal form” but this seems far-fetched. We have already seen that the structure matrix \( N \) may be assumed to be in Smith Normal Form. If \( N \) has this special form, then one can perform certain elementary row transformations on the coordinate matrix \( M \) without changing the group as we will see next. By an elementary row transformation we mean exclusively a transformation that adds an integral multiple of some row to another row.

**Lemma 2.2.** Let \( X = A + \mathbb{Z} N^{-1} M v^i \) be an almost completely decomposable group in standard description.

1. Suppose that there exist matrices \( P, Q \in \mathbb{M}_k(\mathbb{Z}) \) such that \( p \) is invertible and \( PN = NQ \). Then \( X = A + \mathbb{Z} N^{-1} (PM) v^i \).

2. Suppose that \( N \) is in Smith Normal Form, so \( N = \text{diag}(d_1, \ldots, d_k) \) and \( d_i \) divides \( d_{i+1} \) for \( i = 1, \ldots, k - 1 \). If \( M' \) is obtained from \( M \) by adding an integral multiple of row \( j \) to row \( i \), then \( X = A + \mathbb{Z} N^{-1} M' v^i \) provided that one of the following two cases occurs: \( i < j \) or else \( i > j \) and \( d_i = d_j \).

3. If \( N \) is a scalar matrix, i.e., \( d_1 = \cdots = d_k \), then \( X = \mathbb{Z} N^{-1} M' v^i \) where \( M' \) is any matrix obtained from \( M \) by elementary row transformations.

**Proof.** (1) Note that

\[
X = A + \mathbb{Z} N^{-1} P^{-1} PM v^i = A + \mathbb{Z} Q^{-1} N^{-1} (PM) v^i = A + \mathbb{Z} N^{-1} (PM) v^i.
\]
(2) We have that \( M' = PM \) where \( P = [p_{ij}] \) is an invertible integral matrix. Consider the identity

\[
PN = NQ \quad \text{where} \quad Q = \begin{bmatrix}
p_{11} & \cdots & \frac{d}{dt} p_{1s} & \cdots & \frac{d}{dt} p_{1k} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{d}{dt} p_{1} & \cdots & p_{s} & \cdots & \frac{d}{dt} p_{sk} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{d}{dt} p_{1} & \cdots & \frac{d}{dt} p_{k1} & \cdots & p_{kk}
\end{bmatrix}.
\]

(2.7)

It is clear that \( Q \) is invertible as well, provided that it is an integral matrix. The matrix \( P \) is upper triangular if a multiple of a later row \( j \) was added to an earlier row \( i \) (i.e., \( i < j \)). In this case \( p_{st} = 0 \) for \( s > t \), the matrix \( Q \) is integral and part (1) of the lemma applies. If \( d_i = d_j \) and a multiple of row \( j \) is added to \( i \), then \( Q \) is again integral and part (1) applies.

(3) Apply (2) repeatedly.

Of course, Lemma 2.2 can be applied repeatedly, and this means loosely speaking that, without changing the group, we may perform elementary row transformations on the coordinate matrix \( M \) upward at any rate, and downward provided that \( d_i = d_j \) where \( i \) and \( j \) are the labels of the rows involved in the transformation.

The following lemma shows how the coordinate matrices can be simplified by taking advantage of legitimate basis changes.

**Lemma 2.3.** Let \( A = \sigma_1 v_1 \oplus \cdots \oplus \sigma_r v_r \). Suppose that

\[
\sigma_1 = \cdots = \sigma_{r(1)} = \tau_1, \\
\sigma_{r(1)+1} = \cdots = \sigma_{r(1)+r(2)} = \tau_2, \\
\vdots \\
\sigma_{\sum r(j)+1} = \cdots = \sigma_{\sum r(j)} = \tau_t,
\]

where \( \tau_1, \ldots, \tau_t \) are the different critical types of \( A \). Let \( X = A + \mathbb{Z} N^{-1} M v^j \) be a finite essential extension of \( A \) and let \( M[\tau_1] \in \mathbb{M}_{k \times r(1)}, \ldots, M[\tau_t] \in \mathbb{M}_{k \times r(t)} \) be the submatrices of \( M \) formed by the columns corresponding to summands of \( A \) of equal types. Then each \( M[\tau_j] \) may be changed by column transformations by changing the decomposition basis accordingly. If the basis \( \mathcal{V} = \{v_1, \ldots, v_r\} \) is a \((\det N)\)-basis, then so is the new basis.

**Proof.** A column transformation of \( M[\tau_j] \) is right multiplication by an invertible integral matrix \( P_i \). Let \( v^j[\tau_j] \) be the sub-column matrix of \( v^j \) consisting of those elements \( v_j \) whose type is \( \tau_j \). Set \( w^j = \text{diag}(P_1, \ldots, P_t)^{-1} v^j \). Then \( \mathcal{W} = \{w_1, \ldots, w_r\} \) is a new decomposition basis of \( A \) and

\[
M v^j = [M[\tau_1]P_1 | \cdots | M[\tau_t]P_t] \text{diag}(P_1, \ldots, P_t)^{-1} v^j \\
= [M[\tau_1]P_1 | \cdots | M[\tau_t]P_t] w^j.
\]
This means that the coordinate matrix with respect to the new basis \( W \) has the reduced form. We leave the verification of the remaining claims to the reader. \( \square \)

The Hermite Normal Form of an integral matrix is an established, well-known concept [18]. We will call an integral matrix column reduced if its transpose is in Hermite Normal Form.

