Topological Methods for Some Boundary Value Problems

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Abstract—We establish existence of solutions for a finite difference approximation to \( y'' = f(x, y, y') \) on \([0, 1]\), subject to nonlinear two-point Sturm-Liouville boundary conditions of the form \( g_i(y(i), y'(i)) = 0, \ i = 0, 1, \) assuming \( f \) satisfies one-sided growth bounds with respect to \( y' \). © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we give existence results for the following finite difference scheme:

\[
\begin{align*}
D^2 y_{k+1} &= f(t_k, y_k, D y_k), \quad k = 1, \ldots, n-1, \\
0 &= g^0(y_0, D y_1), \\
0 &= g^1(y_n, D y_n),
\end{align*}
\]

which provides a discrete approximation to the two-point boundary value problem

\[
\begin{align*}
y'' &= f(x, y, y'), \quad \text{for all } x \in [0, 1], \\
g^i(y(i), y'(i)) &= 0, \quad i = 0, 1,
\end{align*}
\]

where the \( g^i : \mathbb{R}^2 \to \mathbb{R} \) and \( f : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) are continuous functions, the grid size \( h = 1/n, \) \( D y_k = (y_k-y_{k-1})/h, \) for \( k = 1, \ldots, n \) so that \( D^2 y_{k+1} = (y_{k+1}-2y_k+y_{k-1})/h^2, \) for \( k = 1, \ldots, n-1, \) and the grid points \( t_k = kh, \) for \( k = 0, \ldots, n. \)

By a solution to problem (1)-(3), we mean a vector \( \tilde{y} = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1} \) satisfying (1) for all \( k = 1, \ldots, n-1, \) (2), and (3). The value of the \( k^{th} \) component, \( y_k, \) of a solution \( \tilde{y} \) of (1) is expected to approximate \( y(t_k), \) for some solution \( y \) of (4).

By a solution of (4) and (5), we mean a twice continuously differentiable function on \([0, 1]\) satisfying (4) and (5).

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Baxley [1] proved existence results for the boundary value problem (4) and (5). He assumed that \( f \) satisfies one-sided growth bounds with respect to \( y \) and \( y' \), and that the nonlinear boundary conditions defined by the \( g_i \) are naturally occurring generalizations of the usual Sturm-Liouville linear boundary conditions. His proofs are based on shooting with initial values combined with the maximum principle and the Kneser-Hukuhara continuum theorem.

Abadi and Thompson [2] used degree theory to generalize some of Baxley's results by allowing more rapid growth of \( f \) with respect to \( y \) and \( y' \). Under a mild variant of the assumptions of Abadi and Thompson [2], we show that the finite difference scheme (1)-(3) also has solutions which approximate solutions of the continuous problem. Moreover, when the solution of the continuous problem is unique, the approximate solutions converge to it, as the grid size goes to 0.

We adapt the approach of Henderson and Thompson [3]. Henderson and Thompson used degree theory to establish existence results for solutions to boundary value problems for second-order difference equations (1) and

\[
0 = G ((y_0, y_n), (Dy_1, Dy_n)),
\]

where \( G = (g^0, g^1) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad i = 0, 1 \) is continuous and fully nonlinear. These solutions approximate solutions of the two-point boundary value problem (4) and

\[
0 = G ((y(0), y(1)), (y'(0), y'(1))).
\]

Thompson assumed that there exist strict lower and strict upper solutions for (4), that \( f(z, y, z) \) satisfies a two-sided Nagumo growth condition with respect to \( z \), and that \( G \) is compatible with the strict lower and strict upper solutions. Under these assumptions, the corresponding continuous problem (4) and (7) has solutions; see [4].

We present some notation, definitions, and background results in Section 2. We state our main assumptions on the \( g_i \) and \( f \) and present our main existence result in Section 3 and give some applications in Section 4.

Boundary value problems for (4) with nonlinear boundary conditions have not been studied as intensively as those with linear boundary conditions. For a discussion of the literature, see [1,4-8], and the references quoted therein. The literature on difference equations is extensive; see, for example, the books by Agarwal [9] and by Kelley and Peterson [10], together with the references therein. For other papers also employing discrete lower and discrete upper solutions, see [3,11-15].

2. BACKGROUND NOTATION AND DEFINITIONS

In order to state our results, we need some notation.

