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# Adjoint computation for hypersurfaces using formal desingularizations

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ABSTRACT

We show how to use formal desingularizations (defined earlier by the first author) in order to compute the global sections (also called adjoints) of twisted pluricanonical sheaves. These sections define maps that play an important role in the birational classification of schemes.



**Contents**



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## 1. Introduction

Let $X \subset \mathbb{P}^n$ be an algebraic variety embedded in some projective space, and let $\pi : Y \to X$ be a desingularization. Then the sheaves $\pi_*(\omega_Y^{\otimes m})$, where $m \in \mathbb{Z}_+$, are called *pluricanonical*. They do not depend on the choice of the desingularization $(Y, \pi)$, up to isomorphism. The homogeneous components of the associated graded module

$$\Gamma_*(\pi_*(\omega_Y^{\otimes m})) := \bigoplus_{n \geqslant 0} \Gamma(X, \pi_*(\omega_Y^{\otimes m}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))$$

play an important role in birational geometry. The components in degree 0 (i.e. the global sections of the pluricanonical sheaves) define the geometric genus, the plurigenera and the pluricanonical maps. The arithmetic genus can also be computed in terms of the Hilbert function of the module $\Gamma_*(\pi_*(\omega_Y))$ [3]. In case of rational surfaces, some of the components realize the Enriques–Manin reduction to a minimal model [12], i.e. this reduction is the associated map of the corresponding vector space. Other applications may be derived from the theory of adjunction and its relation to minimal models, see [2] and the references therein.

The topic of this paper is the computation of these components. We restrict to characteristic zero and to hypersurfaces. The computation uses the concept of formal desingularization, which has been introduced in [1]. That paper also contains an algorithm for computing these formal desingularizations for surfaces. Both algorithms (for computing formal desingularizations and for computing adjoints) have been implemented in the computer algebra system Magma [4] for the surface case. It turns out that the computation of formal desingularization is by far cheaper than the computation of a desingularization $\pi : Y \to X$.

The paper is structured as follows: In Section 2 we start by recalling the definition of formal desingularizations for the convenience of the reader. Much of this section is a literal copy of [1]. In Section 3 we define the sheaf of $m$-adjoints on $X$ by a property involving formal prime divisors and show that it is isomorphic to $\pi_*(\omega_Y^{\otimes m})$. In particular it is independent of $Y$. In Section 4 we find a super-sheaf of $\pi_*(\omega_Y^{\otimes m})$. Then the sheaf of $m$-adjoints on $X$ is equal to the subsheaf of sections satisfying conditions related to a formal desingularizations of $X$. This immediately yields Algorithm 1 given in Section 5. We close with an example.

Before we proceed we recall and fix some notions. Let $\mathbb{E}$ be a field of characteristic zero and $X$ and $Y$ *integral* $\mathbb{E}$*-schemes*. All (rational) maps are relative over $\operatorname{Spec}\mathbb{E}$. By $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ we denote the respective function fields. A *rational map* $\pi : Y \dashrightarrow X$ is given by a tuple $(V, \pi)$ such that $V \subseteq Y$ is open and $\pi : V \to X$ is a regular morphism. Note that we do not restrict to schemes of finite type here. In particular all regular morphisms are rational maps. Two tuples $(V_1, \pi_1)$ and $(V_2, \pi_2)$ are *equivalent*, or define the same rational map, if $\pi_1|_{V_1 \cap V_2} = \pi_2|_{V_1 \cap V_2}$.

Assume that two maps send the generic point of $Y$ to $p \in X$ (its image is always defined for rational maps). Then $(V_1, \pi_1)$ and $(V_2, \pi_2)$ are equivalent iff the induced inclusions of fields $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p} \hookrightarrow \mathbb{E}(Y)$ are the same (where $\mathfrak{m}_{X,p} \subset \mathcal{O}_{X,p}$ is the maximal ideal). In particular if $\pi$ is *dense*, i.e., $p$ is the generic point of $X$, we get an inclusion $\mathbb{E}(X) \hookrightarrow \mathbb{E}(Y)$ determining $\pi$.

Note, however, that not all such field inclusions yield rational maps under our assumption since we have not yet restricted to schemes of finite type over $\mathbb{E}$. E.g., let $X := \operatorname{Spec}\mathbb{E}[x]$, $Y := \operatorname{Spec}\mathbb{E}[x]_{\langle x \rangle}$ and $\pi : Y \to X$ be the morphism induced by localization. Then $\pi$ induces an isomorphism of function fields $\mathbb{E}(X) \cong \mathbb{E}(Y)$. Nevertheless $\pi$ has no rational inverse. A rational map with inverse is called *birational* (or also a *birational transformation*).

Further it is easy to see that dense rational maps may be composed. A rational map has a *domain of definition*, which is the maximal open set on which it can be defined (equivalently, the union of all such open sets).

## 2. Definition of formal desingularizations

From now on $X$ and $Y$ will denote *separated, integral schemes of finite type over* $\mathbb{E}$ and they will have the same dimension $l$. Let $(A, \mathfrak{m})$ be a valuation ring of $\mathbb{E}(X)$ over $\mathbb{E}$ (where $\mathfrak{m}$ is the maximal ideal). If $A$ is discrete of rank 1 and the transcendence degree of $A/\mathfrak{m}$ over $\mathbb{E}$ is $n-1$ then it is called a *divisorial valuation ring of* $\mathbb{E}(X)$ *over* $\mathbb{E}$ or a *prime divisor of* $\mathbb{E}(X)$ (see, e.g., [13, Definition 2.6]). It is an *essentially finite*, regular, local $\mathbb{E}$-algebra of Krull-dimension 1 (i.e., the localization of a finitely generated $\mathbb{E}$-algebra at a prime ideal, see [14, Theorem VI.14.31]).

Let $(A, \mathfrak{m})$ be a divisorial valuation ring of $\mathbb{E}(X)$ over $\mathbb{E}$. By [7, Lemma II.4.4.] the inclusion $A \subset \mathbb{E}(X)$ defines a unique morphism $\operatorname{Spec} \mathcal{Q}(A) \to X$ and therefore a rational map $\operatorname{Spec} A \dashrightarrow X$ sending generic point to generic point. Composing this with the morphism obtained by the $\mathfrak{m}$-adic completion $A \to \widehat{A}$ we get a rational map $\operatorname{Spec} \widehat{A} \dashrightarrow X$ in a natural way.

**Definition 2.1** *(Formal prime divisor).* Let $(A, \mathfrak{m})$ be a divisorial valuation ring of $\mathbb{E}(X)$ over $\mathbb{E}$. Assume that the rational map $\operatorname{Spec} \widehat{A} \dashrightarrow X$ (as above) is actually a *morphism* $\varphi : \operatorname{Spec} \widehat{A} \to X$ (i.e., defined also at the closed point). Then $\varphi$ is a representative for a class of schemes up to $X$-isomorphism. This class (and, by abuse of notation, any representative) will be called a *formal prime divisor* on $X$.