**Corollary 2.4.** The submatrices \( M[\tau_1], \ldots, M[\tau_t] \) may be assumed to be in column reduced form.

Special forms of the coordinate matrix give valuable information. For example, if the coordinate matrix has a column of zeros, then the group has a rank-one summand. Similarly, one can recognize other direct decompositions. A matrix \( M = [m_{ij}] \in \mathbb{M}_{k \times r}(\mathbb{Z}) \) decomposes if there are partitions \( \{1, \ldots, k\} = K_1 \cup K_2 \) and \( \{1, \ldots, r\} = L_1 \cup L_2 \) such that 
\[
m_{ij} = 0 \text{ if } i \in K_1 \text{ and } j \in L_2 \text{ or } i \in K_2 \text{ and } j \in L_1.
\]
The prototype for a decomposing matrix \( M \) is the case 
\[
K_1 = \{1, \ldots, k_1\}; K_2 = \{k_1 + 1, \ldots, k\}; L_1 = \{1, \ldots, r_1\}; L_2 = \{r_1 + 1, \ldots, r\}
\]
in which case the matrix is
\[
M = \begin{bmatrix}
M[1, \ldots, k_1\backslash1, \ldots, r_1] & 0 \\
0 & M[k_1 + 1, \ldots, k\backslash r_1 + 1, \ldots, r]
\end{bmatrix}.
\]

It is easy to see that a group \( X = A + \mathbb{Z}N^{-1}M\nu \) decomposes if the coordinate matrix \( M \) decomposes. The converse is also true if \( A \) is the regulating regulator of \( X \).

**Lemma 2.5.** Assume that \( A \) is the regulating regulator of \( X \). Then \( X \) decomposes if and only if the coordinate matrix \( M \) of \( X = A + \mathbb{Z}N^{-1}M\nu \) decomposes for some \( p \)-basis \( \nu = \{v_1, \ldots, v_r\} \) of \( A \).

**Proof.** Suppose that \( X = Y \oplus Z \) is a proper decomposition of \( X \). Since \( X \) has a regulating regulator, we have \( A = R(X) = \sum_{\rho \in \text{Tor}(X)} X(\rho) \) and consequently \( A = R(Y) \oplus R(Z) \). A \( p \)-basis of \( A \) is obtained by forming the union of \( p \)-bases of \( R(Y) \) and of \( R(Z) \). The coordinate matrix with respect to this basis decomposes. \( \square \)

The following lemma connects “relatively prime” with “\( p \)-independent”.

**Lemma 2.6.** Let \( X = A + \mathbb{Z}N^{-1}a' \) be such that \( \gcd^d(N,a') = 1 \), and \( N = \text{diag}(p^{d_1}, \ldots, p^{d_k}) \) with \( 1 \leq d_i \) for all \( i \). Write \( a' = [a_1, \ldots, a_k]^\text{tr} \) and assume that \( \gcd^d(p,a_i) = 1 \) for each \( i \). Then \( \{a_1, \ldots, a_k\} \) is a \( p \)-independent set in \( A \).

**Proof.** The hypotheses mean that
\[
\frac{X}{A} = \mathbb{Z}(p^{-d_1}a_1 + A) \oplus \cdots \oplus \mathbb{Z}(p^{-d_k}a_k + A), \quad \text{ord}(p^{-d_i}a_i + A) = p^{d_i} \geq p.
\]
Suppose that \( n_1a_1 + \cdots + n_ka_k \in pA \). Then
\[
n_1p^{d_1-1}(p^{-d_1}a_1) + \cdots + n_kp^{d_k-1}(p^{-d_k}a_k) \in A
\]
and it follows by considering orders that $p^{d_i}$ divides $n_i p^{d_i - 1}$. Hence $p$ divides $n_i$ and this proves $p$-independence in a torsion-free group. □

Lemma 2.6 is just a special case of Corollary 2.8 below of the following characterization of relatively prime objects $N$ and $a'$.

**Proposition 2.7** (Benabdallah and Mader [2, Lemma 3.5]). Let $N \in \mathbb{M}_k(\mathbb{Z})$ be non-singular, $A$ a torsion-free group and $a' \in A'$. Then $N$ and $a'$ are relatively prime if and only if

$$\bar{x} N \in p \mathbb{Z}, \quad \bar{x} a' \in p A \Rightarrow \bar{x} \in p \mathbb{Z},$$

where $p$ is a prime and $\bar{x} \in \mathbb{Z}$.

The generalization of Lemma 2.6 is almost immediate.

**Corollary 2.8.** Let $N \in \mathbb{M}_k(\mathbb{Z})$ be non-singular, $A$ a torsion-free group, and $a' \in \mathbb{M}_k(\mathbb{Z})\times_{G} 1(A)$. Suppose that $\gcd(a', a') = 1$. Then the entries of $a'$ form a $p$-independent set in $A$ for each prime factor $p$ of the first invariant factor of $N$.

Recall that the first invariant factor of the matrix $N$ is the first diagonal entry of its Smith Normal Form and agrees with the greatest common divisor of all the entries $N[i,j]$ of $N$.

**Proof.** Let $S = \text{diag}(d_1, \ldots, d_k)$ be the Smith Normal Form of $N$ and write $N = PSQ$ where $P, Q$ are invertible integral matrices. Suppose that $p$ divides $d_1$ and that $\bar{x} a' \in pA$. It is to show that $\bar{x} \in p \mathbb{Z}$. But $\bar{x} N = \bar{x} PSQ \in p \mathbb{Z} Q = p \mathbb{Z}$ is automatic and an application of Proposition 2.7 establishes the claim. □

A few more observations concerning $p$-independence and matrices are needed later.

**Lemma 2.9.** (1) Let $A$ be a torsion-free group and $a', b' \in A' = \mathbb{M}_{r \times 1}(A)$. Suppose that $a' = M b'$ for some $M \in \mathbb{M}_r(\mathbb{Z})$ and that the entries of $b'$ are $p$-independent. Then the entries of $a'$ are $p$-independent if and only if the rows of $M$ are $p$-independent in $\mathbb{Z}$.

(2) Let $M \in \mathbb{M}_{k \times r}(\mathbb{Z})$ with $k \leq r$. Then the rows of $M$ are $p$-independent in $\mathbb{Z}$ if and only if $M$ has a submatrix $K \in \mathbb{M}_k(\mathbb{Z})$ such that $\gcd(p, \det K) = 1$.

