Kite-free Distance-regular Graphs

PAUL TERWILLIGER

Let $\Gamma$ denote a distance-regular graph with classical parameters $(d, b, \alpha, \beta)$. Suppose that the diameter $d \geq 3$, and suppose that the parameter $b$ (which is known to be an integer) satisfies $b < -1$. Then we show that $\Gamma$ does not contain vertices $x, y, z$ and $u$ such that $x, y, z$ are mutually adjacent, and such that $u$ is at distance $\delta(u, y) = \delta(u, z) = \delta(u, x) = i$. We conclude that the Hermitean forms graph $\text{Her}(d, q)$ is uniquely determined by its intersection numbers if $d \geq 3$.

© 1995 Academic Press Limited

1. INTRODUCTION

Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ (see formal definitions below), and pick any integer $i$ ($2 \leq i \leq d$). By a kite of length $i$ in $\Gamma$, we mean a 4-tuple $xyzu$ of vertices of $\Gamma$ such that $x, y$ and $z$ are mutually adjacent, and $u$ is at distance $\delta(u, x) = i, \delta(u, y) = i - 1, \delta(u, z) = i - 1$.

Pick any 3-tuple $xyz$ of mutually adjacent vertices in $\Gamma$, and let $e_i(xyz)$ denote the inverse of the intersection number $\pi_i$, times the number of vertices $u$ of $\Gamma$ such that $xyzu$ is a kite of length $i$. In general, $e_i(xyz)$ depends on the choice of $x, y$ and $z$ as well as $i$. In this paper, we assume that $\Gamma$ has a $Q$-polynomial structure, and show that

$$e_i(xyz) = \alpha_i e_2(xyz) + \beta_i \quad (2 \leq i \leq d)$$

for appropriate real scalars $\alpha_i$ and $\beta_i$ that do not depend on $x, y$ and $z$. In fact, $\alpha_i$ and $\beta_i$ are rational expressions in the dual eigenvalues.

We show, in a certain case, that the dependencies in (1) imply that $\Gamma$ has no kites. Indeed, suppose that $\Gamma$ has classical parameters $(d, b, \alpha, \beta)$. Then it turns out that $\alpha_i = 1 + b + b^2 + \cdots + b^{i-1}$ and $\beta_i = 0$ ($2 \leq i \leq d$). It is known that $b$ is an integer with $b < -1$ or $b > 0$ [2, p. 195]. In the case $b < -1$, the fact that $e_2(xyz)$ and $e_3(xyz) = e_2(xyz)(1 + b)$ are non-negative implies that $e_2(xyz) = 0$, forcing $e_i(xyz) = 0$ ($2 \leq i \leq d$) by (1), from which we conclude that $\Gamma$ has no kites. We use this to strengthen a result of Ivanov and Shpectorov concerning the Hermitean forms graph $\text{Her}(d, q)$. In [3], Ivanov and Shpectorov showed that, for diameter $d \geq 3$, the graph $\text{Her}(d, q)$ is characterized by (i) its intersection numbers and (ii) the assumption that there are no kites of length 2. Our results show that condition (ii) is redundant, since $\text{Her}(d, q)$ has classical parameters satisfying $b = -q \leq -2$. Hence $\text{Her}(d, q)$ is characterized by its intersection numbers if the diameter $d \geq 3$. The main results of this paper are Theorems 2.6, 2.7, 2.11, 2.12 and Corollary 2.13.

For the rest of this section, we review some definitions and basic concepts. See the books of Bannai and Ito [1], or Brouwer, Cohen and Neumaier [2], for more background information. The expert may wish to skip directly to Section 2.

Throughout this paper, $\Gamma$ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, path length distance function
\[ \delta, \text{ and diameter } d := \max \{ \delta(x, y) \mid x, y \in V \Gamma \}. \] 
\( \Gamma \) is said to be distance-regular whenever, for all integers \( h, i, j \) \((0 \leq h, i, j \leq d)\), and all \( x, y \in V \Gamma \) with \( \delta(x, y) = h \), the number

\[ p^h_{ij} := \left| \{ z \mid z \in V \Gamma, \delta(x, z) = i, \delta(y, z) = j \} \right| \]

is independent of \( x \) and \( y \). The constants \( p^h_{ij} \) \((0 \leq h, i, j \leq d)\) are known as the intersection numbers of \( \Gamma \). For convenience, set \( c_i := p^1_{i-1} \) \((1 \leq i \leq d)\), \( a_i := p^1_{i} \) \((0 \leq i \leq d - 1)\), \( b_i := p^1_{i+1} \) \((0 \leq i \leq d - 1)\), \( c_0 = 0 \) and \( b_d = 0 \).