Hence we may compose a representative $\varphi$ with an isomorphism $\operatorname{Spec} B \to \operatorname{Spec} \widehat{A}$ to get another representative for the same formal prime divisor. By the *Cohen Structure Theorem* (see, e.g., [5, Theorem 7.7] with $I = 0$) we know that $\widehat{A} \cong \mathbb{F}_\varphi[\![t]\!]$ with $\mathbb{F}_\varphi := A/\mathfrak{m} \cong \widehat{A}/\mathfrak{m}\widehat{A}$. Therefore we will sometimes assume that $\varphi$ is of the form $\operatorname{Spec} \mathbb{F}_\varphi[\![t]\!] \to X$.

Formal prime divisors provide an algorithmic way for dealing with certain valuations; a formal prime divisor yields an inclusion of function fields $\mathbb{E}(X) \hookrightarrow \mathbb{F}_\varphi((t))$. Vice versa, by what was said above, $\varphi$ is determined by this inclusion. Composing this inclusion with the order function $\operatorname{ord}_t : \mathbb{F}_\varphi((t)) \to \mathbb{Z}$ we get the corresponding divisorial valuation (see Definition 4.4 below). We want to single out a special class of formal prime divisors.

**Definition 2.2** *(Realized formal prime divisors).* Let $p \in X$ be a regular point of codimension 1. The formal prime divisor

$$\operatorname{Spec} \widehat{\mathcal{O}_{X,p}} \to X$$

(given by composing the canonic morphism $\operatorname{Spec} \mathcal{O}_{X,p} \to X$ with the morphism induced by the completion $\mathcal{O}_{X,p} \to \widehat{\mathcal{O}_{X,p}}$) is called *realized*.

If $X$ is *normal* then all generic points of closed subsets of codimension 1 are necessarily regular [7, Theorem II.8.22A]. Therefore there is a one–one correspondence of realized formal prime divisors and prime Weil divisors. Another important fact is that we can match the formal prime divisors of birationally equivalent schemes under certain conditions.

**Lemma 2.3** *(Pullback along proper morphisms). Let $\pi : Y \to X$ be a proper, birational morphism. A formal prime divisor $\varphi : \operatorname{Spec} \mathbb{F}_\varphi[\![t]\!] \to X$ lifts to a unique formal prime divisor $\pi^*\varphi : \operatorname{Spec} \mathbb{F}_\varphi[\![t]\!] \to Y$ such that $\pi \circ (\pi^*\varphi) = \varphi$. Vice versa, a formal prime divisor on $Y$ extends to a unique formal prime divisor on $X$, hence $\pi^* : \operatorname{Div} X \to \operatorname{Div} Y$ is a bijection.*

**Proof.** See [1, Corollary 2.4]. □

We will apply the operator $\pi^*$ also to sets of formal prime divisors.

**Definition 2.4** *(Center and support).* Let $\varphi : \operatorname{Spec} \mathbb{F}_\varphi[\![t]\!] \to X$ be a formal prime divisor. We define its *center*, in symbols $\operatorname{center}(\varphi) \in X$, to be the image of the closed point. Further the *support* of a *finite* set of formal prime divisors $\mathcal{S}$ is defined as $\operatorname{supp}(\mathcal{S}) := \overline{\{\operatorname{center}(\varphi) \mid \varphi \in \mathcal{S}\}}$, i.e., the closure of the set of all centers.

We note that the closure of the center of a formal prime divisor $\varphi$ need not be of codimension 1. It is of codimension 1 if and only if $\varphi$ is realized.

Now we are in the situation to define formal desingularizations.

**Definition 2.5** *(Formal description of a desingularization).* Let $\pi : Y \to X$ be a *desingularization*, i.e., $\pi$ is proper, birational and $Y$ is regular. Let $\mathcal{S}$ be a *finite* set of formal prime divisors on $X$. We say that $\mathcal{S}$ is a *formal description* of $\pi$ iff

(1) all divisors in $\pi^*\mathcal{S}$ are realized,
(2) $\pi^{-1}(\operatorname{supp}(\mathcal{S})) = \operatorname{supp}(\pi^*\mathcal{S})$ and
(3) the restricted morphism $Y \setminus \operatorname{supp}(\pi^*\mathcal{S}) \to X \setminus \operatorname{supp}(\mathcal{S})$ is an isomorphism.

The set $\mathcal{S}$ itself consists of formal prime divisors on $X$ and makes no reference to the morphism $\pi$. By another definition we can *avoid* mentioning any *explicit* $\pi$.

**Definition 2.6** *(Formal desingularization).* Let $\mathcal{S}$ be a finite set of formal prime divisors on $X$. Then $\mathcal{S}$ is called a *formal desingularization* of $X$ iff there *exists some* desingularization $\pi$ such that $\mathcal{S}$ is a formal description of it.

In [1, Theorem 2.9] it is shown that such $\pi$ is unique up to $X$-isomorphism if $X$ is a surface. In this case we also have an efficient algorithm to compute a set $\mathcal{S}$. Informally speaking $\mathcal{S}$ makes it possible to work with invertible sheaves on $Y$, although $Y$ is not constructed explicitly.

**Remark 2.7** *(Formal desingularizations in higher dimensions).* This paper deals with projective hypersurfaces of arbitrary dimension $l$. Also in this case, formal desingularizations exist but are not so easy to compute. We want to indicate how a formal desingularization could be obtained by an ad hoc method (modulo a means to represent algebraic power series).

First compute a desingularization $\pi : Y \to X$, for example, by VILLAMAYOR's algorithm [6]. Let $Z \subset X$ be the singular locus of $X$. The algorithm will produce $\pi$ such that $\pi^{-1}(Z) \subset Y$ is a normal crossing divisor and $\pi$ restricts to an isomorphism on $Y \setminus \pi^{-1}(Z)$. Let $\{p_1, \ldots, p_r\} \subset Y$ be the *finitely many* generic points of the irreducible components of $\pi^{-1}(Z)$.

Next we have to compute isomorphisms $\widehat{\mathcal{O}_{Y,p_i}} \to \mathbb{F}_i[\![t]\!]$ where $\mathbb{F}_i := \mathcal{O}_{Y,p_i}/\mathfrak{m}_{Y,p_i}$. Therefore let $U_i \cong \operatorname{Spec} \mathbb{E}[x_{i,1}, \ldots, x_{i,m_i}]/\langle f_{i,1}, \ldots, f_{i,n_i}\rangle$ be an affine neighborhood of $p_i$. Constructing the isomorphism involves finding certain "minimal" algebraic power series $X_{i,1}, \ldots, X_{i,m_i} \in \mathbb{F}_i[\![t]\!]$ (compare [1, Corollary A.2]) that simultaneously solve $f_{i,1}, \ldots, f_{i,n_i}$, essentially, computing a Taylor expansion. These power series together with $\pi$ and the inclusions $U_i \hookrightarrow Y$ can be used to represent a formal prime divisor $\varphi_i$ via, for example, the induced embedding of function fields $\mathbb{E}(X) \hookrightarrow \mathbb{F}_i(\!(t)\!)$. Set $\mathcal{S} := \{\varphi_1, \ldots, \varphi_r\}$.

This approach of course is not practical. It suffers from the huge computational overhead that the general resolution machinery involves. Also we are not very flexible with regard to the representation of the blown up schemes, thus annihilating the benefits of formal descriptions.

