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Ruled cubic surfaces in P$G(4,q)$, Baer subplanes of PG$(2,q^2)$ and Hermitian curves

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Abstract

In PG$(2,q^2)$ let ℓ_∞ denote a fixed line, then the Baer subplanes which intersect ℓ_∞ in $q+1$ points are called *affine* Baer subplanes. Call a Baer subplane of PG$(2,q^2)$ *non-affine* if it intersects ℓ_∞ in a unique point. It is shown by Vincenti (Boll. Un. Mat. Ital. Suppl. 2 (1980) 31) and Bose et al. (Utilitas Math. 17 (1980) 65) that non-affine Baer subplanes of PG$(2,q^2)$ are represented by certain ruled cubic surfaces in the André/Bruck and Bose representation of PG$(2,q^2)$ in PG$(4,q)$ (Math. Z. 60 (1954) 156; J. Algebra 1 (1964) 85; J. Algebra 4 (1966) 117). The André/Bruck and Bose representation of PG$(2,q^2)$ involves a regular spread in PG$(3,q)$. For a fixed regular spread $\mathscr{S}$, it is known that not all ruled cubic surfaces in PG$(4,q)$ correspond to non-affine Baer subplanes of PG$(2,q^2)$ in this manner. In this paper, we prove a characterisation of ruled cubic surfaces in PG$(4,q)$ which represent non-affine Baer subplanes of the Desarguesian plane PG$(2,q^2)$. The characterisation relies on the ruled cubic surfaces satisfying a certain geometric condition. This result and the corollaries obtained are then applied to give a geometric proof of the result of Metsch (London Mathematical Society Lecture Note Series, Vol. 245, Cambridge University Press, Cambridge, 1997, p. 77) regarding Hermitian unitals; a result which was originally proved in a coordinate setting. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let π be a translation plane of order q^2 with kernel containing GF(q); denote by ℓ_∞ a fixed translation line of π. Then by the well-known results of [1,7,8], the plane π corresponds to a certain incidence structure defined in PG$(4,q)$ using a line spread $\mathscr{S}$ in a fixed hyperplane Σ_∞ of PG$(4,q)$; in this representation the points of ℓ_∞ in π correspond to the elements of the spread $\mathscr{S}$. We call this representation the *André/Bruck and Bose representation of* π *in* PG$(4,q)$. The plane π is the Desarguesian plane PG$(2,q^2)$ if and only if the spread $\mathscr{S}$ is regular (see [7]).

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A *Baer subplane* of π is a subplane of order q. Lines of π meet a Baer subplane in 1 or $q+1$ points. The set of $q+1$ points that form the intersection of a line of π with a Baer subplane is called a *Baer subline*.

Consider the case $\pi = \mathrm{PG}(2,q^2)$, so that ℓ_∞ is the fixed line in $\mathrm{PG}(2,q^2)$ corresponding to the regular spread $\mathscr{S}$ in the André/Bruck and Bose representation. It is shown in [22,5] that any Baer subplane of $\mathrm{PG}(2,q^2)$ which intersects ℓ_∞ in a unique point is represented by certain ruled cubic surface in the André/Bruck and Bose representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(4,q)$. Such a ruled cubic surface V_2^3 must have line directrix a line P^* in $\mathscr{S}$ and in $\mathrm{PG}(4,q)$ the hyperplane Σ_∞ intersects V_2^3 in exactly the line P^*. However, for a fixed regular spread $\mathscr{S}$, not every ruled cubic surface with these properties corresponds to a Baer subplane in $\mathrm{PG}(2,q^2)$; that is, by counting, there are more such ruled cubic surfaces in $\mathrm{PG}(4,q)$ than there are Baer subplanes of $\mathrm{PG}(2,q^2)$ which intersect ℓ_∞ in a unique point. A ruled cubic surface in $\mathrm{PG}(4,q)$ which does correspond to a Baer subplane of $\mathrm{PG}(2,q^2)$ in the André/Bruck and Bose representation shall be called a *Baer ruled cubic surface*. Recall that for $\mathscr{S}$ a fixed regular spread of Σ_∞, in the quadratic extension of Σ_∞ there exist exactly two lines g and g' which are disjoint from Σ_∞ but which intersect every line in $\mathscr{S}$ (see [6]); call these lines g, g' *generator lines* of $\mathscr{S}$. Note that lines g and g' are conjugate under the Fröbenius map $\mathbf{x} \mapsto \mathbf{x}^q$ on points of $\mathrm{PG}(4,q^2)$ with coordinate vectors $\mathbf{x}$.