**Proof.** (1) This claim follows by a straightforward application of the definitions.

(2) Assume first that $M$ contains a submatrix $K$ whose determinant is relatively prime to $p$ and that $\bar{x} M \in p \mathbb{Z}$ for some $\bar{x} \in \mathbb{Z}$. Then also $\bar{x} K \in p \mathbb{Z}$ and it follows that

$$(\det K) \bar{x} = \bar{x} K(\text{adj} K) \in p \mathbb{Z}(\text{adj} K) \subset p \mathbb{Z}.$$  

Since $\gcd(p, \det K) = 1$ it follows that $\bar{x} \in p \mathbb{Z}$ as needed.
Assume conversely that the rows of $M$ are $p$-independent in $\mathbb{Z}_r$. Then it is possible to enlarge the set of row vectors of $M$ to a maximal $p$-independent set of elements of $\mathbb{Z}_r$. Using these we obtain a block matrix

$$ N = \begin{bmatrix} M \\ M' \end{bmatrix} \in M_r(\mathbb{Z}) $$

whose rows form a $p$-independent set. As a result, multiplication by $N$ induces an isomorphism $N : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ of vector spaces and it follows that $\det N$ is not divisible by $p$. Laplace expansion shows that $M$ has a submatrix of size $k$ whose determinant is not divisible by $p$.

Given that an integral matrix $M \in M_{k \times r}(\mathbb{Z})$ has $kp$-independent rows and hence a $k \times k$ submatrix with determinant relatively prime to $p$, such a submatrix can be found as follows. Compute modulo $p$ and reduce $M$ to Hermite normal form (= row reduced echelon form). There will be $k$ pivots (= non-zero leading coefficients) in columns $j_1, \ldots, j_k$. The submatrix $M[j_1, \ldots, j_k]$ of the original integral matrix will have determinant relatively prime to $p$. This is just the standard procedure of selecting columns that form a basis of the column space of a matrix.

3. Representing matrices

The term “representing matrix” was used by Dugas and Oxford in a special situation [7] without formal definition. We will give a general formal definition. For a torsion-free group $A$ and a positive integer $e$ we define $\text{FEE}(A, e)$ (FEE for “finite essential extension”) to consist of all groups $X$ such that $A \leq X \leq QA$ and $eX \subset A$ where $QA$ is some divisible hull of $A$.

Let $X \in \text{FEE}(A, e)$ with $A$ completely decomposable, and set $\tilde{X} = \mathbb{Z}/e\mathbb{Z}$. Further choose an $e$-basis $\mathcal{V}' = \{v_1, \ldots, v_r\}$ of $A$. Consider the natural epimorphism

$$ \tau : A \to A/eA = \tilde{A}. $$

The quotient group $\tilde{A}$ is a $\mathbb{Z}$-module, and $\tilde{A} = \mathbb{Z} \tilde{v} = \mathbb{Z} \tilde{v}_1 + \cdots + \mathbb{Z} \tilde{v}_r$. Thus $\mathcal{V}' = \{\tilde{v}_1, \ldots, \tilde{v}_r\}$ is a basis of $\tilde{A}$ but $\mathcal{V}$ need not be a free basis. It is a free $\mathbb{Z}$-basis if no $\tau_i$ is $p$-divisible for any prime divisor $p$ of $e$. Since $e\mathcal{X} \cong X/A$ is finitely generated by, say, $k$ elements $\tilde{x} = \{x_1 + A, \ldots, x_k + A\}$, there is a matrix $mX = [m_{ij}] \in M_{k \times r}(\tilde{Z})$ such that $\tau x_i = \sum_{j=1}^k m_{ij} \tilde{v}_j$, $i = 1, \ldots, k$. Then $\tau \mathcal{X} = \tilde{Z} mX \mathcal{V}$. Adopting the terminology of Dugas–Oxford [7] we call $mX$ a representing matrix of $X$ with respect to the $e$-basis $\mathcal{V}'$ and the generating set $\tilde{x}$. Clearly representing matrices are not unique.

We will make the connection between a standard description $X = A + \tilde{Z} N^{-1} M \mathcal{V}$ of an almost completely decomposable group and a representing matrix in $\tilde{A}$. Consistent with previous notation let $\tau : \mathbb{Z} \to \mathbb{Z}/e\mathbb{Z} = \tilde{Z}$ be the natural epimorphism and denote by the same symbol the induced mapping on matrices: $\tau : M_{k \times r}(\mathbb{Z}) \to M_{k \times r}(\tilde{Z})$. 
We single out one step involved.

**Lemma 3.1.** Let \( X = A + \mathbb{Z}N^{-1}a^i \) be a standard description of an almost completely decomposable group \( X \) with \( \gcd^d(N,a^i) = 1 \), and let \( e \) be a positive integer. Then \( eX \in A \) if and only if \( eN^{-1} \) is integral.

**Proof.** First assume that \( eN^{-1} \) is integral. Then \( eX = eA + e\mathbb{Z}N^{-1}a^i = eA + \mathbb{Z}(eN^{-1})a^i \subseteq A \).

Conversely, assume that \( eX \subseteq A \). Let \( S = P\mathbb{Q} = \text{diag}(d_1, \ldots, d_k) \) be the Smith Normal Form of \( N \), where \( P \) and \( Q \) are invertible integral matrices. Then \( d_k = \exp(X/A) \) and therefore \( d_k \) divides \( e \). We conclude that \( eS^{-1} \) is an integral matrix. Hence \( eN^{-1} = QeS^{-1}P \) is integral. \( \square \)

**Proposition 3.2.** Let \( X \in \text{FEE}(A,e) \), \( A \) completely decomposable, and let \( X = A + \mathbb{Z}N^{-1}Mv^i \) be a standard description with integral coordinate matrix \( M \) with respect to an \( e \)-basis \( \mathcal{V} = \{v_1, \ldots, v_r\} \) of \( A \). Assume that \( \gcd^d(N,Mv^i) = 1 \). Then \( eN^{-1} \) is a representing matrix of \( X \) in \( \tilde{A} \) with respect to the \( e \)-basis \( \mathcal{V} \) and the generating set of \( X/A \) implicit in the standard description.