Let \( Y = \mathbb{R}^{n+1} = \mathbb{R} \times \cdots \times \mathbb{R} \simeq [0, h, \ldots, nh] \times \mathbb{R} \). Let \( \tilde{y} = (y_0, \ldots, y_n) \in Y \). We set \( ||\tilde{y}|| = \max \{|y_k| : k = 0, \ldots, n\} \). Further, by abuse of notation, we set \( |D\tilde{y}| = \max \{|Dy_k| : k = 1, \ldots, n\} \) and \( |D^2\tilde{y}| = \max \{|D^2y_{k+1}| : k = 1, \ldots, n-1\} \). These define norms on the appropriate spaces.

We denote the space of continuous functions from \( A \) to \( \beta \) by \( CA; \beta \). If \( B = \mathbb{R} \), then we omit the \( B \). With \( y \in C([0, 1]) \), we will sometimes associate the vector \( \tilde{y} \in Y \) defined by \( \tilde{y} = (y_0, y_1, \ldots, y_n) = (y(0), y(h), \ldots, y(nh)) \). For \( \tilde{y} \) and \( \bar{z} \) in \( Y \), we will write \( \tilde{y} \leq \bar{z} \) if \( y_i \leq z_i \), for all \( i = 0, \ldots, n \).

If \( A \) is a bounded, open subset of \( \mathbb{R}^n \), \( q \in \mathbb{R}^n \), \( F \in C(A; \mathbb{R}^n) \) and \( q \notin F(\partial A) \), we denote the corresponding Brouwer degree of \( F \) on \( A \) by \( d_q(F, A, q) \).

For the convenience of the reader, we recall some definitions.

**Definition 1.** We call \( \alpha \) (\( \beta \)) a strict lower (strict upper) solution for (1) if \( \alpha \) (\( \beta \)) \( \in C^2([0, 1]) \), and there exists \( \gamma > 0 \) such that

\[
\alpha''(x) - f(x, \alpha(x), \alpha'(x)) \geq \gamma, \quad x \in [0, 1],
\]

and

\[
\beta''(x) - f(x, \beta(x), \beta'(x)) \geq \gamma, \quad x \in [0, 1].
\]
We call $\bar{\alpha}$ ($\bar{\beta}$) a strict discrete lower (strict discrete upper) solution for (2) if there is $\gamma > 0$ such that

\[
\mathcal{D}^2\alpha_{k+1} - f(t_k, \alpha_k, \mathcal{D}\alpha_k) \geq \gamma, \quad k = 1, \ldots, n - 1, \tag{10}
\]
\[
(f(t_k, \beta_k, \mathcal{D}\beta_k) - \mathcal{D}^2\beta_{k+1} \geq \gamma, \quad \text{if } k = 1, \ldots, n - 1. \tag{11}
\]

Let $\alpha \leq \beta$ be nondegenerate, strict lower, and strict upper solutions, respectively, for (1).

**DEFINITION 2.** We call the vector field $\Psi = (\psi^0, \psi^1) \in C(\bar{\Delta}; \mathbb{R}^2)$ inwardly pointing on $\Delta$ if for all $(C, D) \in \partial\Delta$

\[
\psi^0(\alpha(0), D) \geq \alpha'(0), \quad \psi^0(\beta(0), D) \leq \beta'(0), \quad \text{and} \quad \psi^1(C, \alpha(1)) \leq \alpha'(1), \quad \psi^1(C, \beta(1)) \geq \beta'(1). \tag{12}
\]

**DEFINITION 3.** Let $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$. We say $G$ is very strongly compatible with $\alpha$ and $\beta$ if for all inwardly pointing $\Psi$ on $\Delta$

\[
G(C, D) \neq 0, \quad \text{for all } (C, D) \in \partial\Delta, \quad \text{and} \quad d(G, \Delta, 0) \neq 0, \tag{13}
\]

where

\[
G(C, D) = G((C, D); \Psi(C, D)), \quad \text{for all } (C, D) \in \bar{\Delta}. \tag{14}
\]

To state our main results, we need the following assumptions on $f(x, y, z)$.