Continuing with the above notation, our main result (proved in Section 3) is:

Theorem 3.1. *Let V_2^3 be a ruled cubic surface in $\mathrm{PG}(4,q)$ with line directrix P^*, such that P^* is an element of the regular spread $\mathscr{S}$ in Σ_∞ and in $\mathrm{PG}(4,q)$ the hyperplane Σ_∞ intersects V_2^3 in exactly the line P^*. Let $\bar{V}_2^3$ be the ruled cubic surface in $\mathrm{PG}(4,q^2)$ which is the quadratic extension of V_2^3. Then V_2^3 is a Baer ruled cubic surface in this André/Bruck and Bose representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(4,q)$ if and only if in $\mathrm{PG}(4,q^2)$ the generator lines g, g' of $\mathscr{S}$ lie on the ruled cubic surface $\bar{V}_2^3$.*

In Section 2, we review in detail the properties of a ruled cubic surface and provide an alternative proof of the following result.

Theorem 1.1 (Bernasconi and Vincenti [4]). *The ruled cubic surfaces V_2^3 in $\mathrm{PG}(4,q)$ represent Baer subplanes of π in the André/Bruck and Bose representation if and only if π is the Desarguesian plane $\mathrm{PG}(2,q^2)$.*

In Section 4, we apply Theorem 3.1 to obtain a geometric proof of the recent result of [18] concerning Hermitian unitals which was originally proved in the coordinate setting. (See Section 4 for the definitions of *unital*, *orthogonal Buekenhout–Metz unital* and *Hermitian unital*.)

Theorem 4.3 (Metsch [18]). *Let $\mathscr{U}$ be an orthogonal Buekenhout–Metz unital in $\mathrm{PG}(2,q^2)$, defined in $\mathrm{PG}(4,q)$ by use of a regular spread $\mathscr{S}$ in a hyperplane Σ_∞*

and an elliptic quadric cone $\mathscr{U}^*$ *given by a quadratic form* $f(x)=0$. *If g is a generator line of* $\mathscr{S}$ *in* $\mathrm{PG}(4,q^2)$, *then* $\mathscr{U}$ *is Hermitian if and only if g lies on the hyperbolic cone* $\mathscr{U}^*$ *of* $\mathrm{PG}(4,q^2)$ *corresponding to* f.

2. Preliminary results

Firstly, we review the André/Bruck and Bose representation [1,7,8] for a translation plane π of order q^2 with kernel containing $\mathrm{GF}(q)$ and the associated known results concerning Baer subplanes and Baer sublines in π. The following construction is also discussed in [15, Section 17.7]. We shall refer to this representation as the *André/Bruck and Bose representation of* π *in* $\mathrm{PG}(4,q)$.

Let Σ_∞ be a hyperplane of $\mathrm{PG}(4,q)$ and let $\mathscr{S}$ be a spread of Σ_∞. Consider the incidence structure whose *points* are the points of $\mathrm{PG}(4,q)\backslash\Sigma_\infty$, *lines* are the planes of $\mathrm{PG}(4,q)$ which do not lie in Σ_∞ but which meet Σ_∞ in a line of $\mathscr{S}$ and *incidence* is natural. This incidence structure is an affine translation plane and can be completed to a projective translation plane π of order q^2 and kernel containing $\mathrm{GF}(q)$ by adjoining the line at infinity ℓ_∞ whose points are the elements of the spread $\mathscr{S}$. The line ℓ_∞ is a translation line for π. The resulting translation plane is Desarguesian if and only if the spread $\mathscr{S}$ is regular [7].

In what follows, we shall use the phrase *a subspace of* $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ to mean a subspace of $\mathrm{PG}(4,q)$ which is not contained in Σ_∞. Let B be a Baer subplane of π which meets ℓ_∞ in $q+1$ points, then B is called an *affine* Baer subplane of π; the remaining Baer subplanes of π each meet ℓ_∞ in a unique point and shall be called *non-affine* Baer subplanes of π.