**Proof.** By Lemma 3.1 the matrix \( eN^{-1} \) is integral. From \( eX = eA + e\mathbb{Z}N^{-1}Mv^i \subseteq A \) it follows that \( eX = \mathbb{Z}(eN^{-1}M)v^i \). \( \square \)

We reiterate our interest in “normal forms” of coordinate matrices by which we mean special forms that facilitate the treatment and comparison of the groups described by them. By passing to the quotient \( \tilde{A} \) additional simplifications of the coordinate matrix in a standard description are possible. The basis is the following triviality (3.1) together with the fact that a matrix in \( \mathbb{M}_k(\mathbb{Z}) \) whose determinant is relatively prime to \( e \) becomes invertible in \( \mathbb{M}_k(\mathbb{Z}/e\mathbb{Z}) \).

For \( X,Y \in \text{FEE}(A,e) \), \( X = Y \) if and only if \( \overline{eX} = \overline{eY} \). \hspace{1cm} (3.1)

**Lemma 3.3.** Let \( X \in \text{FEE}(A,e) \), and \( A = \sigma_1v_1 + \cdots + \sigma_rv_r \) where \( \mathcal{V} = \{v_1, \ldots, v_r\} \) is an \( e \)-basis of \( A \). Suppose that \( N \) is a structure matrix of \( X \), i.e., \( X = A + \mathbb{Z}N^{-1}a^i \). Let \( mX = eN^{-1}m \) be a representing matrix of \( X \) relative to \( \mathcal{V} \) and the generating set of \( X/A \) implicit in the standard description. If \( M \) is a matrix with \( \tilde{M} = m \), then \( X = A + \mathbb{Z}N^{-1}Mv^i \) is a standard description of \( X \). If \( \gcd^d(N,a^i) = 1 \), then \( \gcd^d(N,Mv^i) = 1 \).

**Proof.** Set \( Y = A + \mathbb{Z}N^{-1}Mv^i \). It is easily checked that \( \overline{eY} = \mathbb{Z}(eN^{-1}Mv^i) = \mathbb{Z}mXv^i = \overline{eX} \) and hence, by (3.1), \( X = Y \). The last claim follows from \( X/A \cong \overline{eX} \) and (2.2). \( \square \)

The next lemma improves on Lemma 2.2.
Lemma 3.4. Let \( X = A + \mathbb{Z}N^{-1}Mv \) be an almost completely decomposable group in standard description. Let \( e = \det N \). Suppose that there exist matrices \( P, Q \in M_k(\mathbb{Z}) \) such that \( \gcd(e, \det P) = 1 \) and \( PN = NQ \). Then \( X = A + \mathbb{Z}N^{-1}PMv \).

Proof. By Lemma 3.1 \( eX \subset A \). Observe that \( \det Q = \det P \) is also relatively prime to \( e \) so that both \( \tilde{P} \) and \( \tilde{Q} \) are invertible matrices in \( M_k(\mathbb{Z}) \). The commutativity relation \( PN = NQ \) implies that \( (eN^{-1})P = Q(eN^{-1}) \). The identity

\[
\mathbb{Z}(eN^{-1})PMv = \mathbb{Z}(eN^{-1})Q(eN^{-1})Mv = eX
\]

shows that \( eN^{-1}PM \) is a representing matrix of \( X \) and the claim follows from Lemma 3.3.

Lemma 3.5. Let \( X \) be a \( p \)-reduced \( p \)-local almost completely decomposable group given in standard description \( X = A + \mathbb{Z}N^{-1}A' \) where \( A' = [a_1, \ldots, a_k]^\mathsf{T} \) with \( a_i \in A \) and \( \gcd(N, A') = 1 \). Let \( V' \) be any \( p \)-basis of \( A \) such that \( X = Mv \) for an integral matrix \( M \in M_{k \times r}(\mathbb{Z}) \). Then there is an ordering \( V = \{v_1, \ldots, v_r\} \) of the \( p \)-basis \( V' \) such that \( M = [E|F] \) where \( E \) is a \( k \times k \)-submatrix of \( M \) that has determinant relatively prime to \( p \).

Proof. Choose any \( p \)-basis \( V' \) of \( A \) such that \( X = Mv \) for an integral matrix \( M \). Then \( V' \) is a \( p \)-independent set in \( A \). Also the set \( \{a_1, \ldots, a_k\} \) is \( p \)-independent in \( A \) since \( \gcd(A', A') = 1 \) (Corollary 2.8). It follows (Lemma 2.9) that the rows \( M[i_1] \) of \( M \) are \( p \)-independent in \( \mathbb{Z}^r \). Therefore (Lemma 2.9) \( M \) contains a submatrix \( E \) such that \( \gcd(p, \det E) = 1 \). By relabeling types (and so permuting the columns of \( M \)) it may be assumed that \( M = [E|F] \) such that \( \gcd(p, \det E) = 1 \).

The following theorem contains the best we can say in general about special forms of representing matrices.