## 3. Adjoint differential forms

Let $l := \dim(X)$. We write

$$\Omega^m_{X,\mathrm{rat}} := \left(\Omega^{\wedge l}_{\mathbb{E}(X)|\mathbb{E}}\right)^{\otimes m}$$

for the $m$-fold tensor power of the rational differential $l$-forms (which is an 1-dimensional $\mathbb{E}(X)$-vector space by [5, Theorem 16.14] and [10, Proposition XIX.1.1 and Corollary XVI.2.4] and can as well be considered a constant sheaf of $\mathcal{O}_X$-modules). (We will always denote by $\Omega^m$ a sheaf of $m$th tensors of volume forms; the sheafs of differential forms of lower degree are denoted by the symbol $\Omega^{\wedge l}$.)

Let $\varphi : \operatorname{Spec} \widehat{A} \to X$ be a formal prime divisor. We define

$$\widetilde{\Omega}^m_{\widehat{A}|\mathbb{E}} := \widehat{A} \otimes_A \left(\Omega^{\wedge l}_{A|\mathbb{E}}\right)^{\otimes m} \quad \left(\text{canonically included in } \widetilde{\Omega}^m_{\mathcal{Q}(\widehat{A})|\mathbb{E}} := \mathcal{Q}(\widehat{A}) \otimes_A \left(\Omega^{\wedge l}_{A|\mathbb{E}}\right)^{\otimes m}\right).$$

By [9, Corollary 12.5], this is the $m$-fold tensor power of the *universally finite module of $l$-differentials of $\widehat{A}$*; it only depends on the $\mathbb{E}$-algebra $\widehat{A}$ and its topology, and is finitely generated by any generating set of $\Omega^{\wedge l}_{\widehat{A}|\mathbb{E}}$. In particular it is independent of the choice of a representative of $\varphi$ up to $X$-isomorphism.

By substituting $\widehat{A}$ by $\mathbb{F}_\varphi[\![t]\!]$, which is isomorphic as an $\mathbb{E}$-algebra by a homeomorphism, we can show that $\widetilde{\Omega}^m_{\widehat{A}|\mathbb{E}}$ is a cyclic free module: if $s_{\varphi,1},\dots,s_{\varphi,l-1}\in\mathbb{F}_\varphi$ is a transcendence basis over $\mathbb{E}$ then $\{\mathrm{d}s_{\varphi,1},\dots,\mathrm{d}s_{\varphi,l-1},\mathrm{d}t\}$ is a free $\mathbb{F}_\varphi[t]_{\langle t\rangle}$-module basis of $\Omega_{\mathbb{F}_\varphi[t]_{\langle t\rangle}|\mathbb{E}}$ by [5, Theorem 16.14] and [11, Theorem 25.1]. It follows that $(\mathrm{d}s_{\varphi,1}\wedge\dots\wedge\mathrm{d}s_{\varphi,l-1}\wedge\mathrm{d}t)^{\otimes m}$ is a free generator of $\Omega^m_{\mathbb{F}_\varphi[t]|\mathbb{E}}$ as an $\mathbb{F}_\varphi[t]$-module, and simultaneously a free generator of $\widetilde{\Omega}^m_{\mathbb{F}_\varphi[\![t]\!]|\mathbb{E}}$ as an $\mathbb{F}_\varphi[\![t]\!]$-module. (Note that the module of Kähler differentials $\Omega_{\mathbb{F}_\varphi[\![t]\!]|\mathbb{E}}$ would not be finitely generated.)

We have seen above that a formal prime divisor $\varphi:\operatorname{Spec}\widehat{A}\to X$ induces an embedding $\varphi^\#:\mathbb{E}(X)\to\mathbb{F}_\varphi(\!(t)\!)$ of function fields. This again induces embeddings $\varphi^\#:\Omega^m_{X,\mathrm{rat}}\to\widetilde{\Omega}^m_{\mathbb{F}_\varphi(\!(t)\!)|\mathbb{E}}$ in the obvious way (i.e., $\mathrm{d}f\mapsto\mathrm{d}\varphi^\#(f)$).

**Definition 3.1** *(Regularity of forms at formal prime divisors).* Let $\varphi:\operatorname{Spec}\mathbb{F}_\varphi[\![t]\!]\to X$ be a formal prime divisor. We say that $\eta\in\Omega^m_{X,\mathrm{rat}}$ is *regular at* $\varphi$ iff $\varphi^\#(\eta)\in\widetilde{\Omega}^m_{\mathbb{F}_\varphi[\![t]\!]|\mathbb{E}}$.

This manner of speaking is justified by Lemma 3.5 below.

**Definition 3.2** *(Sheaf of adjoint forms).* The map

$$U\mapsto\left\{\eta\in\Omega^m_{X,\mathrm{rat}}\;\middle|\;\eta\text{ is regular at all formal prime divisors on }X\text{ centered in }U\right\}$$

for all open subsets $U\subseteq X$ defines a subsheaf which we call the *sheaf of $m$-adjoint forms* (*or just $m$-adjoints*), in symbols $\Omega^m_{X,\mathrm{adj}}$.

By what we have worked out so far we immediately find a nice property of adjoint forms.

**Corollary 3.3** *(Covariance of adjoint forms). The sheaves of adjoints are covariants under proper, birational morphisms: If $\pi:Y\to X$ is a proper, birational morphism then $\pi_*(\Omega^m_{Y,\mathrm{adj}})=\Omega^m_{X,\mathrm{adj}}$ as subsheaves of $\Omega^m_{X,\mathrm{rat}}$.*

**Proof.** Since $\pi$ is birational we get a vector space isomorphism $\pi^\#:\Omega^m_{X,\mathrm{rat}}\to\Omega^m_{Y,\mathrm{rat}}$. With this identification $\pi_*(\Omega^m_{Y,\mathrm{adj}})$ becomes a subsheaf of $\Omega^m_{X,\mathrm{rat}}$. The rest follows from the above definitions and Lemma 2.3. □

By $\Omega^m_{X,\mathrm{reg}}\subset\Omega^m_{X,\mathrm{rat}}$ we denote the subsheaf of regular forms, i.e., all forms locally expressible by sections of $\mathcal{O}_X$. More precisely, if $\Omega_{X|\mathbb{E}}$ is the usual sheaf of Kähler differentials then we mean its image sheaf under the natural map $\iota:(\Omega^{\wedge l}_{X|\mathbb{E}})^{\otimes m}\to\Omega^m_{X,\mathrm{rat}}$. Note that we do not have $(\Omega^{\wedge l}_{X|\mathbb{E}})^{\otimes m}\cong\Omega^m_{X,\mathrm{reg}}$ in general; at singular points the Kähler differentials need not be torsion free (and neither their exterior and tensor products) whereas $\Omega^m_{X,\mathrm{reg}}\subset\Omega^m_{X,\mathrm{rat}}$ always is. At a regular point $p$, on the contrary, $((\Omega^{\wedge l}_{X|E})^{\otimes m})_p$ is free of rank 1 (see [7, Theorem II.8.15]). Therefore $\iota$ is locally an inclusion at $p$. In the next three lemmas we want to explore in detail the relation between the concepts of adjointness and regularity (at points or formal prime divisors) for forms in $\Omega^m_{X,\mathrm{rat}}$.