From now on, assume that \( \Gamma \) is distance-regular, with diameter \( d \geq 3 \).

For each integer \( i \) \((0 \leq i \leq d)\), the \( i \)-th distance matrix \( A_i \) of \( \Gamma \) has rows and columns indexed by \( V \Gamma \), and \( x, y \) entry

\[ (A_i)_{xy} = \begin{cases} 1, & \text{if } \delta(x, y) = i, \\ 0, & \text{if } \delta(x, y) \neq i \end{cases} \]

\((x, y \in V \Gamma)\). \(2\)

Then

\[ A_0 = I, \quad A_i^j = A_i \quad (0 \leq i \leq d), \]

and

\[ A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \quad (0 \leq i, j \leq d). \]

\(3\) \(4\)

By \(3\)–\(5\), the matrices \( A_0, A_1, \ldots, A_d \) form a basis for a commutative semi-simple \( \mathbb{R} \)-algebra \( M \), known as the Bose–Mesner algebra. By \([1, pp. 59, 64]\), \( M \) has a second basis \( E_0, E_1, \ldots, E_d \) such that

\[ E_0 = |V \Gamma|^{-1} I \quad (I = \text{all 1's matrix}), \quad E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d), \]

\(6\) \(7\)

\[ E_0 + E_1 + \cdots + E_d = I, \quad E_i' = E_i \quad (0 \leq i \leq d). \]

\(8\) \(9\)

The \( E_0, E_1, \ldots, E_d \) are known as the primitive idempotents of \( \Gamma \), and \( E_0 \) is known as the trivial idempotent.

Set \( A := A_1 \), and define the real numbers \( \theta_i \) \((0 \leq i \leq d)\) by

\[ A = \sum_{i=0}^{d} \theta_i E_i. \]

\(10\)

The scalars \( \theta_0, \theta_1, \ldots, \theta_d \) are distinct, since \( A \) generates \( M \) \([1, p. 197]\). The \( \theta_0, \theta_1, \ldots, \theta_d \) are known as the eigenvalues of \( \Gamma \).

Let \( E \) denote any primitive idempotent of \( \Gamma \). Then we have

\[ E = |V \Gamma|^{-1} \sum_{i=0}^{d} \theta_i^* A_i \]

\((E, y) = |V \Gamma|^{-1} \theta_i^* (x, y \in V \Gamma), \)

\(11\)

for some \( \theta_0^*, \theta_1^*, \ldots, \theta_d^* \in \mathbb{R} \), called the dual eigenvalues associated with \( E \).

Set \( V = \mathbb{R}^{|V \Gamma|} \) (column vectors), and view the co-ordinates of \( V \) as being indexed by \( V \Gamma \). Then the Bose–Mesner algebra \( M \) acts on \( V \) by left multiplication. We call \( V \) the standard module of \( \Gamma \). For each vertex \( x \in V \Gamma \), set

\[ \xi = (0, 0, \ldots, 0, 1, 0, \ldots, 0)', \]

\(12\)

where the 1 is in co-ordinate \( x \). Also, let \( \langle , \rangle \) denote the dot product

\[ \langle u, v \rangle = u'v \quad (u, v \in V). \]

\(13\)

Then, referring to the primitive idempotent \( E \) in \(11\), we compute from \(9\) and \(11\)–\(13\) that

\[ \langle E \xi, y \rangle = |V \Gamma|^{-1} \theta_i^* \quad (x, y \in V \Gamma), \]

\(14\)
Fix any \( x \in V\Gamma \). For each integer \( i \) (\( 0 \leq i \leq d \)), let \( A^*_{i} = A^*_{i}(x) \) denote the diagonal matrix with rows and columns indexed by \( V\Gamma \), and \( y, y \) entry
\[
(A^*_{i})_{yy} = |V\Gamma| (E_i)_yy \quad (y \in V\Gamma)
\]
\[
= |V\Gamma| \langle E_i x, y \rangle \quad (y \in V\Gamma).
\] (15)

Then, by (6) and the construction, we find that
\[
A^*_{0} = I
\] (16)
and
\[
A^*_{i} A^*_{j} = \sum_{h=0}^{d} q^h_{ij} A^*_{h} \quad (0 \leq i, j \leq d),
\]
where \( q^h_{ij} \) (\( 0 \leq h, i, j \leq d \)) are appropriate real scalars which are independent of \( x \). These scalars are known as the Krein parameters of \( \Gamma \). By a minor extension of [5, Lemma 3.2], we have
\[
[E_i A^*_{j} E_k, E_i A^*_{j} E_{k'}] = \delta_{ij} \delta_{kk'} q^h_{ij} \text{ rank } E_k \quad (0 \leq i, i', j, j', k, k' \leq d),
\] (17)
where \([R, S] = \text{trace } RS'\). In particular,
\[
E_i A^*_{j} E_k = 0 \text{ iff } q^h_{ij} = 0 \quad (0 \leq i, j, k \leq d).
\] (18)