In the above representation, planes of $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ that do not contain a line of the spread $\mathscr{S}$ correspond to affine Baer subplanes of π. Consequently, any line of $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ (which therefore meets Σ_∞ in a unique point) corresponds to a Baer subline of π that meets ℓ_∞ in a point. In the case π is the Desarguesian plane $\mathrm{PG}(2,q^2)$ it can be shown using a counting argument that the converse of these results hold. That is, for a fixed line ℓ_∞ in $\mathrm{PG}(2,q^2)$, the $(q^4+q^3+q^2+q+1)q(q^2+1)$ Baer subplanes of $\mathrm{PG}(2,q^2)$ which have ℓ_∞ as a line correspond to the $(q^4+q^3+q^2+q+1)q(q^2+1)$ planes of $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ in the André/Bruck and Bose representation. Note that for certain non-Desarguesian translation planes π there exist examples of affine Baer subplanes of π which do not correspond to a plane in the André/Bruck and Bose representation in $\mathrm{PG}(4,q)$. In [13], it is shown for certain translation planes of order q^4 that there exists an affine Desarguesian Baer subplane (of order q^2) which corresponds to a Baer subspace of $\mathrm{PG}(4,q^2)$ in the André/Bruck and Bose representation. Indeed for π, a general non-Desarguesian translation plane, the possible André/Bruck and Bose representations of Baer subplanes of π have not been classified. For more examples see the survey article [23].

In the case π is $\mathrm{PG}(2,q^2)$, the André/Bruck and Bose representation in $\mathrm{PG}(4,q)$ of the Baer subplanes and Baer sublines in $\mathrm{PG}(2,q^2)$ is classified and it remains to state the results for non-affine Baer subplanes and for Baer sublines which are disjoint from ℓ_∞.

If X denotes a substructure in π, it will be convenient at times to denote by X^* the substructure in PG$(4,q)$ which is the André/Bruck and Bose representation of X. Conversely, if X^* is a substructure in PG$(4,q)$, we shall denote by X the subset of points and lines of π which corresponds to X^* in the André/Bruck and Bose representation.

Note that every line in PG$(2,q^2)$ is a translation line. Therefore for ℓ_∞ any fixed line in PG$(2,q^2)$, there exists a André/Bruck and Bose representation of PG$(2,q^2)$ in PG$(4,q)$, as above, with $\mathscr{S}$ a regular spread in Σ_∞. To prove our results, we work with the André/Bruck and Bose representation of PG$(2,q^2)$, that is, the representation given above for a given fixed line ℓ_∞ of PG$(2,q^2)$ and a fixed regular spread $\mathscr{S}$ of Σ_∞.

It is well known (see for example [3]) that any Baer subline b, containing no point on ℓ_∞, of a line ℓ in PG$(2,q^2)$ is represented by an irreducible conic C in the plane ℓ^* corresponding to ℓ in the André/Bruck and Bose representation. The conics in PG$(4,q)\backslash\Sigma_\infty$ which represent Baer sublines in PG$(2,q^2)$ in this way shall be called *Baer conics*. Note that a Baer conic is necessarily disjoint from Σ_∞ in PG$(4,q)$.

If B is a Baer subplane in PG$(2,q^2)$ such that B intersects ℓ_∞ in a unique point P, then B is represented by a certain ruled cubic surface B^* in PG$(4,q)$; for such a ruled cubic surface, the spread element P^* is contained in B^* and in PG$(4,q)$ the hyperplane Σ_∞ intersects B^* in exactly P^* (see [5,22] or the alternative proof given in [20]). Note that in [5] the ruled cubic surface is called a *twisted ladder*.