**Lemma 3.4** *(Adjoint forms and regular forms). Let $p\in X$ be a point. Then we have an inclusion $(\Omega^m_{X,\mathrm{reg}})_p\subseteq(\Omega^m_{X,\mathrm{adj}})_p$ of stalks of subsheaves of $\Omega^m_{X,\mathrm{rat}}$.*

**Proof.** Assume $\eta\in(\Omega^m_{X,\mathrm{reg}})_p$. We have $\eta\in(\Omega^m_{X,\mathrm{adj}})_p$ iff $\varphi^\#(\eta)\in\widetilde{\Omega}^m_{\mathbb{F}_\varphi[\![t]\!]|\mathbb{E}}$ for all formal prime divisors $\varphi:\operatorname{Spec}\mathbb{F}_\varphi[\![t]\!]\to X$ centered at some $q$ contained in the intersection of all neighborhoods of $\{p\}$. But for such $\varphi$ we have ring inclusions $\varphi^\#(\mathcal{O}_{X,p}) \subseteq \varphi^\#(\mathcal{O}_{X,q}) \subseteq \mathbb{F}_\varphi[\![t]\!]$ and hence the condition on $\eta$ and $\varphi$ is fulfilled. A fortiori $\eta \in (\Omega^m_{X,\mathrm{adj}})_p$. $\square$

**Lemma 3.5** *(Regularity and realized formal prime divisors). Let $\varphi$ be a realized formal prime divisor on $X$, $p := \mathrm{center}(\varphi)$ and $\eta \in \Omega^m_{X,\mathrm{rat}}$. Then $\eta \in (\Omega^m_{X,\mathrm{reg}})_p$ iff $\eta$ is regular at $\varphi$.*

**Proof.** By Lemma 3.4 it remains to show that regularity at $\varphi$ implies $\eta \in (\Omega^m_{X,\mathrm{reg}})_p$. Since $\varphi$ is realized it is of the form $\mathrm{Spec}\,\widehat{\mathcal{O}_{X,p}} \to X$. Since $p$ is the center of a realized formal prime divisor, it is a regular point and $(\Omega^m_{X,\mathrm{reg}})_p$ is a free cyclic $\mathcal{O}_{X,p}$-module. Let $\gamma$ be a generator of this module. Write $\eta = a/b\,\gamma$ with $a, b \in \mathcal{O}_{X,p}$. Then $\varphi^\#(\eta) \in \widetilde{\Omega}^m_{\widehat{\mathcal{O}_{X,p}}|\mathbb{E}}$ implies $b|a$ in $\widehat{\mathcal{O}_{X,p}}$, in other words $a \in b\widehat{\mathcal{O}_{X,p}}$. But $b\widehat{\mathcal{O}_{X,p}} \cap \mathcal{O}_{X,p} = b\mathcal{O}_{X,p}$ by [11, Theorem 7.5(ii)] (because completion is *faithfully flat* [11, Theorems 7.2 and 8.8]). Therefore $b|a$ in $\mathcal{O}_{X,p}$ and $\eta \in (\Omega^m_{X,\mathrm{reg}})_p$. $\square$

**Lemma 3.6** *(Adjoint forms at regular points). Let $p \in X$ be a* regular *point. Then we have an equality $(\Omega^m_{X,\mathrm{reg}})_p = (\Omega^m_{X,\mathrm{adj}})_p$ of stalks of subsheaves of $\Omega^m_{X,\mathrm{rat}}$.*

**Proof.** By Lemma 3.4 it remains to show $(\Omega^m_{X,\mathrm{adj}})_p \subseteq (\Omega^m_{X,\mathrm{reg}})_p$. Assume indirectly that $\eta \in (\Omega^m_{X,\mathrm{adj}})_p$ but $\eta \notin (\Omega^m_{X,\mathrm{reg}})_p$. Regularity is an open property and $\Omega^m_{X,\mathrm{reg}}$ is free of rank 1 in a neighborhood of $p$. Since $p$ is regular (in particular normal) we must have $\eta \notin (\Omega^m_{X,\mathrm{reg}})_q$ for some point $q$ such that $q$ is regular, of codimension 1 and $p \in \bar{q}$ (see [5, Corollary 11.4]). Consider the realized formal prime divisor $\varphi : \mathrm{Spec}\,\widehat{\mathcal{O}_{X,q}} \to X$. Lemma 3.5 above implies that $\eta$ is not regular at $\varphi$. But $q = \mathrm{center}(\varphi)$ is contained in any open neighborhood of $p$ contradicting $\eta \in (\Omega^m_{X,\mathrm{adj}})_p$. $\square$

Now assume that $Y$ is *regular*. In this situation one has $(\Omega^{\wedge l}_{Y|\mathbb{E}})^{\otimes m} \cong \Omega^m_{Y,\mathrm{reg}}$. In terms of the canonical sheaf this means $\omega_Y^{\otimes m} \cong \Omega^m_{Y,\mathrm{reg}} = \Omega^m_{Y,\mathrm{adj}}$ by the above lemmas. Finally using Corollary 3.3 we get an alternative characterization of the sheaf of $m$-adjoints, in fact, the usual definition when working in a category of desingularizable schemes (e.g., for our case of characteristic zero).

**Corollary 3.7** *(Alternative characterization of adjoints). If $\pi : Y \to X$ is any desingularization then $\Omega^m_{X,\mathrm{adj}} \cong \pi_*(\omega_Y^{\otimes m})$.*

## 4. Computing adjoints

Now let $X \subset \mathbb{P}^{l+1}_{\mathbb{E}}$ be a projective hypersurface with defining homogeneous equation $F \in \mathbb{E}[x_0, \dots, x_{l+1}]$ of degree $d$ (not equal to a coordinate hyperplane). For $0 \leqslant i \leqslant l+1$ we define the open sets $U_i \subset X$ obtained by intersection with the standard open covering sets $x_i \neq 0$ of $\mathbb{P}^{l+1}_{\mathbb{E}}$.

**Definition 4.1** *(Dualizing sheaf).* Following [8], we define the *dualizing sheaf* $\omega^0_X \subseteq \Omega^1_{X,\mathrm{rat}} = \Omega^{\wedge l}_{\mathbb{E}(X)|\mathbb{E}}$ as the invertible sheaf generated on $U_i$ by the form

$$\gamma_i := \sigma_{i,j}\left(\frac{\partial F/\partial x_j}{x_i^{d-1}}\right)^{-1} \mathrm{d}\frac{x_0}{x_i} \wedge \cdots \wedge \widehat{\mathrm{d}\frac{x_i}{x_i}} \wedge \cdots \wedge \widehat{\mathrm{d}\frac{x_j}{x_i}} \wedge \cdots \wedge \mathrm{d}\frac{x_{l+1}}{x_i}$$

for any choice of $j \neq i$ where

$$\sigma_{i,j} := \begin{cases} (-1)^{i+j} & \text{if } j < i, \\ (-1)^{i+j+1} & \text{if } j > i. \end{cases}$$

(The hats here mean that the corresponding terms in the exterior product are to be excluded.)