\( \Gamma \) is said to be Q-polynomial (with respect to the given ordering \( E_0, E_1, \ldots, E_d \) of the primitive idempotents) whenever, for all integers \( h, i, j \) (\( 0 \leq h, i, j \leq d \)), \( q^h_{ij} \neq 0 \) (resp. \( q^h_{ij} = 0 \)) whenever one of \( h, i \) and \( j \) is greater than (resp. equal to) the sum of the other two. Let \( E \) denote any non-trivial primitive idempotent of \( \Gamma \). Then \( \Gamma \) is said to be Q-polynomial with respect to \( E \) whenever there exists an ordering \( E_0, E_1, \ldots, E_d \) of the primitive idempotents of \( \Gamma \), with respect to which \( \Gamma \) is Q-polynomial. If \( \Gamma \) is Q-polynomial with respect to \( E \), then the associated dual eigenvalues are distinct [5, p. 384].

2. Distance-regular Graphs with the Q-polynomial Property

Throughout this section, we will use the following notation.

**Definition 2.1.** Let \( \Gamma \) denote a distance-regular graph with diameter \( d \geq 3 \). For all \( x, y \in V\Gamma \), and all integers \( i \) and \( j \), define
\[
p^h_{ij}(x, y) := \sum_{\substack{z \in V\Gamma \\delta(x, z) = i \\delta(y, z) = j}} \hat{z},
\]
where the \( \hat{z} \) notation is from (12). Furthermore, define
\[
x^+_y := p^h_{1h+1}(x, y), \quad x^0_y := p^h_{1h}(x, y), \quad x^{-}_y := p^h_{1h-1}(x, y),
\]
where \( h = \delta(x, y) \).

In [4, Theorem 1.1], we essentially proved the following result.

**Lemma 2.2.** Let \( \Gamma \) denote a distance-regular graph with diameter \( d \geq 3 \), and suppose that \( \Gamma \) is Q-polynomial with respect to the primitive idempotent
\[
E_1 = |V\Gamma|^{-1} \sum_{h=0}^{d} \theta^h E_h.
\]
Then, for all integers $h$, $i$ and $j$ ($1 \leq h \leq d$, $0 \leq i, j \leq d$), and all $x, y \in V\Gamma$ such that $\delta(x, y) = h$, the vector

$$p_h(x, y) - p_i(x, y) - r_{ij}^h (x - y)$$

is orthogonal to $E_0 V + E_1 V$, where

$$r_{ij}^h = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*}.$$

Our first goal in this section is to prove a 'symmetric' version of the above lemma. Our results are presented in Theorems 2.6 and 2.7.

**Lemma 2.3.** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, and assume that $\Gamma$ is $Q$-polynomial with respect to the ordering $E_0$, $E_1, \ldots, E_d$ of the primitive idempotents. Pick $w \in V\Gamma$, write $A^* = A^*_w(w)$, and pick any matrix $F$ of the form

$$F = e_0 I + e_1 A^*,$$

where $e_0, e_1 \in \mathbb{R}\setminus\{0\}$. Then:

(i) $E_i F E_j = 0$ if and only if $|i - j| > 1$ $(0 \leq i, j \leq d)$.

(ii) Let $\Delta$ denote the vector space

$$\Delta := \text{span}\{mF_n + nF_m \mid m, n \in M\},$$

where $M$ denotes the Bose–Mesner algebra of $\Gamma$. Then $\Delta$ has a basis

$$\{E_i F E_i \mid 0 \leq i \leq d\} \cup \{E_i F E_{i+1} + E_{i+1} F E_i \mid 0 \leq i \leq d - 1\}.$$