A ruled cubic surface V_2^3 in PG$(4,q)$ is constructed as follows: let C be an irreducible conic and let P^* be a line skew to the plane containing C in PG$(4,q)$. Denote by θ,ϕ the non-homogeneous coordinates of points on P^* and C, respectively. Associate θ and ϕ with a projectivity H (so that $\phi=H(\theta)$), join corresponding points and so obtain a set of $q+1$ pairwise disjoint lines in PG$(4,q)$. Call these lines *generator lines* of the resulting surface V_2^3. The line P^* contained in V_2^3 is the unique line incident with each generator line; P^* is called the *line directrix* of V_2^3. The conic C has one point in common with each generator line and is called a *conic directrix* of V_2^3; there exist q^2 conic directrices on the surface V_2^3 in PG$(4,q)$ (see [22,4]). Note that the projectivity $H\in \mathrm{PGL}(2,q)$ is determined uniquely by the images of three distinct points (see [14]), that is if three distinct generator lines are chosen, then the remaining generator lines are uniquely determined. The surface V_2^3 in PG$(4,q)$ naturally extends to a ruled cubic surface $\bar{V}_2^3$ in PG$(4,q^2)$ by taking $\theta\in \mathrm{GF}(q^2)\cup\{\infty\}$ and therefore $\phi\in\mathrm{GF}(q^2)\cup\{\infty\}$. Then V_2^3 is properly contained in $\bar{V}_2^3$ and on restriction of θ to GF$(q)\cup\{\infty\}$ we obtain V_2^3 from $\bar{V}_2^3$. The Fröbenius automorphism of GF(q^2) given by $x\mapsto x^q$ induces a collineation σ of PG$(4,q^2)$ which fixes PG$(4,q)$ pointwise and hence V_2^3 is fixed pointwise. Since GF$(q^2)\backslash$GF(q) is permuted by the Fröbenius automorphism, and $H\in\mathrm{PGL}(2,q)$, the generator lines of $\bar{V}_2^3$ are permuted by the collineation σ and thus $\bar{V}_2^3$ is fixed by σ. Moreover, the generator lines of $\bar{V}_2^3$ not contained in PG$(4,q)$ are partitioned into pairs of conjugate generators with respect to the collineation σ. In particular, note that the generator lines of $\bar{V}_2^3$ are either contained in PG$(4,q)$ or are disjoint from PG$(4,q)$. We call $\bar{V}_2^3$ the *quadratic extension* of V_2^3.

For the fixed regular spread $\mathscr{S}$ in Σ_∞, since there exist irreducible conics in PG$(4,q)\backslash\Sigma_\infty$ which do not represent Baer sublines in the André/Bruck and Bose correspondence (see [19]), it follows that there exist ruled cubic surfaces, with line directrix an element of $\mathscr{S}$, which do not correspond to Baer subplanes of PG$(2,q^2)$; those which do represent Baer subplanes of PG$(2,q^2)$ shall be called *Baer ruled cubics*.

In [4], Theorem 1.1 is proved; we restate this result as Theorem 2.1 below; here we provide an alternative proof.

Theorem 2.1 (Bernasconi and Vincenti [4]). *If V_2^3 is a ruled cubic surface in* PG$(4,q)$ *with line directrix P^* such that in* PG$(4,q)$ *the hyperplane Σ_∞ meets V_2^3 in exactly P^*, then V_2^3 induces a unique spread $\mathscr{S}$ in Σ_∞ such that V_2^3 is a Baer subplane of the translation plane π in the resulting André/Bruck and Bose representation. Moreover, $\mathscr{S}$ is necessarily a regular spread, so that π is necessarily the Desarguesian plane* PG$(2,q^2)$.

Proof. Let $\mathscr{S}$ be the collection of q^2+1 lines in Σ_∞ consisting of P^* and each line $\ell_i=\alpha_i\cap\Sigma_\infty$ $(i=1,\dots,q^2)$, where α_i is the plane of a conic directrix C_i of V_2^3. Since P^* is skew to each plane α_i, the line P^* is skew to each line ℓ_i. Suppose for $i\neq j$, two planes α_i and α_j span a hyperplane Σ, then the generator lines which join points of C_i and points of C_j then lie in Σ and so P^* lies in Σ; a contradiction, as P^* is skew to each plane α_i. It follows that $\mathscr{S}$ is a spread in Σ_∞, the unique spread induced in Σ_∞ by V_2^3.

Let C be a conic directrix of V_2^3 and let ℓ be the line of intersection of the plane of C and Σ_∞. Then ℓ is an external line of C and so meets C in a pair of conjugate points $\{X,X'\}$ in PG$(4,q^2)$. Let $\bar{C}$ denote the (unique) irreducible conic in PG$(4,q^2)$ which contains C. Then $\bar{C}$ is a conic directrix of the ruled cubic surface $\bar{V}_2^3$ in PG$(4,q^2)$ which is the quadratic extension of V_2^3. Let g,g' denote the generator lines of $\bar{V}_2^3$ incident with X and X', respectively. Being generator lines of $\bar{V}_2^3$ we have that g,g' are pairwise skew. Note also that since g,g' each contain points outside PG$(4,q)$, they are disjoint from PG$(4,q)$ by the construction of $\bar{V}_2^3$ mentioned earlier.