**A:** $f(x, y, z)$ is continuous on $[0, 1] \times \mathbb{R}^2$

\[
B_1'(\xi_1): \text{there exists } \eta_1 > 0 \text{ such that } f(x, y, \xi_1) > 0 \text{ for all } (x, y) \in [0, 1] \times [\eta_1, \infty);
\]

\[
B_2'(\xi_2): \text{there exists } \eta_2 < 0 \text{ such that } f(x, y, \xi_2) < 0 \text{ for all } (x, y) \in [0, 1] \times (-\infty, \eta_2];
\]

\[
C_1'(\xi_1): \text{there exist } \eta_1 > 0, \bar{h}_1 : [\xi_1, \infty) \to (0, \infty) \text{ such that}
\]

\[
f(x, y, z) \geq -\bar{h}_1(z), \quad \int_{\xi_1}^{\infty} \frac{dt}{\bar{h}_1(t)} = \infty,
\]

for all $(x, y, z) \in [0, 1] \times [\eta_1, \infty) \times [\xi_1, \infty)$;

\[
C_2'(\xi_2): \text{there exist } \eta_2 < 0, \bar{h}_2 : [\xi_2, \infty) \to (0, \infty) \text{ such that}
\]

\[
f(x, y, z) \leq \bar{h}_2(|z|), \quad \int_{\xi_2}^{\infty} \frac{dt}{\bar{h}_2(t)} = \infty,
\]

for all $(x, y, z) \in [0, 1] \times (0, \infty) \times (-\infty, \xi_2]$;

\[
D_1'': \text{given } \eta_2 < \eta_1, \text{ there exist } s_1(\eta_2, \eta_1) > 0 \text{ and a continuous function } h_1 : [s_1, \infty) \to (0, \infty) \text{ such that}
\]

\[
f(x, y, z) \geq -h_1(z), \quad \int_{s_1}^{\infty} \frac{t \, dt}{h_1(t)} = \infty,
\]

for all $(x, y, z) \in [0, 1] \times [\eta_2, \eta_1] \times [s_1, \infty)$;

\[
D_2'': \text{given } \eta_2 < \eta_1, \text{ there exist } s_2(\eta_2, \eta_1) < 0 \text{ and a continuous function } h_2 : [s_2, \infty) \to (0, \infty) \text{ such that}
\]

\[
f(x, y, z) \leq h_2(|z|), \quad \int_{s_2}^{\infty} \frac{t \, dt}{h_2(t)} = \infty,
\]

for all $(x, y, z) \in [0, 1] \times [\eta_2, \eta_1] \times (-\infty, s_2]$;

\[
D: \text{given } \eta_2 < \eta_1, \text{ there exist } \mathcal{Q}(\eta_2, \eta_1) > 0 \text{ and a continuous function } h : [\mathcal{Q}, \infty) \to (0, \infty) \text{ such that}
\]

\[
|f(x, y, z)| \leq h(|z|), \quad \int_{\mathcal{Q}}^{\infty} \frac{t \, dt}{h(t)} = \infty,
\]

for all $(x, y, z) \in [0, 1] \times [\eta_2, \eta_1] \times \{(-\infty, -\mathcal{Q}] \cup [\mathcal{Q}, \infty)\}$. 

We consider the following nonlinear Sturm-Liouville boundary conditions introduced by Baxley.

SL0: The graph of \( g^0(y, z) = 0 \) contains a (continuous) curve which can be parameterized as \( y = p(\gamma), z = q(\gamma) \), for \(-\infty < \gamma < \infty\), where \( p, q \) are continuous and

\[
\lim_{\gamma \to -\infty} \sup p(\gamma) < +\infty, \quad \lim_{\gamma \to +\infty} \inf p(\gamma) > -\infty, \quad \lim_{\gamma \to +\infty} q(\gamma) = -\infty, \quad \lim_{\gamma \to -\infty} q(\gamma) = +\infty. \quad (15) \]

SL1: \( g^1(u, v) \) is continuous on \( \mathbb{R}^2 \) and given \( u_0 \in \mathbb{R} \), there exist \( v_1, v_2 \in \mathbb{R} \) so that \( g^1(u, v) > 0 \) for \( u > u_0, v > v_1 \), and \( g^1(u, v) < 0 \) for \( u < u_0, v \leq v_2 \).