In particular,

$$\dim \Delta = 2d + 1.$$

**Proof of (i).** Let the integers $i$ and $j$ be given. Observe by (16) and (21) that

$$E_i F E_j = e_0 E_i A^*_j E_j + e_1 E_i A^* E_j,$$ and observe by (17) that $E_i A^*_j E_j$ and $E_i A^* E_j$ are orthogonal with respect to the inner product $[,]$. It follows by (18) and the definition of $Q$-polynomial that

$$E_i F E_j = 0$$

iff

$$E_i A^*_j E_j = 0 \quad \text{and} \quad E_i A^* E_j = 0$$

iff

$$q_{ij}^0 = 0 \quad \text{and} \quad q_{ij}^1 = 0$$

iff

$$|i - j| > 1.$$

**Proof of (ii).** The matrices in (24) span $\Delta$. Indeed, $M$ is spanned by $E_0, E_1, \ldots, E_d$, so

$$\Delta = \text{span}\{mF_n + nF_m \mid m, n \in M\} = \text{span}\{E_i F E_j + E_j F E_i \mid 0 \leq i, j \leq d\} = \text{span}\{E_i F E_i \mid 0 \leq i \leq d\} + \text{span}\{E_i F E_{i+1} + E_{i+1} F E_i \mid 0 \leq i \leq d - 1\}.$$
by part (i) above. The matrices in (24) are linearly independent. Indeed, since $E_i = E_iA^*E_i$, we observe by (17) that the matrices
\[
\{E_i \mid 0 \leq i \leq d\} \cup \{E_iA^*E_i \mid 0 \leq i \leq d\}
\]
\[
\cup \{E_iA^*E_{i+1} \mid 0 \leq i \leq d - 1\} \cup \{E_{i+1}A^*E_i \mid 0 \leq i \leq d - 1\}
\]
are mutually orthogonal with respect to the inner product $[,]$. It follows from this and the construction of $F$ that the matrices in (24) are mutually orthogonal with respect to $[,]$. Also, it follows from the above orthogonality and (22) that each matrix in (24) is non-zero. It follows that the matrices in (24) are linearly independent, as desired.

**Lemma 2.4.** With the notation of Lemma 2.3,
\[
\Delta = \{mF + Fm \mid m \in M\} + \{mFA + AIm \mid m \in M\}. \tag{25}
\]
Moreover, $\Delta$ has a basis
\[
\{AhF + FAh \mid 0 \leq h \leq d\} \cup \{AhFA + AFAh \mid 1 \leq h \leq d\}. \tag{26}
\]

**Proof.** For notational convenience, set $E_{-1} = 0$ and $E_{d+1} = 0$, and let $\theta_{-1}$ and $\theta_{d+1}$ denote indeterminants. Let $\Delta_1$ denote the right-hand side of (25), and observe $\Delta_1 \subseteq \Delta$ by (23). To show that $\Delta \subseteq \Delta_1$, by Lemma 2.3 (ii) it suffices to show that
\[
E_iFE_i, \quad E_iFE_{i+1} + E_{i+1}FE_i \in \Delta_1 \tag{27}
\]
for all integers $i$ ($0 \leq i \leq d$). Let the integer $i$ be given, and by induction assume that (27) holds for all $i' < i$. Observe by (8) and (22) that
\[
E_iF + FE_i = E_i\left(\sum_{j=0}^{d} E_j\right) + \left(\sum_{j=0}^{d} E_j\right)FE_i
\]
\[
= E_iFE_{i-1} + E_{i-1}FE_i + 2E_iFE_i + E_iFE_{i+1} + E_{i+1}FE_i.
\]
Since $E_iF + FE_i$ is contained in $\Delta_1$ by construction, and since $E_iFE_{i-1} + E_{i-1}FE_i$ is contained in $\Delta_1$ by induction, we find that
\[
2E_iFE_i + E_iFE_{i+1} + E_{i+1}FE_i \in \Delta_1. \tag{28}
\]
Similarly, by (10) we have
\[
E_iFA + AFE_i = E_i\left(\sum_{j=0}^{d} \theta_jE_j\right) + \left(\sum_{j=0}^{d} \theta_jE_j\right)FE_i
\]
\[
= \theta_{i-1}(E_iFE_{i-1} + E_{i-1}FE_i) + \theta_i(2E_iFE_i)
\]
\[
+ \theta_{i+1}(E_iFE_{i+1} + E_{i+1}FE_i),
\]
yielding
\[
\theta_i(2E_iFE_i) + \theta_{i+1}(E_iFE_{i+1} + E_{i+1}FE_i) \in \Delta_1. \tag{29}
\]
Since $\theta_i \neq \theta_{i+1}$, we may solve (28) and (29) to obtain (27). This proves that $\Delta \subseteq \Delta_1$, so in fact $\Delta = \Delta_1$. Now observe that the $2d + 1$ matrices in (26) span the $(2d + 1)$-dimensional space $\Delta_1 = \Delta$, and so they form a basis for that space, by elementary linear algebra. This proves Lemma 2.4.