Theorem 3.6. Let \( p \) be a prime and \( X \) a finite essential extension of a completely decomposable group \( A \) such that

\[
\frac{X}{A} \cong \mathbb{Z}(p^{d_1}) \oplus \cdots \oplus \mathbb{Z}(p^{d_k}) \quad \text{with} \quad 1 \leq d_1 \leq \cdots \leq d_k = d.
\]

Then there is a \( p \)-basis \( V' = \{v_1, \ldots, v_r\} \) of \( A \) such that \( X \) has a representing matrix in \( \tilde{A} = A/p^dA \) of the form

\[
m_X = D[E|F] \quad \text{where} \quad D = \text{diag}(p^{d-d_1}, \ldots, p^{d-d_k})
\]

and

\[
E = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
m_{21} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{k-1,1} & m_{k-1,2} & \cdots & 1 & 0 \\
m_{k,1} & m_{k,2} & \cdots & m_{k,k-1} & 1
\end{bmatrix}, \quad m_{ij} \in \mathbb{Z} = \mathbb{Z}/p^d\mathbb{Z}
\]
and \( m_{ij} = 0 \) for \( j < i \) if \( d_i = d_j \). In particular, when \( d_1 = \cdots = d_k = d \), there is a \( p \)-basis \( \mathcal{V} = \{ v_1, \ldots, v_r \} \) of \( A \) such that \( X \) has a representing matrix of the form \( m_X = [E|F] \) where \( E \) is the \( k \times k \) identity matrix.

**Proof.** According to Lemma 3.5 we may assume that \( X = A + \frac{1}{p} N^{-1} M \tilde{v} \) for some \( p \)-basis \( \mathcal{V} \) of \( A \) where \( M = [E'|F'] \) and \( \gcd(p, \det E') = 1 \). It may be assumed further that \( N = \text{diag}(p^{d_1}, \ldots, p^{d_k}) \) with \( 1 \leq d_1 \leq \cdots \leq d_k \). By Proposition 3.2 the matrix \( \frac{1}{p^d} N^{-1} [E'[F']] \) is a representing matrix of \( X \) in \( \tilde{A} = A/p^d A \). The matrix \( E' \in \mathcal{M}_k(\mathbb{Z}) \) is now invertible. There is an upper triangular matrix \( P \in \mathcal{M}_k(\mathbb{Z}) \) such that \( PE' = E \) has the form (3.2). Set \( D = \frac{1}{p^d} N^{-1} \). Then there is another invertible matrix \( Q \in \mathcal{M}_k(\mathbb{Z}) \) such that \( DP = QD \) (cf. with (2.7)) and we obtain

\[
\frac{1}{p} p^d N^{-1} \tilde{M} \tilde{v} = \frac{1}{p} DPP^{-1} [E'|F'] \tilde{v} = \frac{1}{p} Q^{-1} D[PE'|PF'] \tilde{v} = \frac{1}{p} D[E|F] \tilde{v}.
\]

This says that \( m_X = D[E|F] \) is a representing matrix of \( X \) with respect to \( \mathcal{V} \). \qed

The results on representing matrices have consequences for the standard description of an almost completely decomposable group.

**Theorem 3.7.** Let \( A \) be a completely decomposable group with \( p \)-basis \( \mathcal{V} \) and let \( X = A + \frac{1}{p} N^{-1} M \tilde{v} \) with \( N = \text{diag}(p^{d_1}, \ldots, p^{d_k}) \) in Smith Normal Form and \( M \in \mathcal{M}_{k \times r}(\mathbb{Z}) \). Then there exists an ordering \( \mathcal{V} = \{ v_1, \ldots, v_r \} \) such that \( X = A + \frac{1}{p} N^{-1} [E|F] \tilde{v} \) for an integral matrix \( [E|F] \) where \( E \) has the form (3.2) with \( m_{ij} = 0 \) if \( j < i \), \( d_i = d_j \). In particular, if \( X \) is uniform, i.e., \( d_1 = \cdots = d_k \), then \( E \) may be chosen to be the identity matrix.

**Proof.** Apply Theorem 3.6 and view the matrix \( [E|F] \) in that theorem as integral matrices with entries reduced modulo \( e \). Lemma 3.3 does the rest. \( \square \)

### 4. Two applications

As a first application we prove the following well-known result on almost completely decomposable groups with two critical types that has become the standard test for new methods. We will use the following part of the so-called Purification Lemma [13, Lemma 11.4.1]) of [2].

**Lemma 4.1** (Purification Lemma). Assume that \( A = B \oplus C \) is an arbitrary torsion-free abelian group, \( a' = b' + c' \), where \( b' \in B' \) and \( c' \in C' \). Let \( X = A + \frac{1}{p} N^{-1} a' \) be a finite essential extension of \( A \).

1. Let \( N_B = \gcd^3(N, c') \). Then \( N_B \) is non-singular and \( B^X \) is a finite essential extension of \( B \) having the description \( B^X = B + \frac{1}{p} N_B^{-1} b' \).
2. \( X = (B_1^X \oplus C) + \mathbb{Z}(N_0^{-1} N)^{-1}(N_0^{-1} b^i + N_0^{-1} c^i) \), where \( N_0^{-1} b^i \in (B_1^X)^i \), and \( N_0^{-1} c^i \in C^i \).

3. If \( \gcd(A, a') = 1 \), then \( \gcd(A, b') = 1 \) and \( \gcd(B_1^X \oplus C (N_0^{-1} N, N_0^{-1} b^i + N_0^{-1} c^i)) = 1 \).

**Proposition 4.2** (Arnold, Lewis). Let \( X \) be a \( p \)-local almost completely decomposable group with two critical types \( T_{cr}(X) = \{\tau_1, \tau_2\} \). Then \( X \) is the direct sum of groups of rank one and two.

**Proof.** Let \( T_{cr}(X) = \{\tau_1, \tau_2\} \). The case of comparable types being clear, we assume that \( \tau_1 \) and \( \tau_2 \) are incomparable. Let \( A = A_{\tau_1} \oplus A_{\tau_2} \) be a homogeneous decomposition of the regulating regulator \( A = R(X) \) of \( X \). We assume, without loss of generality, that \( X \) is clipped, i.e., \( X \) has no direct summands of rank one. This implies in particular that neither \( \tau_1 \) nor \( \tau_2 \) is \( p \)-divisible. Without loss of generality, we can write \( X = A + \mathbb{Z} N^{-1} M \) such that \( Y = \{v_1, \ldots, v_r\} \) is a \( p \)-basis of \( A \), and \( N = \text{diag}(p^{d_1}, \ldots, p^{d_k}) \) with \( 1 \leq d_1 \leq \cdots \leq d_k \). Since \( X \) is clipped it follows from [15, Lemma 3.7] that \( rk A_{\tau_i} = k_i \).