If \( g^1 \) satisfies SL1, we let \( T_1(u_0) \) (respectively, \( T_2(u_0) \)) denote the infimum (respectively, supremum) of the set of all such values \( v_1 \) (respectively, \( v_2 \)). Clearly, \( T_1(u) \), \( T_2(u) \) are both nonincreasing functions of \( u \).

Later we will construct a strict lower solution, \( \alpha \), using the one-sided inequality \( B_1'(\xi_1) \), respectively, the one-sided growth assumption \( C_1'(\xi_1) \) on \( f \). Moreover, we will construct a strict upper solution \( \beta \) satisfying \( \alpha \leq \beta \), on \([0, 1]\) using the one-sided inequality \( B_1'(\xi_1) \), respectively, the one-sided growth assumption \( C_1'(\xi_1) \) on \( f \).

Our Assumptions \( B_1'(\xi_1) \), and \( B_2'(\xi_2) \), are strengthened versions of Assumptions \( B_1(\xi_1) \), and \( B_2(\xi_2) \), respectively, of Abadi and Thompson [2] obtained by replacing the weak inequalities in [2] by strict inequalities. Our one-sided Nagumo conditions \( D_1'' \) and \( D_2'' \) are variants of Assumptions \( D_1' \) and \( D_2' \), respectively, of Abadi and Thompson [2] obtained by requiring the functions \( h_1 \) and \( h_2 \), respectively, to be continuous. They are used to establish a priori bounds on \( |D\tilde{y}| \) depending on \( ||\tilde{y}|| \) but not \( h \), for solutions, \( \tilde{y} \), of problem (1)-(3). To obtain these a priori bounds [3], we used a mean value style argument to bound \( |D\tilde{y}| \) at some point and then used \( D \) to bound on \( ||D\tilde{y}|| \). By contrast, the current paper uses SL1 to bound \( |D\tilde{y}_1| \) and then uses \( D_1' \) and \( D_2' \) to bound \( ||D\tilde{y}|| \). The question arises as to whether SL1 is needed for the current bound, and if not to produce a counterexample.

Clearly \( D \) implies \( D_1'' \) and \( D_2'' \), but the converse is not true as can be seen from the example in [2].

For a discussion of the relationship between Abadi and Thompson's [2] growth conditions and those of Baxley, see [1].

### 3. THE MAIN RESULTS

Now, we are ready to state our results. We begin with our main theorem.

**Theorem 3.1.** Let \( g^0 \) satisfy SL0, \( g^1 \) satisfy SL1, and \( T_1 \) and \( T_2 \) be given by SL1. Let \( f \) satisfy

1. Assumption A;
2. either Assumption \( B_1'(\xi_1) \) for some \( \xi_1 > \lim_{n \to \infty} T_1(u) \), or Assumption \( C_1'(\xi_1) \) for some \( \xi_2 < -1; \)
3. either Assumption \( B_2'(\xi_2) \) for some \( \xi_2 < \lim_{n \to \infty} T_2(u) \), or Assumption \( C_2'(\xi_2) \) for some \( \xi_2 < -1; \) and
4. Assumptions \( D_1'' \) and \( D_2'' \).

Then there is \( \delta > 0 \), such that for \( 0 < h < \delta \) there exists a solution \( \tilde{y} \) of (1)-(3).

The proof relies on the next theorem and the lemmas following it.

The following theorem is a variant of Henderson and Thompson [3, Theorem 3].

**Theorem 3.2.** Assume that \( f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) is a continuous function. Let \( \alpha \) and \( \beta \) be nondegenerate, strict lower, and strict upper solutions, respectively, for (4) and \( G = (g^0, g^1) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) be continuous and very strongly compatible with \( \alpha \) and \( \beta \). Assume there exists a real number \( N > 0 \), independent of \( h \), such that all solutions \( \tilde{y} \) of (1) lying between \( \alpha \) and \( \beta \)
satisfy \( \|D\tilde{y}\| \leq N \). Then there is \( \delta > 0 \) such that problem (1)-(3) has a solution \( y \) satisfying \( \tilde{\alpha} \leq \tilde{y} \leq \tilde{\beta} \), for \( 0 < h < \delta \).

The proof follows from the proof of Henderson and Thompson [3, Theorem 3] and is omitted.