Using the rules of calculus and the fact that

$$0 = \mathrm{d}\frac{F}{x_i^d} = \sum_{0 \leqslant k \leqslant l+1,\, k \neq i} \frac{\partial F/\partial x_k}{x_i^{d-1}} \mathrm{d}\frac{x_k}{x_i}$$

holds in $\Omega^1_{X,\mathrm{rat}}$ one proves that the definition is indeed independent of the choice of $j$. Because of local freeness we also have $(\omega^0_X)^{\otimes m} \subseteq \Omega^m_{X,\mathrm{rat}}$ (meaning the natural map is an embedding).

The next lemma shows that the definition above coincides with the definition of the dualizing sheaf in [7]. (Its advantage is that it also gives an embedding into the sheaf of rational differential forms.)

**Lemma 4.2** *(Properties of the dualizing sheaf). For the dualizing sheaf we have*:

- $\omega^0_X \cong \mathcal{O}_X(d-l-2)$,
- $(\omega^0_X)_p = (\Omega^1_{X,\mathrm{reg}})_p = (\Omega^1_{X,\mathrm{adj}})_p$ *at all regular points* $p$,
- $\Omega^m_{X,\mathrm{adj}} \subseteq (\omega^0_X)^{\otimes m}$ *as subsheaves of* $\Omega^m_{X,\mathrm{rat}}$.

**Proof.** To prove the first statement one just shows, using the rules of calculus, that $\gamma_{i_1} = (x_{i_1}/x_{i_2})^{d-l-2}\gamma_{i_2}$. The same relation is fulfilled by the local generators $x_{i_1}^{d-l-2}$ and $x_{i_2}^{d-l-2}$ of the invertible sheaf $\mathcal{O}_X(d-l-2)$.

We check the second statement for points $p \in U_i$. The forms

$$\mathrm{d}\frac{x_0}{x_i} \wedge \cdots \wedge \widehat{\mathrm{d}\frac{x_i}{x_i}} \wedge \cdots \wedge \widehat{\mathrm{d}\frac{x_j}{x_i}} \wedge \cdots \wedge \mathrm{d}\frac{x_{l+1}}{x_i}$$

with $j \neq i$, each of them being a $\Gamma(U_i, \mathcal{O}_X)$-multiple of $\gamma_i$, clearly generate $\Gamma(U_i, \Omega^1_{X,\mathrm{reg}})$. Therefore we have $(\Omega^1_{X,\mathrm{reg}})_p \subseteq (\omega^0_X)_p$ at all points. Assuming moreover that $p$ is a regular point, there must be $j \neq i$ such that $(\partial F/\partial x_j)/x_i^{d-1} \notin \mathfrak{m}_{X,p}$ and therefore $(\partial F/\partial x_j)/x_i^{d-1}$ is invertible in $\mathcal{O}_{X,p}$. Choosing this $j$ in Definition 4.1 one immediately sees that $\gamma_i$ is regular at $p$.

For the last statement we consider a generic projection $\pi : X \to Z$ to a hyperplane $Z \subseteq \mathbb{P}^{l+1}_{\mathbb{E}}$. We may assume that $Z$ is given by $x_0 = 0$ and that $F$ is monic in $x_0$. In this situation one can define a trace $\sigma_{X|Z} : \Omega^m_{X,\mathrm{rat}} \to \Omega^m_{Z,\mathrm{rat}}$ obtained from the trace of the field extension $\mathbb{E}(Z) \subseteq \mathbb{E}(X)$. By [8, Satz 2.14] and Lemma 3.6 we know that $(\omega^0_X)^{\otimes m}$ is isomorphic to the "complementary sheaf" whose stalks at $p \in X$ consist of all $\alpha$ such that for all regular functions $f \in \mathcal{O}_{X,p}$, $\sigma_{X|Z}(f\alpha) \in (\Omega^m_{Z,\mathrm{reg}})_{Z,\pi(p)}$. We have to show that the elements in the stalk $(\Omega^m_{X,\mathrm{adj}})_p$ fulfill this condition.

Let $R \subset \mathbb{E}(Z)$ be a divisorial valuation ring containing $\pi(p)$. Let $S \subset \mathbb{E}(X)$ be an extension of $R$ in $\mathbb{E}(X)$ containing $p$. By Definition 3.2 and by the fact that completion is faithfully flat, $f\alpha$ is regular at $S$. Satz 2.15 in [8] states that for regular local rings, the module of regular differential forms is equal to the "complementary module" of forms $\beta$ such that for all regular $g$, the trace of $g\beta$ is regular. In particular $\sigma_{X|Z}(1 \cdot (f\alpha))$ is regular at $R$. Since $R$ was an arbitrary divisorial valuation ring containing $\pi(p)$, and $\Omega^m_{Z,\mathrm{reg}}$ is an invertible sheaf, it follows that $\sigma_{X|Z}(f\alpha)$ does not have denominators and is therefore regular. $\square$

We want to see that, under certain additional assumptions, checking for adjointness involves only finitely many formal prime divisors.

**Lemma 4.3** *(Adjointness by formal desingularizations). Let $\mathcal{S}$ be a formal desingularization of $X$ and $U \subset X$ an open subset. For $\eta \in \Omega^m_{X,\mathrm{rat}}$ the following are equivalent*:

- $\eta \in \Gamma(U, \Omega^m_{X,\mathrm{adj}})$,
- $\eta \in \Gamma(U \setminus \mathrm{supp}(\mathcal{S}), \Omega^m_{X,\mathrm{adj}})$ *and* $\eta$ *is regular at all* $\varphi \in \mathcal{S}$ *with* $\mathrm{center}(\varphi) \in U$.

**Proof.** The first implication is trivial, so assume that the second condition is true. Let $\pi : Y \to X$ be a desingularization that is described by $\mathcal{S}$ and set $V := \pi^{-1}(U)$.

By Corollary 3.3 and Lemma 3.6 we have $\pi^{\#}(\Gamma(U, \Omega^m_{X,\mathrm{adj}})) = \Gamma(V, \Omega^m_{Y,\mathrm{adj}}) = \Gamma(V, \Omega^m_{Y,\mathrm{reg}})$. Since $\pi$ induces an isomorphism $V \setminus \mathrm{supp}(\pi^*(\mathcal{S})) \cong U \setminus \mathrm{supp}(\mathcal{S})$ it remains to check that $\pi^{\#}(\eta)$ is regular in $\mathrm{supp}(\pi^*(\mathcal{S})) \cap V$. Since $Y$ is regular the locus of non-regularity of $\pi^{\#}(\eta)$ has pure codimension 1. Hence, by Lemma 3.5, it is sufficient to check regularity of $\pi^{\#}(\eta)$ at the formal prime divisors in $\pi^*(\mathcal{S})$ with center in $V$. Equivalently, working on $X$, we have to check regularity of $\eta$ at the corresponding formal prime divisors in $\mathcal{S}$. $\square$

In the following definition we assume that we have chosen for each formal prime divisor $\varphi : \mathrm{Spec}\,\mathbb{F}_\varphi[\![t]\!] \to X$ a transcendence basis $s_{\varphi,1}, \ldots, s_{\varphi,l-1}$ of $\mathbb{F}_\varphi$ over $\mathbb{E}$. Then $\omega_{\varphi,m} := (\mathrm{d}s_{\varphi,1} \wedge \cdots \wedge \mathrm{d}s_{\varphi,l-1} \wedge \mathrm{d}t)^{\otimes m}$ is a free generator of $\widetilde{\Omega}^m_{\mathbb{F}_\varphi[\![t]\!]|\mathbb{E}}$ as an $\mathbb{F}_\varphi[\![t]\!]$-module.