We now assume that the basis elements \( v_i \) are so listed that the first \( k \) form a \( p \)-basis of \( A_{\tau_1} \) and the second batch of \( k \) elements forms a \( p \)-basis of \( A_{\tau_2} \). Let \( M[\tau_1] \) be the submatrix of \( M \) consisting of the first \( k \) columns, and let \( M[\tau_2] \) be the submatrix of \( M \) consisting of the last \( k \) columns of \( M \), so that \( M = [M[\tau_1] | M[\tau_2]] \). Further set \( v[\tau_1] = [v_1, \ldots, v_k]^t \) and \( v[\tau_2] = [v_{k+1}, \ldots, v_{2k}]^t \). With this notation

\[
X = A + \mathbb{Z} N^{-1}[M[\tau_1] | M[\tau_2]] \begin{bmatrix} v[\tau_1] \\ v[\tau_2] \end{bmatrix}.
\]

Since \( A \) is regulating in \( X \) the homogeneous summand \( A_{\tau_2} \) is pure in \( X \). By the Purification Lemma 4.1 this means that \( \gcd(N, M[\tau_2]) = 1 \). Hence there exist matrices \( U, V \in \mathcal{M}_k(\mathbb{Z}) \) such that \( I = NU + M[\tau_2]V \). Modulo \( p \) we have therefore \( I \equiv M[\tau_2]V \) mod \( p \). It follows that \( \gcd(p, \det M[\tau_2]) = 1 \). By symmetry \( \gcd(p, \det M[\tau_1]) = 1 \) also. Now consider the group \( Y = A + \mathbb{Z} N^{-1} I_k | I_k \) where \( I_k \) is the \( k \times k \) identity matrix. The group \( Y \) is evidently a direct sum of rank-two groups. Also note that

\[
\bar{e}^i = \begin{bmatrix} M[\tau_1] & 0 \\ 0 & M[\tau_1] \end{bmatrix} \begin{bmatrix} v[\tau_1] \\ v[\tau_2] \end{bmatrix}
\]

defines a type automorphism of \( \bar{A} \). We now compute that

\[
\bar{e} Y \xi = (\mathbb{Z} e N^{-1} [I_k | I_k] \bar{e}^i) \xi = \mathbb{Z} e N^{-1} [I_k | I_k] \begin{bmatrix} M[\tau_1] & 0 \\ 0 & M[\tau_1] \end{bmatrix} \begin{bmatrix} v[\tau_1] \\ v[\tau_2] \end{bmatrix} = \mathbb{Z} e N^{-1} [M[\tau_1] | M[\tau_2]] \bar{e}^i = \bar{X}.
\]

This says that \( X \) and \( Y \) are type isomorphic and therefore near-isomorphic. By a well-known fundamental theorem of Arnold this means that \( X \), just as \( Y \), is a direct sum of rank-two groups. \( \square \)

Arnold and Dugas [1, Theorem 1.8] state that there are indecomposable \( p \)-local almost completely decomposable groups of arbitrarily large rank whose critical typeset
contains merely three types $\tau_1, \tau_2, \tau_3$ ordered as . In contrast, we will show that an indecomposable group with critical typeset . and homocyclic regulating quotient (a direct sum of cyclic groups all of the same order) has rank at most 3. Note that such a group $X$ has a regulating regulator, or, equivalently, $\beta_1^X = \beta_2^X = \beta_3^X = 1$.

Hypothesis. Let $p$ be a prime, $\tau_1, \tau_2, \tau_3$ rational groups not containing $1/p$ and ordered as types as shown. Further assume that $\tau_2 \subset \tau_3$. Let

$$A = \sigma_1 v_1 + \cdots + \sigma_r v_r = A_{\tau_1} + A_{\tau_2} + A_{\tau_3},$$

where

$$\sigma_1 = \cdots = \sigma_k = \tau_1, \quad \sigma_{k+1} = \cdots = \sigma_l = \tau_2, \quad \sigma_{l+1} = \cdots = \sigma_r = \tau_3.$$

Finally, let $X = A + \mathbb{Z} N^{-1} M v^j$ be an almost completely decomposable group given in standard description where $N = \text{diag}(p^{d_1}, \ldots, p^{d_k}), \ 1 \leq d_1 \leq \cdots \leq d_k$, $M \in \mathbb{M}_{k \times r}(\mathbb{Z})$, and $\gcd(N, M) = I$. Further set

$$M[\tau_1] = M[\tau_1], \quad M[\tau_2] = M[\tau_2], \quad M[\tau_3] = M[\tau_3]$$

and

$$v[\tau_1] = [v_1, \ldots, v_k]^T, \quad v[\tau_2] = [v_{k+1}, \ldots, v_l]^T, \quad v[\tau_3] = [v_{l+1}, \ldots, v_r]^T.$$

With this notation

$$A_{\tau_1} = \tau_1 v[\tau_1], \quad A_{\tau_2} = \tau_2 v[\tau_2], \quad A_{\tau_3} = \tau_3 v[\tau_3],$$

$$M = [M[\tau_1] | M[\tau_2] | M[\tau_3]] \quad \text{and} \quad v' = \begin{bmatrix} v[\tau_1] \\ v[\tau_2] \\ v[\tau_3] \end{bmatrix}.$$

The following two examples are assumed to have critical typesets .

Example 4.3. Let $A = \tau_1 v_1 \oplus \tau_2 v_2 \oplus \tau_3 v_3$ and $n \geq 2$. Then the group

$$X = A + \mathbb{Z} p^{-n} [1 \ p^{m} \ 1] v^j,$$

is indecomposable.