Let \( \alpha \leq \beta \) be the nondegenerate, strict lower and strict upper solutions, respectively, from Theorem 3.2. We set

\[
\beta_M = \max\{\beta(x) : x \in [0, 1]\}, \\
\alpha_m = \min\{\alpha(x) : x \in [0, 1]\}.
\]

The next two Nagumo style lemmas give a priori bounds on difference quotients for solutions of (1)-(3). They are a variant of [15, Lemma 5.2] (see also [3. Theorem 1]).

**Lemma 3.3.** Let \( f \) satisfy Assumptions A and D\( f' \) and let \( \alpha \) and \( \beta \) be nondegenerate strict lower and strict upper solutions, respectively, for (4). Let \( N_1 > M_1 \geq s_1(\eta_2, \eta_1) \geq 0 \) satisfy

\[
\int_{M_1}^{N_1} \frac{t \, dt}{h_1(t)} > \beta_M - \alpha_m,
\]

where \( s_1(\eta_2, \eta_1) \) is given in D\( f' \). Let \( K \) be a constant independent of \( h \) and \( \eta_2 \leq \alpha_m < \beta_M \leq \eta_1 \). If \( \tilde{y} \) is a solution of (1) satisfying \( \tilde{\alpha} \leq \tilde{y} \leq \tilde{\beta} \), \( |D^2\tilde{y}| \leq K \), and \( D^2\tilde{y} \leq M_1 \), then there is \( \delta_2 > 0 \) such that \( D^2\tilde{y} \leq N_1 \) when \( 0 < h < \delta_2 \) and \( 1 \leq k \leq n \).

**Proof.** First, since \( s/h_1(s) \) is continuous there is \( \delta_3 > 0 \) such that

\[
\left| \int_{M_1}^{N_1} \frac{t \, dt}{h_1(t)} - \sum_{i=1}^{\lfloor n/2 \rfloor} s_i \frac{h_1(s_i)(b_i - a_i)}{2} \right| < \frac{(\beta_M - \alpha_m + \int_{M_1}^{N_1} \frac{t \, dt}{h_1(t)})}{2}
\]

whenever \( [a, b] \subseteq [c, d] \), the \((a_i, b_i)\) are pairwise disjoint, \( b_i - a_i < \delta_3 \), and \( |s_i - (a_i + b_i)/2| < \delta_3 \). Moreover, since \( s/h_1(s) > 0 \) it follows that

\[
\beta_M - \alpha_m < \frac{(\beta_M - \alpha_m + \int_{M_1}^{N_1} \frac{t \, dt}{h_1(t)})}{2}
\]

whenever \( [u_1, b_1] \subseteq [c_1, d_1] \), \( u_1 - c_1 < \delta_3 \), and \( s_i \subseteq [c_i, d_i] \), for all \( i \).

To simplify notation, we set \( D^2\tilde{y}_k = v_k \) for \( k = 1, \ldots, n \). Set \( \delta_2 = \delta_3/(2K) \) and choose \( h < \delta_2 \). Thus, \( |h(v_{k+1} - v_k)| = |D^2\tilde{y}_{k+1}| \leq K \) for \( k = 1, \ldots, n-1 \).

Assume the result is false so that there exists \( k_1, 1 \leq k_1 < n \) with \( v_{k_1} > N_1 \). Since \( v_n \leq M_1 \), we may choose \( k_2, k_3 \) such that \( k_1 < k_2 < k_3 \leq n \), \( v_{k_2} \leq M_1 < v_{k_3} \leq v_n \), for \( k_2 < k < k_3 \).

Let \( \mathcal{W} = \{k : k_2 \leq k < k_3, D^2\tilde{y}_{k+1} = h(v_{k+1} - v_k) < 0\} \). Thus, \( \mathcal{W} \) is not empty and \( [M_1, N_1] \subseteq \bigcup_{k \in \mathcal{W}} [v_{k+1}, v_k] \). Let \( \mathcal{W}_m \) be a minimal subset of \( \mathcal{W} \) (under the order induced by set theoretic inclusion) with the property that \( [M_1, N_1] \subseteq \bigcup_{k \in \mathcal{W}_m} [v_{k+1}, v_k] \). After relabeling, \( \mathcal{W}_m = \{l_1, l_2, \ldots, l_s\} \) for some natural number \( s \), where the labels are chosen so that \( l_i \) increases with \( i \). Moreover, \( l_1 + 1 = k_3 \). Hence, \( v_{l_i} > v_{l_{i+1}} \geq v_{l_{i+1}} \) for \( i = 1, \ldots, s \),