Similarly, for any conic directrix C_i of V_2^3, C_i is contained in a conic directrix $\bar{C}_i$ of $\bar{V}_2^3$ in PG$(4,q^2)$. Since a conic directrix of $\bar{V}_2^3$ meets each generator line of $\bar{V}_2^3$, we have that $\bar{C}_i$ $(i=1,\dots,q^2)$ is incident with generators g and g' in PG$(4,q^2)$. Note that g and g' are incident with P^* (since P^* is the line directrix of V_2^3) and are incident with ℓ (from above) and so g and g' are contained in the quadratic extension $\bar{\Sigma}_\infty$ of Σ_∞. Hence for C_i any conic directrix of V_2^3, we have $\bar{C}_i\cap\bar{\Sigma}_\infty=\bar{\ell}_i\cap\{g,g'\}$. Thus, the lines g,g', which are conjugate with respect to the quadratic extension, each meet every line of $\mathscr{S}$. By [6], $\mathscr{S}$ is a regular spread and g,g' are the unique pair of generator lines of $\mathscr{S}$.

$\mathcal{S}$ therefore defines a André/Bruck and Bose representation of $\mathrm{PG}(2,q^2)$ [7] and V_2^3 corresponds to a collection B of q^2+q affine points of $\mathrm{PG}(2,q^2)$ together with a unique point P on ℓ_∞. By counting the lines incident with points in B, it follows that B is a (q^2+q+1)—blocking set in $\mathrm{PG}(2,q^2)$ and therefore B is a Baer suplane [14]. $\square$

3. A characterisation of Baer ruled cubic surfaces

Continuing with the notation of Section 2, let $\mathrm{PG}(2,q^2)$ have André/Bruck and Bose representation for a fixed line ℓ_∞ and a fixed regular spread $\mathcal{S}$ in Σ_∞. Let $\{g,g'\}$ (lines in the quadratic extension $\bar{\Sigma}_\infty$ of Σ_∞) denote the generator lines of $\mathcal{S}$.

Theorem 3.1. *Let V_2^3 be a ruled cubic surface in $\mathrm{PG}(4,q)$ with line directrix P^*, such that P^* is an element of a fixed regular spread $\mathcal{S}$ in Σ_∞ and in $\mathrm{PG}(4,q)$ the hyperplane Σ_∞ intersects V_2^3 in exactly the line P^*. Let $\bar{V}_2^3$ be the ruled cubic surface in $\mathrm{PG}(4,q^2)$ which is the quadratic extension of V_2^3. Then V_2^3 is a Baer ruled cubic surface in this André/Bruck and Bose representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(4,q)$ if and only if in $\mathrm{PG}(4,q^2)$ the generator lines g,g' of S lie on the ruled cubic surface $\bar{V}_2^3$.*

Proof. By Theorem 1.1 (see also the proof of Theorem 2.1), it remains to be proved that the number of ruled cubic surfaces in $\mathrm{PG}(4,q)$ which have line directrix, a line of the fixed regular spread $\mathcal{S}$ and which contain g and g' (the generator lines of $\mathcal{S}$) as generator lines of the ruled cubic surface in the quadratic extension, is equal to the number of non-affine Baer subplanes of $\mathrm{PG}(2,q^2)$. The number of non-affine Baer subplanes of $\mathrm{PG}(2,q^2)$, is $q^4(q^4-1)$.

Choose an element P^* of the spread $\mathcal{S}$; there are q^2+1 choices for P^*. Choose a plane α in $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ about a spread element distinct from P^*; there are q^4 choices. Choose an irreducible conic C in α, such that C contains the two points $\alpha\cap\{g,g'\}$ in $\mathrm{PG}(4,q^2)$; there are q^3-q^2 choices. It remains to determine the projectivity associating the non-homogeneous points of P^* in $\mathrm{PG}(4,q)$ with those of C. Choose a line m in $\mathrm{PG}(4,q)$ joining a fixed point of P^* to a point of C; there are $q+1$ choices. Then, the three distinct generators g,g',m determine a unique projectivity $H\in\mathrm{PGL}(2,q^2)$ which therefore determines a unique ruled cubic surface with line directrix $\bar{P}^*$ and conic directrix $\bar{C}$ in $\mathrm{PG}(4,q^2)$; moreover, since g and g' are conjugate with respect to the quadratic extension, the set $\{g,g',m\}$ is fixed by the collineation induced by the Fröbenius automorphism. Hence $H\in\mathrm{PGL}(2,q)$, so that the ruled cubic surface so determined is a V_2^3 contained in $\mathrm{PG}(4,q)$. Note that a ruled cubic surface in $\mathrm{PG}(4,q)$ contains q^2 conic directrices; thus, q^2 such conics will determine the same ruled cubic surface. Hence, we have the required number of ruled cubic surfaces. $\square$