**Lemma 2.5.** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, and suppose
that $\Gamma$ is $Q$-polynomial with respect to the primitive idempotent $E_1$. Then, for all integers $i, j$ ($0 \leq i, j \leq d$), there exist real scalars

\begin{equation}
\begin{aligned}
s_{ij}^h & \quad (0 \leq h \leq d), \\
t_{ij}^h & \quad (1 \leq h \leq d)
\end{aligned}
\end{equation}

such that

\begin{equation}
\begin{aligned}
A_iA^*_jA_j + A_jA^*_iA_i &= \sum_{h=0}^{d} s_{ij}^h(A_hA^*_j + A^*_iA_h) \\
&\quad + \sum_{h=1}^{d} t_{ij}^h(A_hA^*_iA + AA^*_iA_h) \\
&\quad (A^*_i = A^*_i(w))
\end{aligned}
\end{equation}

for all vertices $w \in V\Gamma$ and all $t \in \{0, 1\}$. Moreover, each $s_{ij}^h$, $t_{ij}^h$ is uniquely determined by $h$, $i$, and $j$ (once the $Q$-polynomial structure is fixed).

**Proof.** Let the integers $i$ and $j$ be given. Concerning existence, pick $w \in V\Gamma$, pick $\epsilon_0, \epsilon_1 \in \mathbb{R}\setminus\{0\}$, and let $F$ be as in (21). Observe that $A_iF \epsilon_j + A_jF \epsilon_i \in \Delta$ by (23) and so, by (26), there exist scalars $s_{ij}^h \in \mathbb{R}$ $(0 \leq h \leq d)$, and scalars $t_{ij}^h \in \mathbb{R}$ $(1 \leq h \leq d)$, such that

\begin{equation}
\begin{aligned}
A_iF \epsilon_j + A_jF \epsilon_i &= \sum_{h=0}^{d} s_{ij}^h(A_hF + FA_h) + \sum_{h=1}^{d} t_{ij}^h(A_hF + AFA_h).
\end{aligned}
\end{equation}

Observe that the scalars $s_{ij}^h$ and $t_{ij}^h$ in (32) do not depend on $\epsilon_0, \epsilon_1$ or $w$. To see this, first write $A_iF \epsilon_j + A_jF \epsilon_i$ as a linear combination of the matrices in (24), and observe that the coefficients do not depend on $\epsilon_0, \epsilon_1$ or $w$. Second, write each matrix in (24) as a linear combination of

\begin{equation}
\{E_hF + FE_h \mid 0 \leq h \leq d\} \cup \{E_hF + AFE_h \mid 0 \leq h \leq d\}.
\end{equation}

This is done implicitly in the proof of Lemma 2.4, in such a way that the coefficients do not depend on $\epsilon_0, \epsilon_1$ or $w$. Third, write each matrix in (33) as a linear combination of

\begin{equation}
\{E_hF + FE_h \mid 0 \leq h \leq d\} \cup \{E_hF + AFE_h \mid 0 \leq h \leq d\}.
\end{equation}

Our above observation is now immediate.

Choosing $(\epsilon_0, \epsilon_1) = (1, 1)$, $(1, 2)$ in (21), we obtain two equations from (32). Combining these equations, we find that (31) holds for $t \in \{0, 1\}$, as desired.

Concerning uniqueness, pick any $w \in V\Gamma$, pick any $\epsilon_0, \epsilon_1 \in \mathbb{R}\setminus\{0\}$, and let the matrix $F$ be as in (21). Observe that if (31) holds for $t = 0$ and $t = 1$ then (32) holds. The uniqueness of each $s_{ij}^h$, $t_{ij}^h$ now follows from the second assertion in Lemma 2.4. This proves Lemma 2.5.

For notational convenience, for all integers $h$, $i$ and $j$, set

\begin{equation}
t_{ij}^h = 0 \quad \text{if} \quad h \leq 0 \quad \text{or} \quad h > d.
\end{equation}

**Theorem 2.6.** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, suppose that $\Gamma$ is $Q$-polynomial with respect to the primitive idempotent $E_1$, and let the scalars $s_{ij}^h$ and $t_{ij}^h$ be as in Lemma 2.5 and expression (34). Then, for all integers $h$, $i$ and $j$ $(0 \leq h, i, j \leq d)$, and all $x$, $y \in V\Gamma$ such that $\delta(x, y) = h$, the vector

\begin{equation}
p_i(x, y) + p_i(x, y) - s_{ij}^h(x^-_y + y^-_x) - t_{ij}^{h-1}(x^+_y + y^+_x) - t_{ij}^h(x^+_y + y^+_x) - t_{ij}^{h+1}(x^+_y + y^+_x) \quad (35)
\end{equation}

is orthogonal to $E_0V + E_1V$.