\[
\bigcup_{i=1}^{s} [v_{l_{i+1}}, v_{l_i}] \supseteq \bigcup_{i=1}^{s} \{[v_{l_{i+1}}, v_{l_{i}}] \cap [M_1, N_1]\} \supseteq [M_1, N_1],
\]

the \((v_{l_{i+1}}, v_{l_i})\) are pairwise disjoint, \([v_{l_{i+1}}, v_{l_i}] \subset [v_{l_{i+1}}, v_{l_{i}}]\), and

\[
v_{l_i} - v_{l_{i+1}} \leq v_{l_i} - v_{l_{i+1}} < \delta_3.
\]
Thus,

\[
\beta_M - \alpha_m < \frac{1}{2} \left( \beta_M - \alpha_m + \int_{M_1}^{N_1} (s/h(s)) \, ds \right)
\]

\[
\leq \sum_{i=1}^{n-1} v_i, \quad \text{setting } s_i = v_i \text{ in (18),}
\]

\[
= \sum_{i=1}^{n-1} \frac{-h D^2 y_i + 1}{h (D y_i)},
\]

\[
< \sum_{i=1}^{n-1} \left( -h D^2 y_i + 1 \right) \frac{1}{h (D y_i)}, \quad \text{since } 0 < \frac{1}{h (D y_i)} \leq 1 \text{ and } v_i \geq 0,
\]

\[
< \sum_{i=1}^{n-1} h v_i, \quad \text{since } v_i \geq M_1 > 0,
\]

\[
- y_{k-1} - y_{k-1}, \quad \text{setting } s_i = v_i \text{ in (18),}
\]

\[
\leq \beta_M - \alpha_m,
\]

a contradiction. The result follows.

**Lemma 3.4.** Let \( f \) satisfy Assumptions A and \( D^2_{\beta} \) and let \( \alpha \) and \( \beta \) be nondegenerate, strict lower and strict upper solutions, respectively, for (4). Let \( -N_2 < -M_2 \leq s_2(\eta_2, \eta_1) \leq 0 \) satisfy

\[
\int_{N_2}^{N_2} \frac{t \, dt}{k^2(t)} > \beta_M - \alpha_m,
\]

where \( s_2(\eta_2, \eta_1) \) is given in \( D^2_{\beta} \). Let \( K \) be a constant independent of \( h \) and \( \eta_2 \leq \alpha_m < \beta_{kl} < \eta_1 \).

If \( \bar{y} \) is a solution of (1) satisfying \( \bar{\alpha} < \bar{y} < \bar{\beta} \), \( |D^2 \bar{y}| \leq K \), and \( D y_k \geq -M_2 \), then there is \( \delta_4 > 0 \)

such that \( \bar{y} \) satisfies (4) and is independent of \( h \) when \( 0 < h < \delta_4 \) and \( 1 \leq k < n \).

The proof is similar to that of Lemma 3.3 and hence is omitted.

We now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( g^0 \) satisfy SL0, \( g^1 \) satisfy SL1, and \( T_1 \) and \( T_2 \) be given by SL1. Let \( f: [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) be continuous and satisfy:

1. either Assumption \( D^1_{\beta}(\xi_1) \) for some \( \xi_1 > \lim_{u \to -\infty} T_1(u) \), or Assumption \( C^1_{\beta}(\xi_1) \) for some \( \xi_1 > 1 \);
2. either Assumption \( D^2_{\beta}(\xi_2) \) for some \( \xi_2 < \lim_{u \to -\infty} T_2(u) \), or Assumption \( C^2_{\beta}(\xi_2) \) for some \( \xi_2 < -1 \);
3. Assumptions \( D^1_{\alpha} \) and \( D^2_{\alpha} \).

From the proof of [2, Lemma 3.2], there exist nondegenerate strict lower and strict upper solutions \( \alpha \) and \( \beta \), respectively, for (4) and there exists \( G = (g^0, g^1) \in C(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}) \) which is very strongly compatible with \( \alpha \) and \( \beta \) and such that \( g^0(y, z) = 0 \) when \( g^0(y, z) = 0 \).