**Definition 4.4** *(Valuations associated to formal prime divisors).* Let $\varphi : \mathrm{Spec}\,\mathbb{F}_\varphi[\![t]\!] \to X$ be a formal prime divisor. Define $\kappa_\varphi := \mathrm{ord}_t \circ \varphi^{\#} : \mathbb{E}(X) \to \mathbb{Z}$, which is a *divisorial valuation*. We "extend the valuation" to $\Omega^m_{X,\mathrm{rat}}$ as follows: If $\eta \in \Omega^m_{X,\mathrm{rat}}$ and $\varphi^{\#}(\eta) = f\omega_{\varphi,m}$, then $\kappa_\varphi(\eta) := \kappa_\varphi(f)$. Finally we can also define "valuations" $\kappa_\varphi : \Gamma(X, \mathcal{O}_X(k)) \to \mathbb{Z}$ for $k \in \mathbb{Z}$ by setting $\kappa_\varphi(f) := \kappa_\varphi(f/x_i^k)$ for any index $i$ such that $\mathrm{center}(\varphi) \in U_i$.

The map from $\Omega^m_{X,\mathrm{rat}}$ is obviously well defined because two free generators can differ only by a unit in $\mathbb{F}_\varphi[\![t]\!]$ which has order 0. We should also make sure that the definition for the map from $\Gamma(X, \mathcal{O}_X(k))$ does not depend on the choice of the index $i$. Assume that $j \neq i$ is another index with $\mathrm{center}(\varphi) \in U_j$. Then $f/x_i^k = (x_j/x_i)^k f/x_j^k$ and hence $\kappa_\varphi(f/x_i^k) = k\kappa_\varphi(x_j/x_i) + \kappa_\varphi(f/x_j^k)$. Since $\mathrm{center}(\varphi) \in U_i \cap U_j$ we have that $x_j/x_i \in \mathcal{O}_{X,\mathrm{center}(\varphi)}$ is invertible and so is $\varphi^{\#}(x_j/x_i) \in \mathbb{F}_\varphi[\![t]\!]$. But then again $\kappa_\varphi(x_j/x_i) = \mathrm{ord}_t(\varphi^{\#}(x_j/x_i)) = 0$.

**Definition 4.5** *(Adjoint order).* Let $\varphi : \mathrm{Spec}\,\mathbb{F}_\varphi[\![t]\!] \to X$ be a formal prime divisor and $0 \leqslant i \leqslant l+1$ an index such that $\mathrm{center}(\varphi) \in U_i$. We define the *adjoint order* at $\varphi$ as $\alpha_\varphi := -\kappa_\varphi(\gamma_i)$.

This definition is again independent of the index $i$ by an analogous reasoning as above.

**Theorem 4.6** *(Global sections of twisted pluricanonical sheaves). Let* $\mathcal{S}$ *be a formal desingularization of* $X$*. Then*

$$\Gamma\big(X, \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \Omega^m_{X,\mathrm{adj}}\big) \cong \big\{ f \in \Gamma\big(X, \mathcal{O}_X\big(n + m(d-l-2)\big)\big) \;\big|\; \kappa_\varphi(f) \geqslant m\alpha_\varphi \textit{ for all } \varphi \in \mathcal{S} \big\}.$$

**Proof.** By Lemma 4.2 we can view $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \Omega^m_{X,\mathrm{adj}}$ as a subsheaf of $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} (\omega^0_X)^{\otimes m} \cong \mathcal{O}_X(n + m(d-l-2))$. Let $f \in \Gamma(X, \mathcal{O}_X(n + m(d-l-2)))$ be a global section and $\eta$ its preimage in $\Gamma(X, \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} (\omega^0_X)^{\otimes m})$ via this isomorphism. Projecting $\eta$ to the sections over $U_i$ we find

$$\eta \mapsto x_i^n \otimes \frac{f}{x_i^{n+m(d-l-2)}} \gamma_i^{\otimes m}.$$

We have to check whether $f/x_i^{n+m(d-l-2)} \gamma_i^{\otimes m} \in \Gamma(U_i, \Omega^m_{X,\mathrm{adj}})$ for all $i$. Again by Lemma 4.2 this form is adjoint at all regular points. Applying now Lemma 4.3 it is equivalent to check that

$$\kappa_\varphi\big(f/x_i^{n+m(d-l-2)} \gamma_i^{\otimes m}\big) \geqslant 0$$

for all $i$ and any formal prime divisor $\varphi \in \mathcal{S}$ with center$(\varphi) \in U_i$. This again is equivalent to

$$\kappa_\varphi\big(f/x_i^{n+m(d-l-2)}\big) \geqslant -m\kappa_\varphi(\gamma_i) = m\alpha_\varphi$$

for all $\varphi \in \mathcal{S}$. $\square$

## 5. The algorithm

We first give a few remarks that are relevant for the computation and then give an explicit algorithm. First since a hypersurface $X$ is in particular a complete intersection we know that

$$\Gamma\big(X, \mathcal{O}_X\big(n+m(d-l-2)\big)\big) \cong \big(\mathbb{E}[x_0,\ldots,x_{l+1}]/\langle F\rangle\big)_{n+m(d-l-2)}$$

(see, for example, [7, Exercise III.5.5.(a)]). If $\prec$ is some well-ordering on exponents compatible with addition and $\mu_0$ is the leading exponent of $F$ w.r.t. $\prec$ and $|\mu|$ is the total degree of $\underline{x}^\mu$ then we can write

$$\big(\mathbb{E}[x_0,\ldots,x_{l+1}]/\langle F\rangle\big)_{n+m(d-l-2)} \cong \big\langle \underline{x}^\mu \bigm| |\mu| = n+m(d-l-2) \text{ and } \mu \prec \mu_0\big\rangle_{\mathbb{E}},$$

because every class module $F$ can be uniquely written as a sum of monomials smaller than $F$.

Second we want to comment on the computation of adjoint orders (see Definition 4.5). Therefore we have to determine $\kappa_\varphi(\gamma_i)$. More generally let $\eta \in \Omega^1_{X,\mathrm{rat}}$ be arbitrary; if we can compute $\kappa_\varphi(\eta)$ then we in particular determine the adjoint orders. Let $u_1,\ldots,u_l \in \mathbb{E}(X)$ be a transcendence basis over $\mathbb{E}$. As a generator of $\widetilde{\Omega}^1_{\mathbb{F}_\varphi[\![t]\!]|\mathbb{E}}$ we choose as before $\omega_{\varphi,1} = \mathrm{d}s_{\varphi,1} \wedge \cdots \wedge \mathrm{d}s_{\varphi,l-1} \wedge \mathrm{d}t$, where $s_{\varphi,1},\ldots,s_{\varphi,l-1} \in \mathbb{F}_\varphi$ is a transcendence basis of $\mathbb{F}_\varphi[\![t]\!]$ over $\mathbb{E}$. We can write $\eta = f\,\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_l$ for some $f \in \mathbb{E}(X)$. Then by the rules of calculus

$$\varphi^\#(\eta) = \varphi^\#(f)\left|\frac{\partial(\varphi^\#(u_1),\ldots,\varphi^\#(u_l))}{\partial(s_{\varphi,1},\ldots,s_{\varphi,l-1},t)}\right|\mathrm{d}s_{\varphi,1} \wedge \cdots \wedge \mathrm{d}s_{\varphi,l-1} \wedge \mathrm{d}t$$

and hence

$$\kappa_\varphi(\eta) = \kappa_\varphi(f) + \mathrm{ord}_t\left|\frac{\partial(\varphi^\#(u_1),\ldots,\varphi^\#(u_l))}{\partial(s_{\varphi,1},\ldots,s_{\varphi,l-1},t)}\right|.$$