Proof. The reader may check this directly, or use that there are no sharp types [17].

Example 4.4. Let $2 \leq d_1 \leq d_2$. The group

$$X = A + \mathbb{Z} \begin{bmatrix} p^{d_1} & 0 \\ 0 & p^{d_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} v^j$$
\[ (v_1 + pv_3 + v_4) - p^{d_2 - d_1} (v_2 + pv_3 + v_5) = \frac{1}{p^{d_1}}(v_1 - v_2 + v_4 - v_5) \]

The decomposition in Example 4.4 can also be recognized by means of a manipulation of the coordinate matrix

\[
M = \begin{bmatrix} 1 & 0 & p & 1 & 0 \\ 0 & 1 & p & 0 & 1 \end{bmatrix}
\]

using Lemmas 2.2 and 2.3 as follows.

\[
X = A + \mathbb{Z} \begin{bmatrix} p^{d_1} & 0 \\ 0 & p^{d_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & p & 0 & 1 \end{bmatrix} v^i
\]

\[
= A + \mathbb{Z} \begin{bmatrix} p^{d_1} & 0 \\ 0 & p^{d_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & p & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & v[\tau_1] \\ 0 & 1 & v[\tau_2] \\ 0 & 1 & v[\tau_3] \end{bmatrix}
\]

\[
= (\tau_1 (v_1 - v_2) + \tau_3 (v_4 - v_5)) + \mathbb{Z} \frac{1}{p^{d_1}}((v_1 - v_2) + (v_4 - v_5))
\]

\[
\oplus \left( (\tau_1 v_2 + \tau_2 v_3 + \tau_3 v_5) + \mathbb{Z} \frac{1}{p^{d_2}}(v_2 + pv_3 + v_5) \right).
\]

We explore next what it means for the description of \( X \) if \( X \) is clipped. Recall that the width of a finite abelian group \( G \) is defined by

\[
\text{width}(G) = \max \{ \dim G[p] : p \in \mathbb{P} \}.
\]

If \( G \) is written in principal divisor form \( G = G_1 \oplus \cdots \oplus G_w \) where \( G_i \) is cyclic of order \( d_i \) and \( d_i \) divides \( d_{i+1} \) for each \( i \), then \( w = \text{width}(G) \). In particular, if \( X = A + \mathbb{Z} N^{-1} a' \) is a finite essential extension of a torsion-free group \( A \), \( \gcd(N, a') = 1 \), and \( N \) is in Smith Normal Form, then \( \text{width}(X/A) \) is the number of entries of \( N \) that are \( > 1 \).

According to Mader and Nongxa [15, Theorem 3.10] the group \( X \) is clipped if and only if the widths of the quotients

\[
\frac{X^s[\tau_1]}{X(\tau_1) + X[\tau_1]} = \frac{X}{A},
\]

\[
\frac{X^s[\tau_2]}{X(\tau_2) + X[\tau_2]} = \frac{X}{A_{\tau_1} \oplus (A_{\tau_1} \oplus A_{\tau_1})^X},
\]

(4.1) (4.2)
and let

\[ \frac{X^\mathcal{D}[\tau_3]}{X(\tau_3) + X[\tau_3]} = (A_{\tau_1} \oplus A_{\tau_3})^Y_{\tau_1 \oplus A_{\tau_3}}. \]

(4.3)

equal the ranks \( \text{rk} A_{\tau_1}, \text{rk} A_{\tau_2}, \text{rk} A_{\tau_3} \) respectively.

For a non-singular integral matrix \( N \) we define the width of \( N \), denoted \( \text{width}(N) \) to be the number of non-one invariant factors, or equivalently, the number of entries \( > 1 \) in the Smith Normal Form of \( N \).

**Lemma 4.5.** Assume \( \mathcal{D} \) and let \( D = \gcd(N,M[\tau_2]) \). Then \( X \) is clipped if and only if \( \text{rk} A_{\tau_1} = \text{width}(N) = k \), \( \text{rk} A_{\tau_2} = \text{width}(D) \), and \( \text{rk} A_{\tau_3} = \text{width}(D^{-1}N) \).

**Proof.** By the Purification Lemma 4.1

\[(A_{\tau_1} \oplus A_{\tau_3})^X_{\tau_1} = (X(\tau_1) + X(\tau_3))^X_{\tau_1},\]

and

\[X = ((A_{\tau_1} \oplus A_{\tau_3})^X_{\tau_1} \oplus A_{\tau_2})\]

\[+ \mathbb{Z}(D^{-1}N)^{-1} \left( D^{-1}[M[\tau_1] | M[\tau_3]] \begin{bmatrix} v[\tau_1] \\ v[\tau_3] \end{bmatrix} - D^{-1}M[\tau_2]v[\tau_2] \right).\]

By Lemma 4.1(3) we have \( \gcd(D,[M[\tau_1] | M[\tau_3]]) = I \) and

\[\gcd((A_{\tau_1} \oplus A_{\tau_3})^X_{\tau_1} \oplus A_{\tau_2}; D^{-1}N,D^{-1}[M[\tau_1] | M[\tau_3]] \begin{bmatrix} v[\tau_1] \\ v[\tau_3] \end{bmatrix} - D^{-1}M[\tau_2]v[\tau_2]) = I.\]

This means that \( X|A \cong \mathbb{Z}/\mathbb{Z} N \), \( (A_{\tau_1} \oplus A_{\tau_3})^X_{\tau_1} / (A_{\tau_1} \oplus A_{\tau_3}) \cong \mathbb{Z}/\mathbb{Z} D \), and, thirdly, \( X'((A_{\tau_1} \oplus A_{\tau_3})^X_{\tau_1} \oplus A_{\tau_2}) \cong \mathbb{Z}/\mathbb{Z}(D^{-1}N) \). By Mader and Nongxa [15, Theorem 3.10] it now follows from (4.1) that \( X \) has no rank-one summand of type \( \tau_1 \) if and only if \( \text{rk} A_{\tau_1} = \text{width}(N) \), it follows from (4.2) that \( X \) has no rank-one summand of type \( \tau_2 \) if and only if \( \text{rk} A_{\tau_2} = \text{width}(D^{-1}N) \), and it follows from (4.3) that \( X \) has no rank-one summand of type \( \tau_3 \) if and only if \( \text{rk} A_{\tau_3} = \text{width}(D) \). \( \square \)

**Theorem 4.6.** Assume \( \mathcal{D} \) and assume further that \( X \) has a homocyclic regulator quotient \( X/R(X) \) of exponent \( p^\mathcal{D} \). Then \( X \) is a direct sum of groups of rank \( \leq 3 \).