**Proof.** This follows from Lemma 2.5. Indeed, suppose that $h$, $i$, $j$, $x$ and $y$ are given,
and let $f$ denote the vector in (35). Pick any $w \in V\Gamma$, and compute the $x, y$ entry of (31) using (15). This computation yields the equation

$$\langle E_t \hat{w}, f \rangle = 0 \quad (t \in \{0, 1\}).$$

(36)

Since $w$ is arbitrary, $f$ must be orthogonal to $E_0 V + E_1 V$, as desired.

**Theorem 2.7.** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, suppose that $\Gamma$ is $Q$-polynomial with respect to the primitive idempotent

$$E_1 = |V\Gamma|^{-1} \sum_{h=0}^{d} \theta_h^* A_h,$$

and let the scalars $s^h_{ij}$ and $t^h_{ij}$ be as in Lemma 2.5 and expression (34). Then

(i) $p^h_{ij} = s^h_{ij} + c_h t^h_{ij}^{i-1} + a_h t^h_{ij}^{i+1} + b_h t^h_{ij} + (0 \leq h, i, j \leq d)$,

(ii) $p^h_{ij}(\theta^* + \theta^*_i) = s^h_{ij}(\theta^*_0 + \theta^*_h) + c_h t^h_{ij}^{i-1}(\theta^*_i + \theta^*_h)$

$$+ a_h t^h_{ij}^{i+1}(\theta^*_i + \theta^*_{h+1}) \quad (0 \leq h, i, j \leq d),$$

(38)

where $\theta^*_{i-1}$ and $\theta^*_{h+1}$ are indeterminants.

**Proof.** Referring to Theorem 2.6, we observe that

$$\langle E_t \hat{x}, f \rangle = 0 \quad (t \in \{0, 1\}),$$

where $f$ denotes the vector in (35). To obtain (37) (resp. (38)), set $t = 0$ (resp. $t = 1$), and evaluate the above inner product using (14).

**Lemma 2.8.** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, suppose that $\Gamma$ is $Q$-polynomial with respect to the primitive idempotent

$$E_1 = |V\Gamma|^{-1} \sum_{h=0}^{d} \theta_h^* A_h,$$

and let the scalars $s^h_{ij}$ and $t^h_{ij}$ be as in Lemma 2.5 and line (34). Then

$$s^0_{ii-1} = 0, \quad s^1_{ii-1} = -p^1_{ii-1} \frac{\theta^*_0 + \theta^*_i - \theta^*_{i-1} - \theta^*_i}{\theta^*_0 - \theta^*_i}$$

$$t^0_{ii-1} = 0, \quad t^1_{ii-1} = p^1_{ii-1} \frac{\theta^*_0 + \theta^*_i - \theta^*_{i-1} - \theta^*_i}{\theta^*_0 - \theta^*_i}$$

(39, 40)

$$0 = s^0_{ii-1} = p^0_{ii-1} \theta^*_0 + b_0 t^0_{ii-1} \theta^*_i.$$ (41, 42)

for all integers $i$ ($1 \leq i \leq d$).

**Proof.** From (37) and (38) (with $h = 0$ and $j = i - 1$), and since $p^0_{ii-1} = 0$, $c_0 = 0$ and $a_0 = 0$, we find that

$$0 = s^0_{ii-1} + b_0 t^0_{ii-1}, \quad 0 = s^0_{ii-1} \theta^*_0 + b_0 t^0_{ii-1} \theta^*_i.$$ (41, 42)

Lines (39) and (41) follow, since $\theta^*_0 \neq \theta^*_i$ and $b_0 \neq 0$. From (37) and (38) (with $h = 1$ and $j = i - 1$), and since $t^1_{ii-1} = 0$ and $t^1_{ii-1} = 0$ by (34) and (41), we find that

$$p^1_{ii-1} = s^1_{ii-1} + b_1 t^1_{ii-1},$$

$$p^1_{ii-1}(\theta^*_{i-1} + \theta^*_i) = s^1_{ii-1}(\theta^*_0 + \theta^*_i) + b_1 t^1_{ii-1}(\theta^*_i + \theta^*_0).$$