In order to compute the order of the Jacobian one can use approximative methods, i.e., compute with truncations of the involved power series of sufficiently high precision. A major problem is to do arithmetic in the field extension $\mathbb{F}_\varphi \mid \mathbb{E}$. This turns out to be computationally expensive with current computer algebra systems. It is therefore preferable to compute the adjoint orders, simultaneously with the formal desingularization. This involves essentially repeated application of the chain rule of differential calculus and is much easier from a computational point of view.

With these remarks and the above notation it is now obvious how to derive an algorithm. Correctness of the following is immediate by Theorem 4.6, and termination is trivial because formal desingularizations can be computed and consist of *finitely many* formal prime divisors.

In step 5, *Trunc*$(\varphi^\#(b), m\alpha_\varphi)$ means the truncation of $\varphi^\#(b)$ at order $m\alpha_\varphi$. In step 6, kernel means kernel from the right, i.e. all vectors $v$ such that $Cv = 0$. It remains to explain the function *AddConstraints*. It is meant to stack new rows on top of the matrix $C$, representing the linear constraints imposed by the formal prime divisor $\varphi$. The main difficulty is that we get constraints with coefficients in $\mathbb{F}_\varphi$ but we want to add rows with coefficients in $\mathbb{E}$.

Assume $\mathbb{E} \subset \mathbb{F}' \subset \mathbb{F}$ is a tower of field extensions where $\mathbb{F}$ over $\mathbb{F}'$ is simple (algebraic or transcendental). Let $a := \sum_{b\in B} y_b c_b \in \mathbb{F}[c_b \mid b \in B]$ be a linear form with coefficients $a_b \in \mathbb{F}$. We want to find values $c_b \in \mathbb{E}$ such that the linear constraint $a = 0$ is fulfilled. We are done if we know how to translate the constraint equivalently to a finite number of linear constraints over the smaller field $\mathbb{F}'$. Using

**Algorithm 1.** *Adjoints*$(F : \mathbb{E}[x_0, \ldots, x_{l+1}], m : \mathbb{N}, n : \mathbb{N})$ : (subset of $\mathbb{E}[x_0, \ldots, x_{l+1}]$)

**Require:** An irreducible, homogeneous polynomial $F$ of degree $d$ (not equal to $x_i$ for any $0 \leqslant i \leqslant l+1$), defining $X \subset \mathbb{P}^{l+1}_{\mathbb{E}}$.
**Ensure:** A basis for the space of global sections of $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \pi_*(\omega_Y^{\otimes m})$ represented by homogeneous polynomials of degree $n + m(d-l-2)$ where $\pi : Y \to X$ is any desingularization.

Let $B \subset \mathbb{E}[x_0, \ldots, x_{l+1}]$ be a set representing a basis of $(\mathbb{E}[x_0, \ldots, x_{l+1}]/\langle F\rangle)_{n+m(d-l-2)}$;
Compute a formal desingularization $\mathcal{S}$ of $X$ and adjoint orders $\alpha_\varphi$ for all $\varphi \in \mathcal{S}$;
$C := 0 \in \mathbb{E}^{\infty \times |B|}$; {a matrix with an undetermined number of rows}
**for** $\varphi \in \mathcal{S}$ **do**
  $A := \sum_{b \in B} c_b \mathit{Trunc}(\varphi^\#(b), m\alpha_\varphi) = \sum_{0 \leqslant j < m\alpha_\varphi} a_j t^j$; {with $a_j$ linear in $\mathbb{F}_\varphi[c_b \mid b \in B]$}
  $C := \mathit{AddConstraints}(C, \{a_j\}_{0 \leqslant j < m\alpha_\varphi})$;
Let $K \subset \mathbb{E}^{\#B}$ be a basis of $\ker(C)$;
**return** $\{\sum_{b \in B} c_b b \mid (c_b)_{b \in B} \in K\}$;

this step recursively and considering the fact that $\mathbb{F}$ over $\mathbb{E}$ is finitely generated, we finally get a set of constraints with coefficients in $\mathbb{E}$. If $\sum_{b \in B} y_b c_b = 0$ is such a constraint, the function *AddConstraints* would stack the row vector $(y_b)_{b \in B}$ on top of the matrix $C$. We distinguish two cases:

- If $\mathbb{F}$ over $\mathbb{F}'$ is algebraic, say, of degree $e+1$, choose a basis $\{f_r\}_{0 \leqslant r \leqslant e}$ of $\mathbb{F}$ as an $\mathbb{F}'$-vector space. Then

$$0 = \sum_{b \in B} y_b c_b = \sum_{b \in B} c_b \sum_{0 \leqslant r \leqslant d} y_{b,r} f_r = \sum_{0 \leqslant r \leqslant e} \left( \sum_{b \in B} y_{b,r} c_b \right) f_r$$

  holds if and only if $\sum_{b \in B} y_{b,r} c_b = 0$ for all $0 \leqslant r \leqslant e$.
- Now assume $\mathbb{F} = \mathbb{F}'(s)$ is transcendental. Without loss we may assume that the $y_b$ are actually polynomials in $\mathbb{F}'[s]$, otherwise multiply the equation by the common denominator. Let $e$ be the maximal degree of all the $y_b$. Then

$$0 = \sum_{b \in B} y_b c_b = \sum_{b \in B} c_b \sum_{0 \leqslant r \leqslant e} y_{b,r} s^r = \sum_{0 \leqslant r \leqslant e} \left( \sum_{b \in B} y_{b,r} c_b \right) s^r$$

  holds again if and only if $\sum_{b \in B} y_{b,r} c_b = 0$ for all $0 \leqslant r \leqslant e$.

## 6. Example

Let $\mathbb{E} := \mathbb{Q}$ and write $x := x_0$, $y := x_1$, $z := x_2$, $w := x_3$. The homogeneous polynomial $F := w^3 y^2 z + (xz + w^2)^3 \in \mathbb{Q}[x, y, z, w]$ of degree $d = 6$ defines a hypersurface $X \subset \mathbb{P}^3_{\mathbb{Q}}$, i.e., $l = \dim(X) = 2$. We compute a formal desingularization $\mathcal{S}$ using Algorithm 1 of [1]. Amongst others, we get a formal prime divisor $\varphi : \operatorname{Spec} \mathbb{F}_\varphi[\![t]\!] \to X$ defined by the $\mathbb{Q}$-algebra homomorphism

$$\varphi^\# : \mathbb{Q}[x, y, z, w]/\langle F\rangle \to \mathbb{F}_\varphi[\![t]\!] : \begin{cases} x \mapsto 1, \\ y \mapsto -\frac{8}{s} t^3, \\ z \mapsto \frac{64}{s} t^6, \\ w \mapsto -\frac{8}{s}\alpha t^3 - \frac{8}{s} t^4 + \frac{4}{s^2}\alpha t^5 + \frac{1}{s^3}\alpha t^7 + \frac{1}{2s^4}\alpha t^9 + O(t^{11}) \end{cases}$$

where $\mathbb{F}_\varphi = \mathbb{Q}(s)[\alpha]$ and $\alpha$ has minimal polynomial $\alpha^2 + s$.