Theorem 3.2. *Let $\mathcal{S}$ be a fixed regular spread in a hyperplane Σ_∞ of $\mathrm{PG}(4,q)$ in the André/Bruck and Bose representation of $\mathrm{PG}(2,q^2)$. Denote by g,g' the pair of* *conjugate generator lines of $\mathscr{S}$ in the quadratic extension of Σ_∞. Let C be an irreducible conic in a plane α of* $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ *about a spread element. Then C is a Baer conic if and only if C contains the two points $\alpha\cap\{g,g'\}$ in* $\mathrm{PG}(4,q^2)$.

Remarks. Recall that the points of the line $\mathrm{PG}(1,q^2)$ together with the Baer sublines of $\mathrm{PG}(1,q^2)$, with incidence given by inclusion, is a Miquelian inversive plane I of order q (see [11] for definitions and fundamental results on inversive planes). For X_∞ a fixed point of I, the points of $I\backslash X_\infty$ together with the circles of I which contain X_∞, with incidence inherited from I, form the points and lines, respectively, of a Desarguesian affine plane aff(α) of order q; this plane is called the *internal plane of I at X_∞*. By the *plane model of inversive planes* (see [21]), the Baer sublines of $\mathrm{PG}(1,q^2)$ which do not contain X_∞ correspond to a certain collection of irreducible conics in aff(α) which contain a conjugate pair of fixed points on the line at infinity of aff(α). By Theorem 3.1 and Corollary 3.2, we can visualise this correspondence for lines in $\mathrm{PG}(2,q^2)$ and the corresponding plane in the André/Bruck and Bose representation; moreover, for a fixed regular spread, precisely which conics in $\mathrm{PG}(4,q)$ are Baer conics has been determined.

4. An application concerning Hermitian curves

A *unital* in a projective plane π of order q^2 is a set $\mathscr{U}$ of q^3+1 points of π such that every line of the plane meets $\mathscr{U}$ in 1 or $q+1$ points. A line is a *tangent line* or a *secant line* of $\mathscr{U}$ if it contains 1 or $q+1$ points of $\mathscr{U}$, respectively.

In the Desarguesian plane $\mathrm{PG}(2,q^2)$, a class of unitals called the *classical unitals* (or *Hermitian curves*) consist of unitals arising from the absolute points and non-absolute lines of a unitary polarity. The classical unitals are projectively equivalent and any classical unital is the image under an element of $P\Gamma L(3,q^2)$ of the set of points (x_0,x_1,x_2) satisfying the equation $x_0^qx_0+x_1^qx_1+x_2^qx_2=0$.

Buekenhout [9] proved the existence of unitals in all translation planes of order q^2 with kernel containing $\mathrm{GF}(q)$. The construction is as follows; we continue with the notation of Section 2. Recall that an ovoid in $\mathrm{PG}(3,q)$ is a set of q^2+1 points none three collinear. In the André/Bruck and Bose representation of a translation plane π, for any spread $\mathscr{S}$ of Σ_∞: Let V and W be points on a line t of $\mathscr{S}$, and let $\mathscr{O}$ be an ovoid in some 3-dimensional subspace of $\mathrm{PG}(4,q)$ not containing V and such that $\mathscr{O}$ meets Σ_∞ in exactly the point W. Let $\mathscr{U}^*$ be the union of the lines joining V to each point of $\mathscr{O}$. Then the set $\mathscr{U}$ of points of π corresponding to $\mathscr{U}^*$ is a *Buekenhout–Metz unital with respect to ℓ_∞*. Note that ℓ_∞ is tangent to $\mathscr{U}$ at a point T. If the base ovoid $\mathscr{O}$ is an elliptic quadric (so $\mathscr{U}^*$ is an elliptic quadric cone) we call $\mathscr{U}$ an *orthogonal Buekenhout–Metz unital with respect to ℓ_∞*.