Since $\theta^*_0 \neq \theta^*_i$ and $b_1 \neq 0$, we may solve these equations to obtain (40) and (42). This proves the lemma.
Lemma 2.9. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, and suppose that $\Gamma$ is $Q$-polynomial with respect to the primitive idempotent

$$E_1 = |V\Gamma|^{-1} \sum_{h=0}^{d} \theta_h^* A_h.$$ 

Then, for any adjacent vertices $x, y \in V\Gamma$, and any integer $i$ ($1 \leq i \leq d$), the vector

$$(p_{i-1}^i)^{-1} p_{i-1}(x, y) - \frac{\theta_i^* - \theta_{i-1}^*}{\theta_0^* - \theta_1^*} (\theta_i^* - \theta_{i-1}^*) - \frac{\theta_i^* - \theta_{i-1}^*}{\theta_0^* - \theta_1^*} (\theta_i^* - \theta_{i-1}^*)$$

is orthogonal to $E_0 V + E_1 V$.

Proof. From (35) (with $h = 1$ and $j = i - 1$), and since $t_{i-1}^1 = 0$ and $t_{i-1}^1 = 0$ by (34) and (41), respectively, we find that the vector

$$p_{i-1}(x, y) + p_{i-1}(x, y) - s_{i-1}^1 (\ell + \eta) - t_{i-1}^1 (x_x^\ell + y_x^\ell)$$

is orthogonal to $E_0 V + E_1 V$, where $s_{i-1}^1$ and $t_{i-1}^1$ are from (40) and (42), respectively. From (19) (with $h = 1$ and $j = i - 1$) we find that the vector

$$p_{i-1}(x, y) - p_{i-1}(x, y) - r_{i-1}^1 (\ell - \eta)$$

is orthogonal to $E_0 V + E_1 V$, where $r_{i-1}^1$ is from (20). Setting $i = 2$ in (44) we find that the vector

$$y_x^\ell - x_x^\ell - r_{21} (\ell - \eta)$$

is orthogonal to $E_0 V + E_1 V$. Eliminating $p_{i-1}(x, y)$ and $x_x^\ell$ in (43), using (44) and (45), we find that the vector

$$2p_{i-1}(x, y) - 2(p_{i-1}^i)^{-1} [u | u \in V\Gamma, xyzu is a kite of length i]$$

is orthogonal to $E_0 V + E_1 V$. The result is now obtained by evaluating the coefficients in the above expression using (20), (40) and (42), and simplifying.

Definition 2.10. A distance-regular graph $\Gamma$ is said to have classical parameters $(d, b, a, \beta)$ whenever the diameter of $\Gamma$ is $d$, and the intersection numbers of $\Gamma$ satisfy

$$c_i = \binom{i}{1}(1 + \alpha \binom{i-1}{1}) \quad (0 \leq i \leq d),$$

$$b_i = \left( \binom{d}{1} - \binom{i}{1} \right) (\beta - \alpha \binom{i}{1}) \quad (0 \leq i \leq d),$$

where

$$\binom{j}{1} := 1 + b + b^2 + \cdots + b^{j-1}. \quad (46)$$

Theorem 2.11. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, and assume that the intersection number $a_1 \neq 0$. Pick any 3-tuple $xyz$ of mutually adjacent vertices in $V\Gamma$, and set

$$e_i(xyz) := (p_{i-1}^i)^{-1} [u | u \in V\Gamma, xyzu is a kite of length i] \quad (2 \leq i \leq d).$$
(i) Suppose that \( \Gamma \) is \( Q \)-polynomial with respect to the primitive idempotent

\[
E_1 = \frac{1}{|V\Gamma|} \sum_{h=0}^{d} \theta_h^* A_h.
\]

Then

\[
e_i(\text{xyz}) = \alpha_i e_2(\text{xyz}) + \beta_i \quad (2 \leq i \leq d),
\]

where

\[
\alpha_i = \frac{(\theta_i^* - \theta_1^*)(\theta_0^* + \theta_1^* - \theta_{i-1}^* - \theta_i^*)}{(\theta_0^* - \theta_1^*)(\theta_{i-1}^* - \theta_i^*)} \quad (2 \leq i \leq d),
\]

\[
\beta_i = \frac{(\theta_0^* - \theta_1^*)(\theta_1^* - \theta_i^*) - (\theta_1^* - \theta_0^*)(\theta_{i-1}^* - \theta_i^*)}{(\theta_0^* - \theta_1^*)(\theta_{i-1}^* - \theta_i^*)} \quad (2 \leq i \leq d).
\]

(ii) Suppose that \( \Gamma \) has classical parameters \((d, b, \alpha, \beta)\), and let \([\ ]\) be as in (46). Then (47) holds, where

\[
\alpha_i = \left[ \begin{array}{c} i-1 \\ 1 \end{array} \right] \quad (2 \leq i \leq d), \quad \beta_i = 0 \quad (2 \leq i \leq d).
\]