First we want to compute the adjoint order of this formal prime divisor. Therefore we consider the rational differential form

$$\frac{x^{d-1}}{\partial F/\partial w}\mathrm{d}\frac{y}{x}\wedge\mathrm{d}\frac{z}{x}=\frac{x^5}{6x^2z^2w+12xzw^3+3y^2zw^2+6w^5}\mathrm{d}\frac{y}{x}\wedge\mathrm{d}\frac{z}{x}.$$

Now we apply the map induced by $\varphi^{\#}$ and find the differential form

$$\frac{1}{\frac{786432}{s^4}\alpha t^{17}+\frac{1572864}{s^4}t^{18}-\frac{1966080}{s^5}\alpha t^{19}+\mathrm{O}(t^{20})}\mathrm{d}\left(-\frac{8}{s}t^3\right)\wedge\mathrm{d}\left(\frac{64}{s}t^6\right).$$

According to Definition 4.4 we rewrite this differential form as a multiple of $\mathrm{d}s\wedge\mathrm{d}t$ and get

$$\frac{1}{\frac{512}{s}\alpha t^9+\frac{1024}{s}t^{10}-\frac{1280}{s^2}\alpha t^{11}+\mathrm{O}(t^{12})}\mathrm{d}s\wedge\mathrm{d}t.$$

The coefficient has order $-9$, so $\kappa_\varphi=9$ as of Definition 4.5.

Assume now, we want to compute the global sections of $\pi_*(\omega_Y)\otimes_{\mathcal{O}_X}\mathcal{O}_X(1)$ (where $\pi:Y\to X$ is any desingularization, not necessarily the one described by $\mathcal{S}$), i.e., we have $m=1$ and $n=1$. We compute $n+m(d-l-2)=1+1(6-2-2)=3$. Therefore we first need a set $B$ projecting bijectively to the component of $\mathbb{Q}[x,y,z,w]/\langle F\rangle$ of homogeneous degree 3. Since the defining equation is of degree 6 we can choose the set of all monomials of degree 3:

$$B:=\{x^3,x^2y,xy^2,y^3,x^2z,xyz,y^2z,xz^2,yz^2,z^3,x^2w,xyw,y^2w,xzw,yzw,z^2w,xw^2,yw^2,zw^2,w^3\}.$$

Applying $\varphi^{\#}$ to the generic form of degree 3 we find:

$$\begin{aligned}\varphi^{\#}\left(\sum_{b\in B}c_bb\right)={}&(c_{x^3})t^0+\left(-\frac{8}{s}c_{x^2y}-\frac{8}{s}\alpha c_{x^2w}\right)t^3+\left(-\frac{8}{s}c_{x^2w}\right)t^4+\left(\frac{4}{s^2}\alpha c_{x^2w}\right)t^5\\&+\left(\frac{64}{s^2}c_{xy^2}+\frac{64}{s}c_{x^2z}+\frac{64}{s^2}\alpha c_{xyw}-\frac{64}{s}c_{xw^2}\right)t^6+\left(\frac{1}{s^3}\alpha c_{x^2w}+\frac{64}{s^2}c_{xyw}+\frac{128}{s^2}\alpha c_{xw^2}\right)t^7\\&+\left(-\frac{32}{s^3}\alpha c_{xyw}+\frac{128}{s^2}c_{xw^2}\right)t^8+\mathrm{O}(t^9).\end{aligned}$$

A form is adjoint iff its $\varphi^{\#}$-image vanishes with order greater or equal to $m\kappa_\varphi=1\cdot 9=9$, i.e., the coefficients of $t^0,\dots,t^8$ have to vanish. Viewing $B$ as an ordered basis we can write this as a matrix in $\mathbb{Q}(s)[\alpha]^{9\times 20}$:

$$\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&-\frac{8}{s}&0&0&0&0&0&0&0&0&-\frac{8}{s}\alpha&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&-\frac{8}{s}&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&\frac{4}{s^2}\alpha&0&0&0&0&0&0&0&0&0\\0&0&\frac{64}{s^2}&0&\frac{64}{s}&0&0&0&0&0&0&\frac{64}{s^2}\alpha&0&0&0&0&-\frac{64}{s}&0&0&0\\0&0&0&0&0&0&0&0&0&0&\frac{1}{s^3}\alpha&\frac{64}{s^2}&0&0&0&0&\frac{128}{s^2}\alpha&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&-\frac{32}{s^3}\alpha&0&0&0&0&\frac{128}{s^2}&0&0&0\end{pmatrix}.$$

The third row from the bottom corresponds to the constraint

$$\frac{64}{s^2}c_{xy^2} + \frac{64}{s}c_{x^2z} + \frac{64}{s^2}\alpha c_{xyw} - \frac{64}{s}c_{xw^2} = 0.$$

Reordering this in terms of the basis $\alpha^0, \alpha^1$ of $\mathbb{Q}(s)[\alpha]$ over $\mathbb{Q}(s)$ we find

$$\left(\frac{64}{s^2}c_{xy^2} + \frac{64}{s}c_{x^2z} - \frac{64}{s}c_{xw^2}\right)\alpha^0 + \left(\frac{64}{s^2}c_{xyw}\right)\alpha^1 = 0$$

and hence the two equivalent constraints

$$\frac{64}{s^2}c_{xy^2} + \frac{64}{s}c_{x^2z} - \frac{64}{s}c_{xw^2} = 0 \quad \text{and} \quad \frac{64}{s^2}c_{xyw} = 0.$$

The second one is clearly equivalent to $64c_{xyw} = 0$ which is already a constraint over $\mathbb{Q}$. The first one can be multiplied by $s^2$ and rewritten in terms of $s^0$ and $s^1$:

$$(64c_{xy^2})s^0 + (64c_{x^2z} - 64c_{xw^2})s^1 = 0.$$

This yields another two constraints

$$64c_{xy^2} = 0 \quad \text{and} \quad 64c_{x^2z} - 64c_{xw^2} = 0$$

over $\mathbb{Q}$. Altogether the third row from the bottom corresponds to the following matrix in $\mathbb{Q}^{3\times 20}$:

$$\begin{pmatrix}
0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -64 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Treating all rows similarly, stacking the computed matrices on top of each other and skipping zero rows we obtain:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -64 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Doing the same computation for all the other formal prime divisors in $\mathcal{S}$ one computes a huge matrix which turns out to have a one-dimensional kernel spanned by

$$(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1).$$

The entries of this vector are the coefficients of the form $xzw + w^3$. So we have

$$\Gamma\left(X, \pi_*(\omega_Y) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)\right) \cong \left\langle xzw + w^3 \right\rangle_{\mathbb{Q}}.$$

## References


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