**Proof.** Note that \( X(\tau_2) \) is completely decomposable since it has a linearly ordered critical typeset. Since \( A \) is the regulating regulator of \( X \), the subgroup \( A_{\tau_1} \oplus A_{\tau_3} \subset X(\tau_2) \) is pure in \( X \) and this means that \( \gcd(N,M[\tau_1]) = I \) (Purification Lemma). Hence there are integral matrices \( U, V \in \mathcal{M}(\mathbb{Z}) \) such that \( I = N \cdot U + M[\tau_1] \cdot V. \) Let \( \overline{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) be the natural epimorphism, and use the same symbol for the map on integral matrices induced by applying the scalar mapping component-wise. We obtain from the Bézout equation above that \( \overline{I} = \overline{M[\tau_1]} \cdot \overline{V}. \) Consequently, \( \det \overline{M[\tau_1]} \neq 0 \), and so \( \gcd(p, \det M[\tau_1]) = 1. \) The assumption that \( X/A \) is homocyclic means that \( N = p^\mathcal{D} I_k \) is a scalar matrix, and consequently it commutes with every matrix in \( \mathcal{M}(\mathbb{Z}). \) Recall
that a row transformation on a matrix is left multiplication by an invertible matrix. By Lemma 2.2 we may perform any row transformation on \( M \), and by Lemma 2.3 any column transformation within the blocks \( M[\tau_1], M[\tau_2], M[\tau_3] \). Hence we may assume, without loss of generality, that \( M[\tau_2] \in \mathbb{M}_{k \times \ell}(\mathbb{Z}) \) is in Smith Normal Form, which, assuming without loss of generality that \( X \) is clipped, is of the form

\[
M[\tau_2] = \begin{bmatrix}
diag(p^{e_1}, \ldots, p^{e_{\ell-k}}) \\
0_{(2k-\ell) \times (\ell-k)}
\end{bmatrix}, \quad 0 \leq e_1 \leq \cdots \leq e_{\ell-k}, \quad e_i < d.
\]

To see this note that \( X \) would have a rank-one summand of type \( \tau_2 \) if \( M[\tau_2] \) had a zero column or if it were true that \( e_i \geq d \). By the theorem of Arnold, \( X \) is a direct sum of groups of rank \( \leq 3 \) if and only if any near-isomorphic group has this property. Recall that \( M[\tau_1] \) is invertible in \( \mathbb{M}_k(\mathbb{Z}) \). By replacing \( X \) by a near-isomorphic group, we may assume without loss of generality that \( M[\tau_1] = I_k \) and, by Lemma 3.3 \( M[\tau_1] = I_k \).

Write \( M = [m_{ij}] \). If \( e_1 = 0 \), then we use the well-defined homomorphism

\[
\phi : \tau_2 v_{k+1} \rightarrow X^F(\tau_2) = A_{\tau_3} : v_{k+1} \phi = m_{1,\ell+1}v_{\ell+1} + \cdots + m_{1,r}v_r,
\]

to replace the basis element \( v_{k+1} \) by the new basis element \( v'_{k+1} = v_{k+1} + m_{1,\ell+1}v_{\ell+1} + \cdots + m_{1,r}v_r \). With respect to the basis \( \{v_1, \ldots, v'_{k+1}, \ldots, v_r\} \), the coordinate matrix is

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 0 & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0
\end{bmatrix}
\]

and this shows that \( X \) has a summand of rank 2, namely \( (\tau_1 v_1 \oplus \tau_2 v'_{k+1}) + \mathbb{Z} p^{-d}(v_1 + v'_{k+1}) \). By induction on rank, the theorem is proved in this case.

We therefore assume that \( e_1 > 0 \). It is now clear that \( D = \gcd(N, M[\tau_2]) = \diag(p^{e_1}, \ldots, p^{e_{\ell-k}}, p^{d_1}, \ldots, p^{d_l}) \) where \( 1 \leq e_1 \leq \cdots \leq p^{d_l} \), and that \( D \) has width \( k \). Since \( X \) is clipped this means that \( \text{rk} A_{\tau_2} = k \), and \( \ell = 2k \). It follows next that \( D^{-1}N = \diag(p^{-e_1}, \ldots, p^{-e_{\ell-k}}, p^{d_2}, \ldots, p^{d_l}) \) which has width \( k \). By Lemma 4.5 \( \text{rk} A_{\tau_1} = k \). The fact that \( A_{\tau_1} \) is pure in \( X \) means that \( \gcd(N, [M[\tau_2], M[\tau_3]]) = 1 \). Since \( M[\tau_2] \) is a left factor of \( N \), it follows that \( \gcd(M[\tau_2], M[\tau_3]) = 1 \). Hence there are matrices \( U, V \) such that \( I = M[\tau_2]U + M[\tau_3]V \). This implies that \( M[\tau_3] \) is invertible modulo \( p \). Again passing to a near-isomorphic group we may assume that

\[
X = A + \mathbb{Z} p^{-d}[I_k | \diag(p^{e_1}, \ldots, p^{e_{\ell-k}})] I_k v_1
\]

This shows that \( X \) is the direct sum of rank-three groups.

References