The classical unital in $\mathrm{PG}(2,q^2)$ can be constructed by the above method of Buekenhout and so is included in the class of Buekenhout–Metz unitals in $\mathrm{PG}(2,q^2)$. The classical unital is Buekenhout–Metz with respect to each of its tangent lines. In [9] it was shown that the above construction gave rise to a family of non-classical unitals in $\mathrm{PG}(2,q^2)$, where $q>2$ is even and a non-square (by taking $\mathcal{O}$ to be the Tits ovoid). Metz [19] in 1979 showed that Buekenhout's construction provides examples of non-classical unitals in $\mathrm{PG}(2,q^2)$ for all $q>2$ by constructing elliptic quadric cones containing conics which were not Baer conics for a fixed regular spread. All known unitals in $\mathrm{PG}(2,q^2)$ arise from this ovoidal cone construction.

Note that in our definition a Buekenhout–Metz unital arises from an ovoidal cone in $\mathrm{PG}(4,q)$ with base *any* ovoid in $\mathrm{PG}(3,q)$. This terminology is in contrast to some authors who reserve the term Buekenhout–Metz unital for only those unitals arising from an ovoidal cone with base an elliptic quadric (which we term as *orthogonal Buekenhout–Metz unitals*).

The classes of classical and Buekenhout–Metz unitals have been characterised in the class of unitals by many authors. We quote three examples (see also [2,10,16,20]).

Theorem 4.1 (Faina and Korchmáros [12]; Lefèvre-Percsy [17]). *Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q^2)$, where $q>2$. If each line of the plane which is secant to $\mathcal{U}$ intersects it in a Baer subline, then $\mathcal{U}$ is classical.*

Lefèvre–Percsy [17] proved a characterisation of the classical unitals among the class of orthogonal Buekenhout–Metz unitals in $\mathrm{PG}(2,q^2)$; this was recently improved by Barwick and Quinn.

Theorem 4.2 (Barwick and Quinn [3]). *Let $\mathcal{U}$ be a Buekenhout–Metz unital with respect to a tangent line ℓ_∞ in $\mathrm{PG}(2,q^2)$. If there is a secant line not through $\mathcal{U}\cap\ell_\infty$ that meets $\mathcal{U}$ in a Baer subline, then $\mathcal{U}$ is classical.*

The following is another characterisation of classical unitals among the class of orthogonal Buekenhout–Metz unitals in $\mathrm{PG}(2,q^2)$ by Metsch [18]; we provide an alternative and geometric proof using our results of Section 3.

Theorem 4.3 (Metsch [18]). *Let $\mathcal{U}$ be an orthogonal Buekenhout–Metz unital in $\mathrm{PG}(2,q^2)$, defined in $\mathrm{PG}(4,q)$ by use of a regular spread $\mathcal{S}$ in a hyperplane Σ_∞ and an elliptic quadric cone $\mathcal{U}^*$ given by a quadratic form $f(x)=0$. If g is a generator line of $\mathcal{S}$ in $\mathrm{PG}(4,q^2)$, then $\mathcal{U}$ is Hermitian if and only if g lies on the hyperbolic cone $\bar{\mathcal{U}}^*$ of $\mathrm{PG}(4,q^2)$ corresponding to f.*

Proof. Suppose $\mathcal{U}$ is a Hermitian (classical) unital, then by Theorem 4.1 secant lines meet $\mathcal{U}$ in Baer sublines. Thus in the André/Bruck and Bose representation the elliptic quadric cone $\mathcal{U}^*$ contains many Baer conics and so by Corollary 3.2, in $\mathrm{PG}(4,q^2)$, the hyperbolic cone $\bar{\mathcal{U}}^*$ contains more than two distinct points of g. It follows that the line g is contained in the quadric cone $\bar{\mathcal{U}}^*$.