Since \( g^1(y, z) \) satisfies SL1, we can find

\[
\nu_- = \sup \{ v \in (-\infty, s_2]; g^1(y, z) < 0, \text{ for all } y \leq \eta_1, z \leq v \}
\]

and

\[
\nu_+ = \inf \{ v \in [s_1, \infty); g^1(y, z) > 0, \text{ for all } y \geq \eta_2, z \geq v \},
\]

where \( s_1 = s_1(\eta_2, \eta_1) \) and \( s_2 = s_2(\eta_2, \eta_1) \) are given in \( D^1_{\alpha} \) and \( D^2_{\alpha} \).

Set \( M_1 = \nu_1 \) and \( M_2 = |\nu_-| \), and let \( N_1 \) and \( N_2 \) be given by (17) and (19), respectively.

Since \( \alpha \) and \( \beta \) are twice continuously differentiable on \([0, 1]\), we may choose \( N > 0 \) independent of \( h \) such that

\[
N - 1 > \max \{ N_1, N_2, |\alpha'(x)|, |\beta'(x)|: 0 \leq x \leq 1 \}.
\]
Set \( f_N(x, y, z) = f(x, y, \max\{\min\{-N, z\}, N\}) \), choose

\[ K > \max\{|f(x, y, z) : 0 \leq x \leq 1, \alpha_m - 1 \leq y \leq \beta_M + 1, |z| \leq N\}, \]

and let \( \tilde{y} \) be a solution of

\[ D^2 y_{k+1} = f_N(t_k, y_k, Dy_k), \]  

(20)

together with

\[ 0 = g^0(y_0, Dy_0) \]

(21)

and (3). It follows from (20) that \( \|D^2 \tilde{y}\| \leq K \). From the choice of \( N \), it follows that \( \alpha \) and \( \beta \) are strict lower and strict upper solutions, respectively, for

\[ y'' = f_N(x, y, y'), \quad 0 \leq x \leq 1. \]

By SL1 and the choice of \( M_1 \) and \( M_2 \), it follows from (3) that \(-M_2 \leq Dy_n \leq M_1 \). If \( \tilde{\alpha} \leq \tilde{y} \leq \tilde{\beta} \), applying Lemmas 3.3 and 3.4 to (20) it follows that there is \( \delta > 0 \) such that \( \|D\tilde{y}\| \leq \max\{N_1, N_2\} < N \), for \( 0 < h < \delta \). Thus, by Theorem 3.2, we may choose \( \delta \) sufficiently small that there is a solution, \( \tilde{y} \), of (20), (21), and (3) satisfying \( \tilde{\alpha} \leq \tilde{y} \leq \tilde{\beta} \), for \( 0 < h < \delta \). Since \( \|D\tilde{y}\| < N \), it follows that \( \tilde{y} \) is a solution of (1). Since \( g^0(y, z) = 0 \) when \( g^0(y, z) = 0 \), \( \tilde{y} \) satisfies (2). Hence, \( \tilde{y} \) is the required solution.

REMARK 1. We note that \( \alpha \) and \( \beta \) can be bounded in terms of the assumptions as can be seen implicitly from their construction in [2, Lemma 3.2]. Since \( \tilde{\alpha} \leq \tilde{y} \leq \tilde{\beta} \), it follows that \( \tilde{y} \) can also be bounded in terms of the assumptions.

QUESTION 2. Under the assumptions of Theorem 3.1, can we bound \( \|D\tilde{y}\| \) in terms of \( \|\tilde{y}\| \) independently of \( h \) for solutions \( \tilde{y} \) of (1)?

We now discuss the sense in which solutions of the difference equation (1) approximate solutions of the continuous problem (4).

THEOREM 3.5. Under the assumptions of Theorem 3.1, given \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \) such that if \( 0 < h < \delta(\epsilon) \) and \( \tilde{y} \) is a solution of (1)–(3), then there is a solution, \( y \), of (4) and (5) such that

\[ \max\{|y(t, \tilde{y}) - y(t)| : 0 \leq t \leq 1\} \leq \epsilon \]

and

\[ \max\{|v(t, \tilde{y}) - y'(t)| : 0 \leq t \leq 1\} \leq \epsilon, \]

where

\[ y(t, \tilde{y})) = y_k + Dy_{k+1}(t - t_k), \quad \text{for} \quad t_k \leq t \leq t_{k+1}, \quad \text{and} \]

\[ v(t, \tilde{y}) = \left\{ \begin{array}{ll}
Dy_k + (t - t_k)D^2 y_{k+1}, & \text{for} \quad t_k \leq t \leq t_{k+1}, \\
Dy_k, & \text{for} \quad 0 \leq t \leq t_1.
\end{array} \right. \]

The proof uses \( f_N \) rather than \( f \). The remainder of the proof follows similar lines to that in [15, Theorem 2.5] and so we omit it. The notation \( y(t, \tilde{y}) \) and \( v(t, \tilde{y}) \) was introduced in [15].