Proof of (i). To obtain (47), compute the inner product of \( E_1 \xi \) and the vector in Lemma 2.9, and set the result equal to 0. The computation is readily carried out once we observe, by (14), that

\[
|V\Gamma| \langle E_1 \xi, p_{i-1}(x, y) \rangle = p_{i-1}^1(e_i(\text{xyz})(\theta_{i-1}^* - \theta_i^*) + \theta_i^*),
\]

\[
|V\Gamma| \langle E_1 \xi, \xi \rangle = \theta_i^*,
\]

\[
|V\Gamma| \langle E_1 \xi, y_1^* \rangle = \theta_i^*,
\]

\[
|V\Gamma| \langle E_1 \xi, y_2^* \rangle = b_1(e_2(\text{xyz})(\theta_i^* - \theta_1^*) + \theta_1^*).
\]

Proof of (ii). \( \Gamma \) is \( Q \)-polynomial with respect to a primitive idempotent

\[
E = \frac{1}{|V\Gamma|} \sum_{h=0}^{d} \theta_h^* A_h,
\]

where

\[
\theta_i^* - \theta_0^* = (\theta_i^* - \theta_0^*) \left[ \begin{array}{c} i \\ 1 \end{array} \right] b^{1-i} \quad (0 \leq i \leq d)
\]

[2, p. 250], so (47)–(49) hold. Expressions (50) and (51) are obtained by eliminating \( \theta_1^* \), \( \theta_{i-1}^* \), and \( \theta_i^* \) in (48) and (49), using (52), and simplifying. This proves Theorem 2.11.

Suppose that \( \Gamma \) has classical parameters \((d, b, \alpha, \beta)\) with \( d \geq 3 \). Then it is known that \( b \) is an integer satisfying \( b > 0 \) or \( b < -1 \) [2, p. 195]. For most of the known examples we have \( b > 0 \), but in some cases \( b < -1 \). This second case occurs, for example (in the notation of [2, p. 194]), for the graph of the dual polar space \( U(2d, r) \), the Hermitian forms graph \( \text{Her}(d, q) \), the triality graph \( ^3D_{4,2}(q) \), the Witt graphs \( M_{23} \) and \( M_{24} \), and the extended ternary Golay code graph [2, p. 194]. These graphs have the following property.

Theorem 2.12. Let \( \Gamma \) denote a distance-regular graph with classical parameters \((d, b, \alpha, \beta)\), such that \( d \geq 3 \) and \( b < -1 \). Then \( \Gamma \) has no kites of any length \( j \) \((2 \leq j \leq d)\).
PROOF. Suppose that $\Gamma$ has a kite $xyzu$ of some length $j$ ($2 \leq j \leq d$). Then certainly $a_1 \neq 0$, and (with the notation of Theorem 2.11),

$$e_j(xyz) > 0.$$ 

But

$$e_j(xyz) = e_2(xyz)\binom{j-1}{1}$$

by Theorem 2.11 (ii), so that

$$e_2(xyz) > 0.$$ 

Now setting $i = 3$ in Theorem 2.11 (ii), we find that

$$e_3(xyz) = e_2(xyz)(b + 1)$$

$$< 0,$$

an impossibility. Hence $\Gamma$ has no kites.

COROLLARY 2.13. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, and intersection numbers of the form

$$b_i = (q^{2d} - q^{2i})/(q + 1) \quad (0 \leq i \leq d),$$

$$c_i = q^{i-1}(q^i - (-1)^i)/(q + 1) \quad (0 \leq i \leq d)$$

for some integer $q \geq 2$. Then $q$ is a prime power and $\Gamma$ is isomorphic to $\text{Her}(d, q)$.

PROOF. In [3, Theorem A(ii)], Ivanov and Shpectorov showed that the conclusion of our corollary holds if it is assumed that $\Gamma$ has no kites of length 2. This assumption is unnecessary, since any distance-regular graph satisfying (53) and (54) has classical parameters $(d, -q, -q - 1, -(-q)^d - 1)$, and therefore has no kites, by Theorem 2.12.

ACKNOWLEDGEMENTS

A. Neumaier pointed out that Theorem 2.12 and Corollary 2.13 could be obtained from Theorem 2.11, and encouraged the author to write this paper. This research was partially supported by NSF grant DMS-8600882.

REFERENCES


Received 18 February 1994 and accepted in revised form 21 September 1994

PAUL TERWILLIGOR

Department of Mathematics,

University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, U.S.A. E-mail: terwilli@math.wisc.edu.