Conversely, suppose that the hyperbolic cone $\bar{\mathcal{U}}^*$ in $\mathrm{PG}(4,q)$ contains the generator line g of the regular spread $\mathcal{S}$. Since $\bar{\mathcal{U}}^*$ is the quadratic extension of elliptic cone

$\mathscr{U}^*$ in PG$(4,q)$, it follows that the conjugate line g' of g in PG$(4,q)$ is also contained in $\bar{\mathscr{U}}^*$. Let α be a plane of PG$(4,q)$ about a spread element ℓ $(\neq t)$. Then $\alpha \cap \mathscr{U}^*$ is an irreducible conic C such that in PG$(4,q^2)$ the points $\bar{\ell} \cap \{g,g'\} \subset \bar{\alpha} \cap \bar{\mathscr{U}}^*$ are contained in C. By Corollary 3.2, C is then a Baer conic. We have that $\mathscr{U}$ is an orthogonal Buekenhout–Metz unital such that there exists a secant line not through T which meets $\mathscr{U}$ in a Baer subline and so by Theorem 4.2, $\mathscr{U}$ is a classical unital. □

References

[1] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe, Math. Z. 60 (1954) 156–186.
[2] S.G. Barwick, A characterization of the classical unital, Geom. Dedicata 52 (1994) 175–180.
[3] S.G. Barwick, C.T. Quinn, Generalising a characterisation of Hermitian curves, J. Geom. 70 (2001) 1–7.
[4] C. Bernasconi, R. Vincenti, Spreads induced by varieties $\mathscr{V}_2^3$ of PG$(4,q)$ and Baer subplanes, Boll. Un. Mat. Ital. B 18 (5) (1981) 821–830.
[5] R.C. Bose, J.W. Freeman, D.G. Glynn, On the intersection of two Baer subplanes in a finite projective plane, Utilitas Math. 17 (1980) 65–77.
[6] R.H. Bruck, Construction problems in finite projective planes, Conference on Combinatorial Mathematics and its Applications, University of North Carolina, 1967, University of North Carolina Press, Chapel Hill, 1969.
[7] R.H. Bruck, R.C. Bose, The construction of translation planes from projective spaces, J. Algebra 1 (1964) 85–102.
[8] R.H. Bruck, R.C. Bose, Linear representations of projective planes in projective spaces, J. Algebra 4 (1966) 117–172.
[9] F. Buekenhout, Existence of unitals in finite translation planes of order q^2 with a kernel of order q, Geom. Dedicata 5 (1976) 189–194.
[10] L.R.A. Casse, C.M. O'Keefe, T. Penttila, Characterizations of Buekenhout–Metz unitals, Geom. Dedicata 59 (1996) 29–42.
[11] P. Dembowski, Finite Geometries, Springer, New York, 1968.
[12] G. Faina, G. Korchmáros, A graphic characterization of Hermitian curves, Ann. Discrete Math. 18 (1983) 335–342.
[13] J.W. Freeman, Reguli and pseudoreguli in PG$(3,s^2)$, Geom. Dedicata 9 (1980) 267–280.
[14] J.W.P. Hirschfeld, Projective Geometries Over Finite Fields, Clarendon Press, Oxford, 1979.
[15] J.W.P. Hirschfeld, Finite Projective Spaces of Three dimensions, Clarendon Press, Oxford, 1985.
[16] C. Lefèvre-Percsy, Characterization of Buekenhout–Metz unitals, Arch. Math. 36 (1981) 565–568.
[17] C. Lefèvre-Percsy, Characterization of Hermitian curves, Arch. Math. 39 (1982) 476–480.
[18] K. Metsch, A note on Buekenhout–Metz unitals, Geometry, combinatorial designs and related structures (Spetses, 1996), London Mathematical Society Lecture Note Series, Vol. 245, Cambridge University Press, Cambridge, 1997, pp. 177–180.
[19] R. Metz, On a class of unitals, Geom. Dedicata 8 (1979) 125–126.
[20] C.T. Quinn, L.R.A. Casse, Concerning a characterisation of Buekenhout–Metz unitals, J. Geom. 52 (1995) 159–167.
[21] J.A. Thas, Solution of a classical problem on finite inversive planes, in: Finite Geometries, Buildings and Related Topics, Pingree Park, CO, 1988, Oxford Sci. Publ., OUP, New York, 1990, pp. 145–159.
[22] R. Vincenti, Alcuni tipi di varieta $\mathscr{V}_2^3$ di $S_{4,q}$ e sottopiani di Baer, Boll. Un. Mat. Ital. Suppl. 2 (1980) 31–44.
[23] R. Vincenti, A survey on varieties of PG$(4,q)$ and Baer subplanes of translation planes, Ann. Discrete Math. 18 (1983) 775–780.