4. APPLICATIONS

In this section, we give some applications of our results.

EXAMPLE 1. Consider the boundary value problem

\[ y'' = y'(y')^{2} \ln \left(1 + y'^{2}\right) + \cos x = f(x, y, y'), \quad 0 \leq x \leq 1, \]

(22)

\[ g^0 = y'(0) - y^2(0) + 2y^2(0) - 1 = 0 = y'(1) + y^2(1) - 5y(1) - 2 = g^1. \]

(23)
It is not difficult to check that \( f(x, y, y') \) satisfies Assumptions \( B_1'(\xi_1) \) with \( y_1 = 1 \) and \( \xi_1 = 2 \), \( B_2'(\xi_2) \) with \( y_2 = -1 \) and \( \xi_2 = -2 \), and \( D_1'' \) and \( D_2'' \), where \( h_1(y') = 2\max\{|y_1|, |y_2|, 1\} \times (y')^2 \ln(1 + y^2) = h_2(y') \) and \( s_1 = 2 = -s_2 \). Moreover, it is not difficult to check that \( g^0 \) satisfies SL0 and \( g^1 \) satisfies SL1. Thus, by Theorem 3.1, there is \( \delta > 0 \) such that for \( 0 < h < \delta \) the finite difference scheme associated with (22) and (23) has a solution.

The next example is from Baxley [1]. It poses an interesting problem.

**Example 2.** (See [1, Example 2].) The equation

\[
y'' = A(x)y^m + a(x)y' = f(x, y, y'), \quad 0 < x < 1,
\]

where \( A(x) > 0 \) and \( a(x) \) is continuous on \([0, 1]\) occurs in the problem of radiation heat transfer for annular fins with boundary conditions

\[
y(0) = 0, \quad y'(1) = 0.
\]

If \( m \) is a positive odd number, then \( f \) satisfies \( B_1'(\xi_1) \) and \( B_2'(\xi_2) \) for suitable \( y_1, y_2, \xi_1 \), and \( \xi_2 \). Moreover, \( f \) satisfies \( D_1'' \) and \( D_2'' \) for \( h_1(z) = h_2(z) = \max\{|a(x)| : 0 \leq x \leq 1\} \). Thus, our Theorem 3.1 applies to show that the finite difference scheme associated with (24) and (25) has a solution. Moreover, it follows from the maximum principle that the solution to continuous problem (24) and (25) is unique. Thus, solutions of the associated difference scheme converge, in the sense of Theorem 3.5, to solutions of continuous problems, as the grid size converges to 0.

It would be interesting to know if solutions to the finite difference scheme are unique for small step size. If two solutions persist as the step size converges to 0, then the point at which the maximum of their difference is attained must converge to an end of the interval as step size converges to 0. Moreover, solutions are unique if \( a \) is nonnegative. This raises the general question as to what, if any, is the connection between uniqueness for boundary value problems for the continuous problem and uniqueness for its associated finite difference approximation.

If \( m \) is a positive even number, then our Theorem 3.1 may not apply. In this case, as in [1], we set

\[
f(x, y, z) = A(x)|y|^m \text{sgn} y + a(x)z,
\]

and then solutions to the boundary value problem (4) together with boundary conditions (25) exist and are nonnegative. Moreover, our Theorem 3.1 applies with this \( f \), and consequently the associated finite difference scheme has a solution which approximates a solution of the corresponding continuous problem when the step size is sufficiently small. It would be interesting to know if solutions to the finite difference schemes are nonnegative for small step size. Solutions are nonnegative if \( a \) is nonnegative.

In a future note, we will extend our results to other boundary conditions of physical interest.

**REFERENCES**


